Exponential Function:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

Properties: (1) $(e^z)' = e^z$

(2) $e^{z_1+z_2} = e^{z_1}e^{z_2}$

(3) For $z = iy$ $\quad e^z = e^{iy} = \cos y + i\sin y$ ↳ Eulers Formula

(4) $|e^z| = |e^x e^{iy}| = |e^x| = e^x \quad (\because |e^{iy}| = 1)$

(5) $|e^{iy}| = \sqrt{\cos^2 y + \sin^2 y} = 1$

(6) $e^{z+2n\pi i} = e^z e^{2n\pi i} = e^z \quad (\because e^{2n\pi i} = 1)$

$\Rightarrow e^z$ is periodic fn with period $2n\pi i$

Ex when can $e^z = 0$? Reason*.

\# $|e^z| = e^x \quad \arg e^z = y \pm 2n\pi \quad n = 0, 1, 2, \ldots$

$\operatorname{Arg} e^z = y \quad -\pi < y \le \pi$.

Ex Find e^z (in form of $u+iv$) and $|e^z|$ if z equals.

(1) $2+3\pi i \quad$ (2) $1+i \quad$ (3) $-\pi i/2$

Soln (1) $z = 2+3\pi i \quad e^z = e^{2+3\pi i} = e^2 e^{3\pi i} = e^2(\cos 3\pi + i\sin 3\pi)$

$|e^z| = e^x = e^2$.

(2) $z = 1+i \quad e^z = e^{1+i} = e(\cos 1 + i\sin 1) \quad |e^z| = e$.

(3) $z = -\pi i/2 \quad e^z = e^{-i\pi/2} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} \quad |e^z| = 1 \quad (e^0 = 1)$

Ex Represent each of the following in exponential polar form (i.e, $z = re^{i\theta}$)

(1) $1+i \quad$ (2) $\sqrt[n]{z} \quad$ (3) $\sqrt{i}$

Soln $z = 1+i \quad r = \sqrt{1^2+1^2} = \sqrt{2} \quad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$

$\therefore z = \sqrt{2}e^{i\pi/4}$ — Answer

* $|e^z| = e^x \neq 0 \quad \therefore e^z \neq 0$

(2) $\sqrt[n]{z} = z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n}$ where $\theta = \tan^{-1}(y/x)$ & $r = \sqrt{x^2+y^2}$

(3) $\sqrt{i}$ $z = \sqrt{i} = (e^{i\pi/2})^{1/2}$ $\quad i = e^{i\pi/2}$

$= e^{i\pi/4}$

Ex Find all solutions of the following equations:

(1) $e^z = 1$ $\quad$ (2) $e^z = -3$ $\quad$ (3) $e^z = 4+3i$

Soln: (1) $e^z = 1$ $\quad$ $1 = e^x \Rightarrow x = \ln 1 = 0$

$e^z = 1 = e^{i(0 \pm 2n\pi i)}$ $\quad \Rightarrow e^x \cdot e^{iy} = 1 \cdot e^{i(0 \pm 2n\pi i)}$

$\Rightarrow e^x = 1 \quad e^{iy} = e^{i(0 \pm 2n\pi i)}$

$\Rightarrow x = \ln 1 = 0 \quad y = \pm 2n\pi i$

$\therefore z = 0 \pm 2n\pi i = \pm 2n\pi i \quad n = 0, 1, 2, \ldots$

$\hookrightarrow$ Ans

(2) $e^z = -3$

$e^z = 3e^{i\pi \pm 2n\pi i}$ $\quad \Rightarrow e^x = 3 \quad e^{iy} = e^{i(\pi \pm 2n\pi)}$

$\Rightarrow x = \ln 3 \quad y = \pi \pm 2n\pi, \quad n = 0, 1, 2, \ldots$

$\therefore z = \ln 3 + i(\pi \pm 2n\pi) \quad n = 0, 1, 2, \ldots$

$\hookrightarrow$ Ans

(3) $e^z = 4+3i$ $\quad$ $r = \sqrt{4^2+3^2} = \sqrt{25} = 5$

$e^z = 5e^{i(\theta \pm 2n\pi)}$ $\quad$ $\theta = \tan^{-1}(3/4)$

$\Rightarrow e^x = 5 \quad e^{iy} = e^{i(\theta \pm 2n\pi)}$ $\quad n = 0, 1, 2, \ldots$

