Entropy → (Average Information)

Entropy is average information of symbols.

Let us consider that we have 'M' different messages. Let these messages be $m_1, m_2 ---- m_M$ and they have probabilities of occurrence as $P_1, P_2 ---- P_m$ respectively.

Suppose that a sequence of 'L' messages is transmitted.

Then if 'L' is very-very large then we may say that

$P_1 L$ messages of m_1 are transmitted.

$P_2 L$,, ,, ,, ,,

⋮

$P_M L$ messages of m_M are transmitted.

Hence the information due to message m_1 will be.

$$I_1 = \log_2\left(\frac{1}{P_1}\right)$$

Since there are $P_1 L$ number of messages of m_1, the total information due to all message of m_1 will be →

$$I_1(\text{total}) = P_1 L \log_2\left(\frac{1}{P_1}\right)$$

Similarly the total information due to all messages m_2 will be

$$I_2(\text{total}) = P_2 L \log_2\left(\frac{1}{P_2}\right)$$

Created by Cam Scanner(Manya Technologies).

Thus the total information carried due to the sequence of 'L' messages will be

$$I_{total} = I_{1total} + I_{2total} + \cdots + I_{m total}$$

So $$I_{total} = P_1 L \log_2\left(\frac{1}{P_1}\right) + P_2 L \log_2\left(\frac{1}{P_2}\right) + \cdots + P_m L \log_2\left(\frac{1}{P_m}\right)$$

Average Information per message will be,

$$\text{Average Information} = \frac{\text{Total Information}}{\text{Number of messages}}$$

$$\text{Average information or Entropy} = \frac{I_{(total)}}{L}$$

or

$$\boxed{\text{Entropy } (H) = \frac{I_{total}}{L}}$$

$$\text{Entropy } (H) = P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + \cdots + P_m \log_2\left(\frac{1}{P_m}\right)$$

$$\boxed{\text{Entropy } (H) = \sum_{k=1}^{M} P_k \log_2\left(\frac{1}{P_k}\right)} \text{ bits/symbol}$$

Properties of Entropy →

1-) Entropy is zero if the event is sure or it is impossible

i.e $H = 0$ if $P_K = 0$ or 1

Proof → (a) When $P = 0$

$$H = \sum_{K=1}^{M} P_K \log_2\left(\frac{1}{P_K}\right)$$

$$\boxed{H = 0}$$

Proved

(b) When $P = 1$

$$H = \sum_{K=1}^{M} P_K \log_2\left(\frac{1}{P_K}\right)$$

$$H = 1 \log_2(1)$$

$$\boxed{H = 0}$$

Thus entropy is zero for both certain & most rare message.

Proved

(2) When $P_K = \frac{1}{M}$ for all 'M' symbols, then the symbols are equally likely, for such source entropy is given as $H = \log_2 M$

Proof → $P_K = \frac{1}{M}$

$$H = \sum_{K=1}^{M} P_K \log_2\left(\frac{1}{P_K}\right)$$

$$H = \sum_{k=1}^{M} \left(\frac{1}{M}\right) \log_2 M \quad \left\{ \text{Since } \sum_{k=1}^{M} \frac{1}{M} = 1 \right.$$

$$\boxed{H = \log_2 M}$$

Proved

Property ③ → The upper bound on entropy is given as $H_{max} \leq \log_2 M$ where 'M' is the number of messages emitted by the source.

Proof → To prove the above property we will use the following property of natural logarithm.

$$\ln x \leq (x-1) \text{ for } x \geq 0 \rightarrow ①$$

Let us consider any two probability distribution $\{P_1, P_2 --- P_m\}$ and $\{q_1, q_2 --- q_M\}$ on the alphabet $X = \{x_1, x_2 --- x_m\}$ of the DMS.

Let us consider the term $\sum_{k=1}^{M} P_k \log_2\left(\frac{q_k}{P_k}\right)$

$$\sum_{k=1}^{M} P_k \log_2\left(\frac{q_k}{P_k}\right) = \sum_{k=1}^{M} P_k \frac{\log_{10}\left(\frac{q_k}{P_k}\right)}{\log_{10} 2}$$

$$= \sum_{k=1}^{M} P_k \frac{\log_{10} e}{\log_{10} e} \times \frac{\log_{10}(q_k/P_k)}{\log_{10} 2}$$

$$= \sum_{k=1}^{M} P_k \frac{\log_{10} e}{\log_{10} 2} \frac{\log_{10}(q_k/P_k)}{\log_{10} e}$$

$$= \sum_{k=1}^{M} P_k \log_2 e \log_e(q_k/P_k)$$

$$= \log_2 e \sum_{k=1}^{M} P_k \log_e(q_k/P_k)$$

from eqn ①; we can write

$\ln\left(\frac{q_K}{p_K}\right) \le \left(\frac{q_K}{p_K} - 1\right)$ Hence from above eqn $\rightarrow$

$$\sum_{K=1}^{M} p_K \log_2\left(\frac{q_K}{p_K}\right) \le \log_2 e \sum_{K=1}^{M} p_K \cdot \left(\frac{q_K}{p_K} - 1\right)$$

$$\le \log_2 e \sum_{K=1}^{M} p_K \frac{(q_K - p_K)}{p_K}$$

$$\le \log_2 e \sum_{K=1}^{M} (q_K - p_K)$$

$$\le \log_2 e \left\{ \sum_{K=1}^{M} q_K - \sum_{K=1}^{M} p_K \right\}$$

$$\left\{ \begin{array}{l} \sum_{K=1}^{M} q_K = 1 \\ \sum_{K=1}^{M} p_K = 1 \end{array} \right.$$

$$\sum_{K=1}^{M} p_K \log_2\left(\frac{q_K}{p_K}\right) \le 0 \longrightarrow (2)$$

Let us consider $q_K = \frac{1}{M}$ {That is all symbols in the alphabet are equally likely.

$$\sum_{K=1}^{M} p_K \left\{ \log_2 q_K + \log_2\left(\frac{1}{p_K}\right) \right\} \le 0$$

$$\sum_{K=1}^{M} p_K \log_2 q_K + \sum_{K=1}^{M} p_K \log_2 \frac{1}{p_K} \le 0$$

$$\sum_{k=1}^{M} P_k \log_2\left(\frac{1}{P_k}\right) \leq - \sum_{k=1}^{M} P_k \log_2 q_k$$

$$\sum_{k=1}^{M} P_k \log_2\left(\frac{1}{P_k}\right) \leq \sum_{k=1}^{M} P_k \log_2\left(\frac{1}{q_k}\right)$$

$$H \leq \sum_{k=1}^{M} P_k \log_2 M$$

$$H \leq \log_2 M \sum_{k=1}^{M} P_k$$

$$\boxed{H \leq \log_2 M} \qquad \left\{ \text{Since } \sum_{k=1}^{M} P_k = 1 \right.$$

Proved

* Maximum value of entropy is →

$$\boxed{H_{max} = \log_2 M}$$

Source Efficiency →

It is defined as the ratio of source entropy to max^{m} Entropy.

$$\text{Source efficiency} = \frac{\text{Entropy of Source}}{\text{Max}^{m}\ \text{Entropy}}$$

$$\boxed{\eta = \frac{H}{H_{max}}}$$

Where $H_{max} = \log_2 M$

Redundancy of Source →

$$\boxed{R_e = 1-\eta}$$

Information Rate → Information rate is represented by R and it is given as →

$$\boxed{\text{Information Rate } (R) = rH} \text{ bits/sec}$$

where 'H' is entropy or avg information.

'r' is rate at which messages are generated.