To find analytic fn $f(z)$ given real or imaginary part in polar form:

C-R eqns in polar form $rU_r = V_\theta$ & $rV_r = -U_\theta$

$f'(z) = \frac{r}{z}[U_r + iV_r]$ $\quad$ ($f(z) = u+iv$)

Ex Find analytic function $f(z) = u+iv$ given:

(1) $u = -r^3 \sin 3\theta$ $\quad$ (2) $v = r^2\cos 2\theta - r\cos\theta$

Solution (1) $u = -r^3\sin 3\theta$. $\quad$ $dv = V_r dr + V_\theta d\theta$

$= \frac{-U_\theta}{r} dr + rU_r d\theta$.

$U_r = -3r^2 \sin 3\theta$

$U_\theta = -3r^3 \cos 3\theta$

$\therefore dv = \frac{3r^{\cancel{3}2}\cos 3\theta}{\cancel{r}} dr + r(-3r^2 \sin 3\theta) d\theta$

$\therefore v = 3\int r^2 \cos 3\theta dr - 3\int r^3 \sin 3\theta d\theta$
neglecting terms in r

$= \frac{\cancel{3}r^3}{\cancel{3}} \cos 3\theta + c$

$\therefore f(z) = u+iv = -r^3\sin 3\theta + ir^3\cos 3\theta + c$

$= ir^3(\cos 3\theta + i\sin 3\theta) + c = ir^3 e^{3i\theta} + c = iz^3 + c$ Ans

(2) $v = r^2\cos 2\theta - r\cos\theta$ $\quad$ $du = U_r dr + U_\theta d\theta$

$= \frac{V_\theta}{r} dr - rV_r d\theta$

$V_r = 2r\cos 2\theta - \cos\theta$

$V_\theta = -2r^2 \sin 2\theta + r\sin\theta$

$\therefore du = \left(\frac{-2r^{\cancel{2}}\sin 2\theta + \cancel{r}\sin\theta}{\cancel{r}}\right) dr - r(2r\cos 2\theta - \cos\theta) d\theta$

$\therefore \int du = -2\int r\sin 2\theta \, dr + \int \sin\theta \, dr - \int r(2r\cos 2\theta - \cos\theta) d\theta$
neglecting terms in r

$u = -r^2 \sin 2\theta + r\sin\theta + c$

$\therefore f(z) = (-r^2\sin 2\theta + r\sin\theta) + i(r^2\cos 2\theta - r\cos\theta) + ic$

$= ir^2(\cos 2\theta + i\sin 2\theta) - ir(\cos\theta + i\sin\theta) + c_1$

$= ir^2 e^{2i\theta} - ire^{i\theta} + c_1 = i(z^2 - z) + c_1$ Ans

Ex. Determine analytic function whose real given

(1) $u=\log\sqrt{x^2+y^2}$ (2) $V=\dfrac{x-y}{x^2+y^2}$

Soln (1) $u=\log\sqrt{x^2+y^2}$ $x=r\cos\theta$, $y=r\sin\theta$

$r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}(y/x)$

$\therefore u=\log r$

$dv = V_r dr + V_\theta d\theta = -\dfrac{u_\theta}{r}dr + r u_r d\theta$

$= 0\,dr + r\cdot\dfrac{1}{r}d\theta = d\theta$

$\therefore v=\int d\theta = \theta + c = \tan^{-1}(y/x)+c$

$\therefore f(z) = u+iv = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x) + c = \ln z + c$ — Ans

(2) $V=\dfrac{x-y}{x^2+y^2} = \dfrac{r(\cos\theta-\sin\theta)}{r^2} = \dfrac{\cos\theta-\sin\theta}{r}$

$du = u_r dr + u_\theta d\theta = \dfrac{V_\theta}{r}dr - rV_r d\theta$

$= \dfrac{1}{r}\left(\dfrac{-\sin\theta-\cos\theta}{r}\right)dr - r\left(\dfrac{\cos\theta-\sin\theta}{-r^2}\right)d\theta$

$u = -(\cos\theta+\sin\theta)\int\dfrac{dr}{r^2} + \int\left(\dfrac{\cos\theta-\sin\theta}{r}\right)d\theta$

neglect terms in r

$= \dfrac{\cos\theta+\sin\theta}{r} + c$

$= \dfrac{r(\cos\theta+\sin\theta)}{r^2} + c = \dfrac{x+y}{x^2+y^2} + c$

$\therefore f(z) = u+iv = \dfrac{x+y}{(x^2+y^2)} + i\left(\dfrac{x-y}{x^2+y^2}\right) + c = \dfrac{(1+i)}{z} + c$ — Ans

$\left[= \dfrac{x-iy}{x^2+y^2} + i\left(\dfrac{x-iy}{x^2+y^2}\right) + c = \dfrac{\bar{z}}{z\cdot\bar{z}} + i\dfrac{\bar{z}}{z\bar{z}} + c\right]$

Ex Find an analytic fn $f(z) = u(r,\theta) + i\,v(r,\theta)$ given

(1) $V = r^2\cos 2\theta - r\cos\theta$ (2) $u = \left(r+\dfrac{1}{r}\right)\cos\theta$

Soln $f'(z) = \dfrac{r}{z}[u_r + iV_r] = \dfrac{r}{z}\left[\dfrac{V_\theta}{r} + iV_r\right]$

$= \dfrac{r}{z}\left[\dfrac{-2r^2\sin 2\theta + r\sin\theta}{r} + i(2r\cos 2\theta - \cos\theta)\right]$

or, $f'(z) = \frac{r}{z}\left[(-2r\sin 2\theta + \sin\theta) + i(2r\cos 2\theta - \cos\theta)\right]$

put $r = z$ & $\theta = 0$.