$\Rightarrow x = \ln 5 \quad y = i(\theta \pm 2n\pi) = i(\tan^{-1}(3/4) \pm 2n\pi)$

$\therefore z = \ln 5 + i(\tan^{-1}(3/4) \pm 2n\pi) \quad n = 0, 1, 2, \ldots$

$\hookrightarrow$ Ans

$= \ln 5 + i(0.6435 \pm 2n\pi)$

(4) $e^z = 0$

$e^x \neq 0 \Rightarrow \cos y + i\sin y = 0 \Rightarrow \cos y = 0$ & $\sin y = 0$

$\hookrightarrow$ not possible

$\therefore$ No solution $\quad [\because e^z = e^x(\cos y + i\sin y)]$

Trigonometric functions

$\cos z = \frac{e^{iz}+e^{-iz}}{2}$ $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ $\cos(iz) = \cosh z$ $\sin(iz) = i\sinh z$

Derivatives & relations between trigonometric functions hold the way they were in real variables.

$$\cos z = \cos(x+iy) = \cos x\cos(iy) - \sin x\sin(iy)$$
$$= \cos x\cosh y - i\sin x\sinh y$$

$$\sin z = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy)$$
$$= \sin x\cosh y + i\cos x\sinh y$$

Ques (1) What will be periodicity of $\cos z$ & $\sin z$?

(2) Are $\sin z$ and $\cos z$ bounded?

(Like $|\sin x| \leq 1$ & $|\cos x| \leq 1$)

Soln: (1) $\cos(z+2\pi) = \cos[(x+2\pi)+iy] = \cos(x+2\pi)\cosh y - i\sin(x+2\pi)\sinh y$

$= \cos x\cosh y - i\sin x\sinh y$ [$\because$ $\cos x$ & $\sin x$ are periodic fns with period 2π)

$= \cos z$

Similarly $\sin(z+2\pi) = \sin z$

$\Rightarrow$ $\cos z$ & $\sin z$ are periodic fns with period 2π (like real case)

(2) $|\sin z|^2 = (\sin x\cosh y)^2 + (\cos x\sinh y)^2$

$= \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$ ($\cosh^2 y = 1+\sinh^2 y$)

$= \sin^2 x(1+\sinh^2 y) + \cos^2 x\sinh^2 y$

$= \sin^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y$

$= \sin^2 x + \sinh^2 y$

$|\sin x| \leq 1$ But $\sinh^2 y \to \infty$ as $y \to \infty$.

$\therefore$ $\sin z$ is not bounded.

Similarly you can prove $\cos z$ is not bounded.

[$\sinh y = \frac{e^y - e^{-y}}{2}$
$= \frac{e^{2y}-1}{2e^y}$

As $y \to \infty$

$\lim_{y\to\infty} \sinh y = \lim_{y\to\infty} \frac{e^{2y}}{2e^y}$
$= \lim_{y\to\infty} e^y = \infty$

(32)

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$
$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_1$$
& $$\cos^2 z + \sin^2 z = 1$$

These hold for Complex variables

Ex Find in form of $u+iv$

(1) $\cos(1+i)$ (2) $\sin\left(\frac{\pi}{2} - \pi i\right)$

Soln (1) $\cos(1+i) = \cancel{\cos 1 \cosh 1 + i \sin 1 \sinh 1}$
$= \cos 1 \cosh 1 - i \sin 1 \sinh 1$

(2) $\sin\left(\frac{\pi}{2} - \pi i\right) = \sin\frac{\pi}{2} \cosh \pi - i \cos\frac{\pi}{2} \sinh \pi$
$= \cosh \pi$ $\left(\because \cos\frac{\pi}{2} = 0\right)$

