Exact Differential Equation

$$Mdx + Ndy = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The solution was

$$\boxed{\int M dx + \int N dy = c}$$

neglecting terms in x

Relation between trigonometric & hyperbolic f^ns

$\sin(i\theta) = i \sinh\theta$

$\cos(i\theta) = \cosh\theta$

$\sinh(i\theta) = i \sin\theta$

$\cosh(i\theta) = \cos\theta$

Ex For what values of z does the function $z = \sin w$ ceases to be analytic?

Soln For function to be analytic, the necessary condition is that it should be differentiable.

$\because \ w = \sin^{-1} z \quad \frac{dw}{dz} = \frac{1}{\sqrt{1-z^2}}$

The derivative is not defined at $z = \pm 1$. Hence, the function is ~~an~~ not analytic at $z = \pm 1$

Orthogonal Functions Real and imaginary parts of analytic function form 2 families of curves which form orthogonal system.

Pf Let $f(z) = u + iv$ Let family of curves be $u(x,y) = c_1$ —(1)

and $v(x,y) = c_2$ —(2)

Differentiate (1), $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = \frac{-u_x}{u_y} = m_1$

Differentiate (2), $\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = \frac{-v_x}{v_y} = m_2$

Now, product of ~~both~~ slopes is

$m_1 \cdot m_2 = \left(\frac{-u_x}{u_y}\right) \cdot \left(\frac{-v_x}{v_y}\right) = \left(\frac{-u_x}{u_y}\right)\left(\frac{u_y}{u_x}\right) = -1$

$(\because \ u_x = v_y \ \& \ u_y = -v_x)$
C-R eqns.

$\Rightarrow u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal //

Methods to find conjugate function.

Method I: Using C-R eqns.

Suppose $u \to$ given and $f(z) = (u+iv)$ is analytic fn

To find: v and hence $f(z)$

(I) From C-R eqn

$V_y = U_x$ → known as u is given

$\therefore V = \int U_x \, dy + \phi(x)$ —①

(II) 2nd C-R eqn is $V_x = -U_y$ — known as u is given

differentiate ① partially w.r.t. 'x'

(III) $\Rightarrow \frac{\partial}{\partial x}\left(\int U_x \, dy\right) + \phi'(x) = -U_y$ → This will give $\phi'(x)$ in terms of x only or constant.

IV Find $\phi(x)$. Put it in ①.

Thus we have v.

$\therefore f(z) = u + iv$

* In a similar manner if v is ~~known~~ given, u can be found.
* You can use C-R eqns in any order.

Ex. ~~Show that the~~ Find harmonic conjugate of following

(1) $u = 2x(1-y)$

First check u is harmonic: $U_x = 2(1-y)$ $U_{xx} = 0$

$U_y = -2x$ $U_{yy} = 0$

$\Rightarrow \nabla^2 u = U_{xx} + U_{yy} = 0$

$\therefore u$ is harmonic

Now, 1st C-R eqn $V_y = U_x = 2(1-y)$

$\Rightarrow \therefore V = \int 2(1-y)\,dy + \phi(x)$

$= 2\left(y - \frac{y^2}{2}\right) + \phi(x)$ —①

$V_x = \phi'(x)$

From 2nd C-R eqn $U_y = -V_x$

$\Rightarrow -2x = -(\phi'(x))$ $\Rightarrow \phi'(x) = 2x$

$\Rightarrow \phi(x) = x^2 + c$

$\therefore \boxed{V = 2\left(y - \frac{y^2}{2}\right) + x^2 + c}$ — Ans

(2) $u = \frac{1}{2}\log(x^2+y^2)$

$u_x = \frac{x}{x^2+y^2}$ $\quad u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad u_y = \frac{y}{x^2+y^2}$ $\quad u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore \nabla^2 u = u_{xx}+u_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{(x^2-y^2)}{(x^2+y^2)^2} = 0 \Rightarrow u$ is harmonic

1st C-R eqn: $v_y = u_x = \frac{x}{x^2+y^2} \Rightarrow v = \int \frac{x}{x^2+y^2}\,dy + \phi(x)$
$= \tan^{-1}(y/x) + \phi(x)$

