Kinetic Energy in Generalized Co-ordinates.

Kinetic energy of a particle of mass m is a homogenous quadratic function of the velocities $\dot{r}_i$

$$T=\frac{1}{2}\sum_{i=1}^{N} m_i \dot{r}_i^2$$

$$=\frac{1}{2}\sum_{i=1}^{N} m_i (\dot{r}_i \cdot \dot{r}_i) \quad \text{— (1)}$$

But $\dot{r}_i = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}$. — (2)

From (1) and (2)

$$T=\frac{1}{2}\sum_i m_i \left[\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right]\cdot\left[\sum_{K=1}^{n} \frac{\partial r_i}{\partial q_K}\dot{q}_K + \frac{\partial r_i}{\partial t}\right]$$

$$=\frac{1}{2}\sum_i\sum_j\sum_K m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_K}\dot{q}_j\dot{q}_K$$

$$+\frac{1}{2}\sum_i m_i \left(2\sum_j \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}\dot{q}_j\right)$$

$$+\frac{1}{2}\sum_i m_i \left(\frac{\partial r_i}{\partial t}\cdot\frac{\partial r_i}{\partial t}\right) \quad \text{— (3)}$$

$$T(q,\dot{q},t) = \sum_j\sum_K a_{jK}\dot{q}_j\dot{q}_K + \sum_j b_j \dot{q}_j + c$$

where

$$a_{jK} = \frac{1}{2}\sum_i m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_K}$$

$$b_j = \sum_i m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_j}{\partial t}$$

$$c = \frac{1}{2}\sum_i m_i \left(\frac{\partial r_i}{\partial t}\right)^2$$

In general, the Kinetic Energy in terms of generalized co-ordinates consists of three distinct terms, the first terms contains quadratic terms, the second contains linear terms, the third term is

Independent of velocities, if

$$\frac{\partial r_i}{\partial q_j} \cdot \frac{\partial r_i}{\partial q_k} = 0 \quad \text{for } j \neq k \qquad — \quad (4)$$

The generalized co-ordinate system in the q_i's is referred to as an <u>orthogonal system</u>.

Generalized Momentum

Consider the motion of a particle of mass m moving along x-axis. Its linear momentum p is $m\dot{x}$ and K.E. $T = (\frac{1}{2})m\dot{x}^2$. Differentiating T w.r. to $\dot{x}$ we have

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} = p.$$

If the potential V is not a function of the velocity $\dot{x}$, since $L = T - V$

$$p = \frac{\partial T}{\partial \dot{x}} = \frac{\partial L}{\partial \dot{x}}$$

Let us use this concept to define generalize momentum. For a system described by a set of generalized co-ordinates $q_1, q_2, \ldots, q_n$, we define <u>generalized momentum p_i</u> corresponding to generalized co-ordinate q_i as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

It is also know as <u>conjugate momentum</u>.

Cyclic co-ordinates

Co-ordinates that do not appear explicitly in the Lagrangian of a System (although it may contain the corresponding generalized velocities) are said to be cyclic or ignorable. If q_i is a cyclic co-ordinate

$$L = L(q_1, q_2, ---, q_{i-1}, q_{i+1}, ---, q_n, \dot{q}_1, -- \dot{q}_n, t)$$

In such a case $(\partial L/\partial q_i) = 0$ and Lagrange's equation reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad \text{or} \quad \frac{\partial L}{\partial \dot{q}_i} = \text{constant } \alpha_i$$

which means that

$$\boxed{\frac{\partial L}{\partial \dot{q}_1} = p_i = \text{constant } \alpha_i} \quad -(1)$$

Equation (1) constitutes a first integral for the Equation of motion.

The generalized momentum conjugate to a cyclic co-ordinate is conserved during the motion.

Homogeneity of Space and Conservation of Linear Momentum

Homogeneity in space means that the mechanical properties of a closed system remain unchanged

by any parallel displacement of the entire system in space.
That means that the Lagrangian is ($\delta L=0$) if the system is displaced by an infinitesimal amount δr: $r_i \to r_i + \delta r_i$.
The change in L due to infinitesimal displacement δr, the velocities remaining fixed, is given by

$$\delta L = \sum_i \frac{\partial L}{\partial r_i} \delta r_i \quad — (1)$$

The second term in this equation vanished as velocities remained constant ($\delta \dot{r}_i = 0$). Since each of the δr_i in Equation (1) in an arbitrary independent displacement, the coefficient of each term is zero separately.
Hence,

$$\frac{\partial L}{\partial r_i} = 0 \quad — (2)$$

with this condition, Lagrange's Equation reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} = 0 \quad \text{or} \quad \frac{\partial L}{\partial \dot{r}_i} = \text{constant} \quad — (3)$$

$$p_i = \text{constant}.$$

As the p_i's are additive, the total linear momentum p of a closed system is a constant. Thus, the homogeneity of space implies that the linear momentum p is a contant of motion.

Homogeneity of Time and Conservation of Energy

Homogeneity in time implies that the Lagrangian of a closed system does not depend explicitly on the time t. That is, $(\partial L/\partial t) = 0$. The total time derivative of the Lagrangian is

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial L} \quad -(1)$$

Use of the condition $\left(\frac{\partial L}{\partial t}\right) = 0$, gives

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \quad -(2)$$

Replacing $(\partial L/\partial q_i)$ using Lagrange's Equation, we have

$$\frac{dL}{dt} = \sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$$

$$= \sum_i \frac{d}{dt}(p_i \dot{q}_i)$$

$$\frac{d}{dt}\left(\sum_i p_i\dot{q}_i - L\right) = 0 \quad -(3)$$

That is, the quantity in parenthesis must be constant in time. Denoting the constant by H, called the Hamiltonian of the system

$$\sum_i p_i\dot{q}_i - L = H \text{ (constant)}. \quad -(4)$$

Here we can show that H is the total energy of the system ⑥

if

(i) the potential Energy V is velocity-independent

(ii) the transformation Equations connecting the rectangular and generalized co-ordinates do not depend on time explicitly.

Therefore, when condition (ii) is satisfied, the kinetic Energy T is a homogeneous quadratic function of the generalized velocities and by Eluer's theorem,

$$\sum_i \frac{\partial T}{\partial \dot{q}_1} \dot{q}_1 = 2T \quad \text{———} \quad (5)$$

Now Equation (4) can be written as

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = \sum_i \frac{\partial (T-V)}{\partial \dot{q}_1} \dot{q}_i - L$$

$$= \sum_i \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i - L = 2T - (T-V)$$

$$H = T + V = E \quad \text{(Total energy)} \text{ ——— } (6)$$

when condition (ii) is not satisfied, the Hamiltonian H is no longer equal to the total Energy of the system. However, the total Energy is still conserved for a conservative system.