$\cos(\pi i)$ $\left(\because \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right)$
$= \cosh \pi$

Ex Find all solutions of following equations:

(1) $\cos z = 3i$ (2) $\sin z = 1000$

Soln (1) $\cos z = 3i \Rightarrow \frac{e^{iz} + e^{-iz}}{2} = 3i \Rightarrow e^{2iz} - 6ie^{iz} + 1 = 0$

$\therefore e^{iz} = \frac{6i \pm \sqrt{-36-4}}{2} = \frac{6i \pm 2i\sqrt{10}}{2} = 3i \pm i\sqrt{10}$

$e^{ix} e^{-y} = i(3 \pm \sqrt{10})$ $\cancel{\pm(3\pm\sqrt{10})}$

Case I: $e^{ix} e^{-y} = i(3+\sqrt{10}) = (3+\sqrt{10})\, e^{(i\pi/2 \pm 2n\pi i)}$
$\left(\because i = e^{i\pi/2} \;\&\; e^{\pm 2n\pi i} = 1\right)$

$\Rightarrow e^{-y} = 3+\sqrt{10}$ $x = \frac{\pi}{2} \pm 2n\pi$ $n = 0, 1, 2, \ldots$

$\Rightarrow y = -\ln(3+\sqrt{10})$ $x = \frac{\pi}{2} \pm 2n\pi$.
$= \ln\left(\frac{1}{3+\sqrt{10}}\right)$ $\left(\because -\ln x = \ln\frac{1}{x}\right)$
$= \ln\left(\frac{3-\sqrt{10}}{(3+\sqrt{10})(3-\sqrt{10})}\right) = \ln\left(\frac{3-\sqrt{10}}{9-10}\right) = \ln(\sqrt{10}-3)$

$\therefore z = \left(\frac{\pi}{2} \pm 2n\pi\right) + i\left(\ln(\sqrt{10}-3)\right)$ $n = 0, 1, 2, \ldots$

Case II $e^{ix}e^{-y} = i(3-\sqrt{10}) = (\sqrt{10}-3)(-i)$ (33) $(\because 3-\sqrt{10}<0)$

$= (\sqrt{10}-3)\, e^{-i\pi/2 \pm 2n\pi i}$ $(\because -i = e^{-i\pi/2})$ $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$

$\therefore e^{-y} = \sqrt{10}-3$ $x = -\frac{\pi}{2} \pm 2n\pi$ $n=0,1,2,\ldots$

$y = -\ln(\sqrt{10}-3) = \ln\left(\frac{1}{\sqrt{10}-3} \times \frac{\sqrt{10}+3}{\sqrt{10}+3}\right) = \ln(\sqrt{10}+3)$

$\therefore z = \left(-\frac{\pi}{2} \pm 2n\pi\right) + i\ln(\sqrt{10}+3)$ $n=0,1,2,\ldots$

(2) $\sin z = 1000 \Rightarrow \frac{e^{iz}-e^{-iz}}{2i} = 1000$ or, $e^{2iz} - 2000i\,e^{iz} - 1 = 0$

$\therefore e^{iz} = \frac{2000i \pm \sqrt{-4000000+4}}{2}$

$\therefore e^{ix}e^{-y} = \frac{2000i \pm i(1999.99)}{2} = 1999.99,\ 0.09$

$= 1999.99\, e^{i(\pm 2n\pi)},\ 0.09\, e^{i(\pm 2n\pi)}$

$\Rightarrow e^{-y} = 1999.99,\ 0.09$ $x = \pm 2n\pi$, $n = 0,1,2,\ldots$

$\therefore y = -7.6008,\ 2.4079$

$\therefore z_1 = \pm 2n\pi - i7.6008$ $z_2 = \pm 2n\pi + i2.4079$ — Ans

Hyperbolic functions: $\cosh^2 z - \sinh^2 z = 1$

$\cosh z = \frac{e^z + e^{-z}}{2}$ $\sinh z = \frac{e^z - e^{-z}}{2}$

$(\cosh z)' = \sinh z$ $(\sinh z)' = \cosh z$ Likewise you can compute for other hyperbolic fns