2nd C-R eqn: $u_y = -v_x$

$\Rightarrow \frac{y}{x^2+y^2} = -\left(-\frac{y}{x^2+y^2} + \phi'(x)\right)$ $\quad \Rightarrow \phi'(x) = 0$
or $\phi(x) = c$

$\therefore \boxed{v = \tan^{-1}(y/x) + c}$

(3) $u = x^3 - 3xy$

Cannot find harmonic conjugate (why?!).
(As fn is not harmonic)

Ex Let $v = \frac{1}{2}\log(x^2+y^2) + x - 2y$ be imaginary part of ~~an~~ analytic fn $f(z) = u+iv$. Find $f(z)$

Soln 1st C-R eqn $u_x = v_y = \frac{y}{x^2+y^2} - 2$

$\therefore u = \int \frac{y}{x^2+y^2}\,dx - 2x + \phi(y) = \left[\tan^{-1}(y/x) + \pi/2\right] - 2x + \phi(y)$ —— (*)

2nd C-R eqn: $u_y = -v_x$

$\frac{-x}{x^2+y^2} + \phi'(y) = -\left[\frac{x}{x^2+y^2} + 1\right]$ $\quad \Rightarrow \phi'(y) = -1$
or, $\phi(y) = -y + c$

$\therefore u = \frac{\pi}{2} - \tan^{-1}(y/x) - 2x - y + c$

$\therefore f(z) = \left[\frac{\pi}{2} - \tan^{-1}(y/x) - 2x - y + c\right] + i\left[\frac{1}{2}\log(x^2+y^2) + x - 2y\right]$

$= i\left[\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x) + (x-2y) + i(2x+y)\right] + \underbrace{\frac{\pi}{2} + c}_{c_1}$

$= i\ln z + z + 2iz + c_1 = i(\ln z + z) + z + c_1$ — Ans

(*) T.P.T $\int \frac{y}{x^2+y^2}dx = \frac{\pi}{2} - \tan^{-1}(y/x)$

$\frac{y}{y^2}\int \frac{dx}{1+\left(\frac{x^2}{y^2}\right)} = \frac{1}{y}\int \frac{y\,dt}{1+t^2}$ where $\frac{x}{y}=t$, $\therefore dx = y\,dt$

$= \tan^{-1}t = \tan^{-1}\left(\frac{x}{y}\right)$

Let $p=\tan^{-1}\left(\frac{x}{y}\right) \Rightarrow \tan p = \frac{x}{y}$ or, $\frac{y}{x} = \cot p = \tan\left(\frac{\pi}{2}-p\right)$

$\Rightarrow \frac{\pi}{2}-p = \tan^{-1}(y/x)$

or, $p = \pi/2 - \tan^{-1}(y/x)$ or, $\boxed{\tan^{-1}\left(\frac{x}{y}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{y}{x}\right)}$ //.

Method II: Given: u To find: v $f(z)=u+iv$ is analytic.

Then $dv = v_x dx + v_y dy$

$= -u_y dx + u_x dy$ (C-R eqns $u_x = v_y$ & $u_y = -v_x$)

$\therefore v = \int -u_y dx + \int u_x dy$ → neglect the terms in x

Similarly if v is given & to find u

$du = u_x dx + u_y dy$

$= v_y dx - v_x dy$

$\therefore u = \int v_y dx - \int v_x dy$

Ex Find analytic fn $f(z)=u+iv$ given

(1) $u = e^{-x}(x\sin y - y\cos y)$

$u_x = -e^{-x}(x\sin y - y\cos y) + e^{-x}\sin y$

$u_y = e^{-x}(x\cos y - \cos y + y\sin y)$

$\therefore v = \int -u_y dx + \int u_x dy = -\int e^{-x}(x\cos y - \cos y + y\sin y)dx$

$+ \int -e^{-x}(x\sin y - y\cos y - \sin y)dy$ (neglect)