$\therefore f'(z) = \frac{z}{z}\left[\cancel{-2z}\, 0 + i(2z-1)\right]$

or, $f(z) = i(z^2 - z) + c$ — Ans

(2) $u = (r + \frac{1}{r})\cos\theta$ $\quad f'(z) = \frac{r}{z}[u_r + iv_r] = \frac{r}{z}\left[u_r - i\frac{u_\theta}{r}\right]$

$= \frac{r}{z}\left[(1 - \frac{1}{r^2})\cos\theta + i\frac{\sin\theta}{r}(r + \frac{1}{r})\right]$

Put $r = z$ & $\theta = 0$

$\therefore f'(z) = (1 - \frac{1}{z^2})$ $\quad$ or $\quad f(z) = z + \frac{1}{z} + c$ — Ans

<u>To find analytic fn $f(z) = u + iv$ given $u+v$ or $u-v$</u>

$f(z) = u + iv$ $\quad if(z) = iu - v$

$\therefore \underbrace{(1+i)f(z)}_{F(z)} = \underbrace{(u-v)}_{U} + i\underbrace{(u+v)}_{V}$ $\quad \Rightarrow f(z) = \frac{F(z)}{1+i}$

Ex Given, $u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$, $f(\pi/2) = 0$. find $f(z)$

Here $U = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ $\quad \therefore U_x = \frac{(-\sin x + \cos x)(\cos x - \cosh y) + \sin x(\cos x + \sin x - e^{-y})}{2(\cos x - \cosh y)^2}$

$U_y = \frac{e^{-y}(\cos x - \cosh y) + \sinh y(\cos x + \sin x - e^{-y})}{2(\cos x - \cosh y)^2}$

Now, $F'(z) = U_x + iU_y$

put $x = z$ & $y = 0$, then

$F'(z) = \frac{(-\sin z + \cos z)(\cos z - 1) + \sin z(\cos z + \sin z - 1)}{2(\cos z - 1)^2}$

$+ i\left[\frac{(\cos z - 1) + 0}{2(\cos z - 1)^2}\right]$ $\quad (\because \sinh y = 0)$

$$= \frac{-\cancel{\sin z}(\cos z - 1) + \cos z(\cos z - 1) + \cancel{\sin z}(\cos z - 1) + \sin^2 z + i(\cos z - 1)}{2(\cos z - 1)^2}$$

$$= \frac{\cos^2 z - \cos z + \sin^2 z + i(\cos z - 1)}{2(\cos z - 1)^2}$$

$$= \frac{(1 - \cos z) - i(1 - \cos z)}{2(1 - \cos z)^2} = \frac{(1-i)}{2(1-\cos z)} = \frac{(1-i)}{4\sin^2 z/2}$$

$(\because \cos z = 1 - 2\sin^2 z/2)$

$$\therefore = \frac{(1-i)}{4} \operatorname{cosec}^2 z/2$$

$$\therefore F(z) = \frac{(1-i)}{4}\left(-\frac{\cot z/2}{1/2}\right) + c = \frac{-(1-i)}{2}\cot z/2 + c$$

$$f(z) = \frac{F(z)}{1+i} = \frac{-(1-i)}{2(1+i)}\cot z/2 + c_1 = \frac{i}{2}\cot z/2 + c$$

$f(\pi/2) = 0$ $(\because \cot \frac{\pi}{4}) = 1$

$\Rightarrow \boxed{c_1 = -\frac{i}{2}}$. $\therefore f(z) = \frac{i}{2}[\cot z/2 - 1]$ — Ans

Ex Given $u + v = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, find $f(z) = u + iv$
V ''   where $f(z) = \frac{F(z)}{1+i}$

$\therefore F'(z) = V_y + iV_x$

$$V_y = \frac{-2\sin 2x(2e^{2y} - 2e^{-2y})}{(e^{2y} + e^{-2y} - 2\cos 2x)^2}$$

$$V_x = \frac{4\cos 2x(e^{2y} + e^{-2y} - 2\cos 2x) - 4\sin 2x(2\sin 2x)}{(e^{2y} + e^{-2y} - 2\cos 2x)^2}$$

put $x = z$ & $y = 0$ in $F'(z)$ we have

$$F'(z) = 0 + 2i\left[\frac{\cos 2z - 1}{(1 - \cos 2z)^2}\right] = \frac{2i}{\cos 2z - 1} = \frac{2i}{-2\sin^2 z} = -i\operatorname{cosec}^2 z$$

or, $F(z) = i\cot z + c$ or, $f(z) = \frac{(1+i)}{2}\cot z + c$ — Ans