$\cosh(iz) = \cos z$ $\sinh(iz) = i\sin z$

$\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

$\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$\cosh(z_1+z_2) = \cos(i(z_1+z_2)) = \cos(iz_1 + iz_2)$

$= \cos(iz_1)\cos(iz_2) - i\sin(iz_1)\sin(iz_2)$

$= \cosh z_1 \cosh z_2 - i[i\sinh z_1\ i\sinh z_2]$

$= \cosh z_1 \cosh z_2 + i\sinh z \sinh z_2$ //

Ex Find in form of $u+iv$

(1) $\cosh(-3-6i)$ (2) $\sinh(4+5i)$

Soln (1) $\cosh(-3-6i) = \cancel{\cosh(3)\cosh(6i)+i\sinh(3)\sinh 6}$

$$= \cosh(3+6i)$$

$$= \cosh 3 \cosh(6i) + \sinh 3 \sinh(6i)$$

$$= \cosh 3 \cos 6 + i \sinh 3 \sin 6$$

$$\boxed{\begin{aligned} \cosh z &= \frac{e^{z}+e^{-z}}{2} \\ \cosh(-z) &= \frac{e^{-z}+e^{z}}{2} \end{aligned}}$$

(2) $\sinh(4+5i) = \sinh 4 \cosh(5i) + \cosh 4 \sinh(5i)$

$$= \sinh 4 \cos 5 + i \cosh 4 \sin 5$$

Ex Find all solutions of

(1) $\cosh z = 0$ (2) $\cancel{\sin z}\ \cosh z = \frac{1}{2}$

Soln (1) $\cosh z = 0 \Rightarrow \frac{e^{z}+e^{-z}}{2} = 0 \Rightarrow e^{2z}+1=0$

$$\Rightarrow e^{2z} = -1 = e^{i\pi \pm 2n\pi i} \quad n=0,1,2,\dots$$

$$\Rightarrow 2z = i(\pi \pm 2n\pi)$$

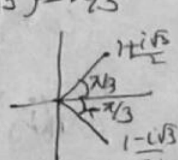
or, $z = \frac{i}{2}(\pi \pm 2n\pi)$ — Ans

(2) $\cosh z = \frac{1}{2} \Rightarrow \frac{e^{z}+e^{-z}}{2} = \frac{1}{2} \Rightarrow e^{2z} - e^{z} + 1 = 0$

$$\Rightarrow e^{z} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$r = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

$\theta = \tan^{-1}(\pm\sqrt{3}) = \pm \pi/3$



$$\Rightarrow e^{x+iy} = 1 \cdot e^{\pm i\pi/3 \pm 2n\pi i}$$

$$\Rightarrow e^{x} = 1 \quad y = \pm\frac{\pi}{3} \pm 2n\pi \quad n=0,1,2,\dots$$

$$\Rightarrow x = \ln 1 = 0$$

$\therefore z = i\left(\pm\frac{\pi}{3} \pm 2n\pi\right) \quad n=0,1,2,\dots$

— Ans

__Logarithm:__ $e^{i\omega} = z \Rightarrow \omega = \ln z$

$$e^{\omega} = e^{u+iv} = re^{i\theta} \Rightarrow e^{u}\cdot e^{iv} = re^{i(\theta \pm 2n\pi)}$$
$$\Rightarrow e^{u} = r \qquad e^{iv} = e^{i(\theta \pm 2n\pi)}$$
$$\Rightarrow u = \ln r \qquad v = \theta \pm 2n\pi \quad n = 0,1,2\ldots$$
$$\therefore \ln z = \ln r + i\theta = \ln\sqrt{x^2+y^2} + i\tan^{-1}(y/x) \pm 2n\pi i$$

$\text{Ln}\, z = \ln|z| + i\,\text{Arg}\, z \rightarrow$ Principal value of $\ln z$