$= -\cos y\int xe^{-x}dx - e^{-x}\cos y + e^{-x}y\sin y$

$= xe^{-x}\cos y + e^{-x}\cos y - e^{-x}\cos y + e^{-x}y\sin y$

∴ $V = e^{-x}(x\cos y + y\sin y)$

$f(z) = u + iv$

$= e^{-x}(x\sin y - y\cos y) + i\,e^{-x}(x\cos y + y\sin y)$

$= e^{-x}\sin y\,(x+iy) + e^{-x}\cos y\,(-y+ix)$

$= e^{-x}\sin y\cdot z + i\,e^{-x}\cos y\,\underbrace{(x+iy)}_{z}$ $\quad (i^2=-1)$

$= z e^{-x}(i\cos y + \sin y) = z e^{-x} i(\cos y - i\sin y)$

$= z e^{-x} i e^{-iy} = iz(e^{-x-iy}) = iz e^{-z}$ — Ans

(2) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$u_x = 6xy + 4x \qquad u_y = 3x^2 - 3y^2 - 4y$

$V = \int -u_y\,dx + \int u_x\,dy$ neglect terms in x

$= -\int(3x^2-3y^2-4y)\,dx + \int(\cancel{6xy+4x})\,dy$ neglect

$= -(x^3 - 3y^2x - 4xy)$

∴ $f(z) = u+iv = (3x^2y+2x^2-y^3-2y^2) + i(-x^3+3y^2x+4xy)$

$= -iz^3 + 2z^2$ — Ans ∴ (Put $x = z$ & $y = 0$)

(3) $V = \sinh x\cos y \qquad V_x = \cosh x\cos y \qquad V_y = -\sinh x\sin y$

$u = \int V_y\,dx - \int V_x\,dy$ neglect terms in x

$= \int -\sinh x\sin y\,dx \int \cosh x\cos y\,dy$ neglect

$= -\cosh x\sin y$

$f(z) = -\cosh x\sin y + i\sinh x\cos y$

$= i\sinh z$ (Put $x = z$ & $y = 0$)

$= \sin(iz)$ — Ans

Ex ~~u or v is given and~~

Milne Thomson Method: This method is used to find $f(z)$ from u or v [i.e. without finding the other part]

Step I: Write $f'(z) = u_x - iu_y$ or $f'(z) = v_y + iv_x$ depending on u is given or v is given

Step II: Replace x by z & y by zero $\therefore dx = dz$, $dy = 0$

Step III: Integrate both sides w.r.t z

Step IV: $f(z)$ is obtained

Ex Determine analytic function $f(z)$ whose given

(1) $v = \dfrac{x-y}{x^2+y^2}$ $\quad v_x = \dfrac{y^2-x^2+2xy}{(x^2+y^2)^2}$ $\quad v_y = \dfrac{y^2-x^2-2xy}{(x^2+y^2)^2}$

$\therefore f'(z) = v_y - iv_x = \left(\dfrac{y^2-x^2-2xy}{(x^2+y^2)^2}\right) + i\left(\dfrac{y^2-x^2+2xy}{(x^2+y^2)^2}\right)$

Put $x = z$, $y = 0$, $dx = dz$ $dy = 0$

$\therefore f'(z) = \dfrac{-z^2}{z^4} + i\left(\dfrac{-z^2}{z^4}\right) = -(1+i)\dfrac{1}{z^2}$

On integration we have

$f(z) = (1+i)\dfrac{1}{z} + c$ — Ans

(2) $v = x^2 - y^2 + 4y$ $\quad v_x = 2x$ $\quad v_y = -2y+4$

$\therefore f'(z) = v_y + iv_x = -2y + 4 + i2x$

Put $x = z$ & $y = 0$

$\therefore f'(z) = i2z + 4 \Rightarrow f(z) = iz^2 + 4z + c$

(3) $u = e^{-x}(x\cos y + y\sin y)$ $\quad u_x = -e^{-x}(x\cos y + y\sin y) + e^{-x}\cos y$

$u_y = e^{-x}(-x\sin y + \sin y + y\cos y)$

$f'(z) = u_x - iu_y$

$= [-e^{-z}(z-1)] - ie^{-z}(0)$

$\therefore f(z) = -[-ze^{-z} - e^{-z} - e^{-z}] + c$

$= e^{-z}(z+2) + c$ — Ans