$\ln z = \ln|z| + i \arg z \rightarrow$ General value of $\ln z$

__Relation between principal & general value:__

$$\boxed{\ln z = \text{Ln}\, z \pm 2n\pi i} \qquad n = 0,1,2,\ldots$$

__Note:__ $\ln x$ is not defined for -ive values.

__Lets check for $\ln z$.__

(i) Let $z = -4 \Rightarrow z = 4(-1) = 4e^{i\pi}$

$\therefore \text{Ln}\, z = \ln 4 + i\pi$ & $\ln z = \ln 4 + i(\pi \pm 2n\pi)$ $n = 0,1,2,\ldots$

(Concl) $\Rightarrow$ Complex valued fn $\ln z$ is defined for negative values

__Ex__ Find both general and principal values of

(1) $z = -1$ $\quad z = e^{i\pi}$ $\quad \text{Ln}\, z = i\pi$ $\quad \ln z = i\pi \pm 2n\pi i$

(2) $z = i$ $\quad z = e^{i\pi/2}$ $\quad \text{Ln}\, z = i\frac{\pi}{2}$ $\quad \ln z = i\frac{\pi}{2} \pm 2n\pi i$

(3) $z = -4i$ $\quad z = 4(-i) = 4e^{-i\pi/2}$ $\quad \therefore \text{Ln}\, z = -i\frac{\pi}{2} + \ln 4$

$\ln z = -\frac{i\pi}{2} \pm 2n\pi i + \ln 4$

(4) $z = -e^{-i}$ $\quad z = -(\cos 1 - i\sin 1)$

$z = -\cos 1 + i\sin 1$ $\quad r = \sqrt{\cos^2 1 + \sin^2 1} = 1$ $\quad \theta_1 = \tan^{-1}\left(\frac{\sin 1}{\cos 1}\right) = \tan^{-1}(\tan 1) = 1$

θ is in 2^{nd} quadrant

$\therefore \theta = \pi - \theta_1 = \pi - 1$

$\therefore \text{Ln}\, z = \overset{0}{\ln 1} + i(\pi - 1) = i(\pi - 1)$ $\quad \ln z = i(\pi - 1) \pm 2n\pi i$ – __Ans__

(5) $z = 4+3i$ $\quad |z| = r = \sqrt{4^2+3^2} = 5 \quad \theta = \tan^{-1}(3/4)$

$\therefore \; \mathrm{Ln}\, z = \ln 5 + i\tan^{-1}(3/4) \quad \ln z = \ln 5 + i\tan^{-1}\left(\frac{3}{4}\right) \pm 2n\pi i$

$n = 0, 1, 2, \ldots$

Ex solve for z

(1) $\ln z = -(\pi/2) i$ (2) $\ln z = -2 - \frac{3}{2} i$ (3) $\ln z = e - \pi i$

Soln (1) $\ln z = -\frac{\pi i}{2} \Rightarrow z = e^{-i\pi/2} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$

(2) $\ln z = -2 - \frac{3}{2} i \Rightarrow z = e^{-2-\frac{3}{2}i} = e^{-2}\left(\cos\frac{3}{2} - i\sin\frac{3}{2}\right)$

(3) $\ln z = e - \pi i \Rightarrow z = e^{e-\pi i} = e^{e}(\cos\pi - i\sin\pi) = -e^{e}$ $\left(\because \begin{array}{l}\cos\pi = -1 \\ \sin\pi = 0\end{array}\right)$

$\boxed{\text{Note}}$: Consider z^c where $z = x+iy$ & c is any number complex or real

Then $z = e^{c\ln z} \rightarrow$ general value

& $z^c = e^{c\,\mathrm{Ln}\, z} \rightarrow$ principal value.

Ex Find the principal value of given expression.

(1) $(2i)^{2i}$ (2) 3^{4-i} (3) $(1+i)^{1-i}$ (4) $i^{1/2}$

(5) $(-3)^{3-i}$

Soln (1) $(2i)^{2i} = e^{2i\,\mathrm{Ln}(2i)}$ $\qquad \mathrm{Ln}(2i) = \ln\sqrt{0^2+2^2} + i\frac{\pi}{2}$

$= \ln 2 + i\frac{\pi}{2}$

$= e^{2i(\ln 2 + i\pi/2)}$

$= e^{-\pi + i\ln 4}$ $\qquad (\because 2\ln 2 = \ln 2^2 = \ln 4)$

$= e^{-\pi}(\cos(\ln 4) + i\sin(\ln 4))$

(2) $3^{4-i} = e^{(4-i)\mathrm{Ln}\, 3}$ $\qquad \mathrm{Ln}\, 3 = \ln 3 + i0$

$= e^{(4-i)(\ln 3)} = e^{4\ln 3 - i\ln 3}$

$= e^{\ln 81}(\cos(\ln 3) - i\sin(\ln 3)) = 81(\cos(\ln 3) - i\sin(\ln 3))$ — Ans

$(\because e^{\ln a} = a$ for real variables$)$

(3) $(1+i)^{(1-i)} = e^{(1-i)\,Ln(1+i)}$ $Ln(1+i) = \ln\sqrt{1^2+1^2} + i\tan^{-1}\left(\frac{1}{1}\right)$

$= e^{(1-i)\left[\frac{1}{2}\ln 2 + i\frac{\pi}{4}\right]}$ $= \frac{1}{2}\ln 2 + i\frac{\pi}{4}$

$= e^{\left(\frac{1}{2}\ln 2 + \pi/4\right) + i\left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)}$ $(\because \ln x^a = a\ln x)$

$= e^{(\ln\sqrt{2} + \pi/4)}\left[\cos\left(\frac{\pi}{4} - \ln\sqrt{2}\right) + i\sin\left(\frac{\pi}{4} - \ln\sqrt{2}\right)\right]$

$\cancel{= e^{(\ln\sqrt{2} + \pi/4)}\left[\sin(\ln\sqrt{2}) + i\cos(\ln\sqrt{2})\right]}$

$\cancel{= \sqrt{2}e^{\pi/4}\left[\sin(\ln\sqrt{2}) + i\cos(\ln\sqrt{2})\right]}$ ~~-Ans~~

$= \sqrt{2}\,e^{\pi/4}\left[\cos\left(\frac{\pi}{4} - \ln\sqrt{2}\right) + i\sin\left(\frac{\pi}{4} - \ln\sqrt{2}\right)\right]$ -Ans

(4) $i^{1/2} = e^{\frac{1}{2}Ln\,i} = e^{i\pi/4}$ $Ln\,i = i\pi/2$

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$

$Ln(-3) = \ln 3 + i\pi$

(5) $(-3)^{3-i} = e^{(3-i)Ln(-3)}$

$= e^{(3-i)(\ln 3 + i\pi)}$

$= e^{(3\ln 3 + \pi) + i(3\pi - \ln 3)}$

$= e^{3\ln 3}\cdot e^{\pi}\,e^{i(3\pi - \ln 3)} = 27e^{\pi}\left[\cos(3\pi - \ln 3) + i\sin(3\pi - \ln 3)\right]$

$= 27e^{\pi}\left[-\cos\ln 3 + i\sin\ln 3\right]$ -Ans

$e^{\ln z} = z$

$e^{\ln z} = e^{(\ln|z| + i\arg z)}$

$= e^{\ln|z|}\,e^{i(\theta \pm 2n\pi)}$ θ - principal value

$= |z|\,e^{i\theta}\,e^{\pm 2n\pi i}$ $=1$

$= |z|e^{i\theta} = z$. //

$\ln(e^z) = z \pm 2n\pi i$

$\ln(e^z) = \ln(e^x\cos y + ie^x\sin y)$

$= \ln\sqrt{(e^x\cos y)^2 + (e^x\sin y)^2} + i\left[\tan^{-1}\left(\frac{\sin y}{\cos y}\right)\right] \pm 2n\pi i$

$= \ln e^x + iy \pm 2n\pi i$ $n = 0, 1, 2, \ldots$

$= x + iy \pm 2n\pi i$

$= z \pm 2n\pi i$