Frequency domain analysis and Fourier Transform

How to Represent Signals?

- Option 1: Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

Jean Baptiste Joseph Fourier (1768-1830)

- Had crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don’t believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it’s true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of x:

$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(\omega)$$

- For every ω from 0 to inf, $F(\omega)$ holds the amplitude A and phase ϕ of the corresponding sine

$$A\sin(\omega x + \phi)$$

  – How can F hold both? Complex number trick!

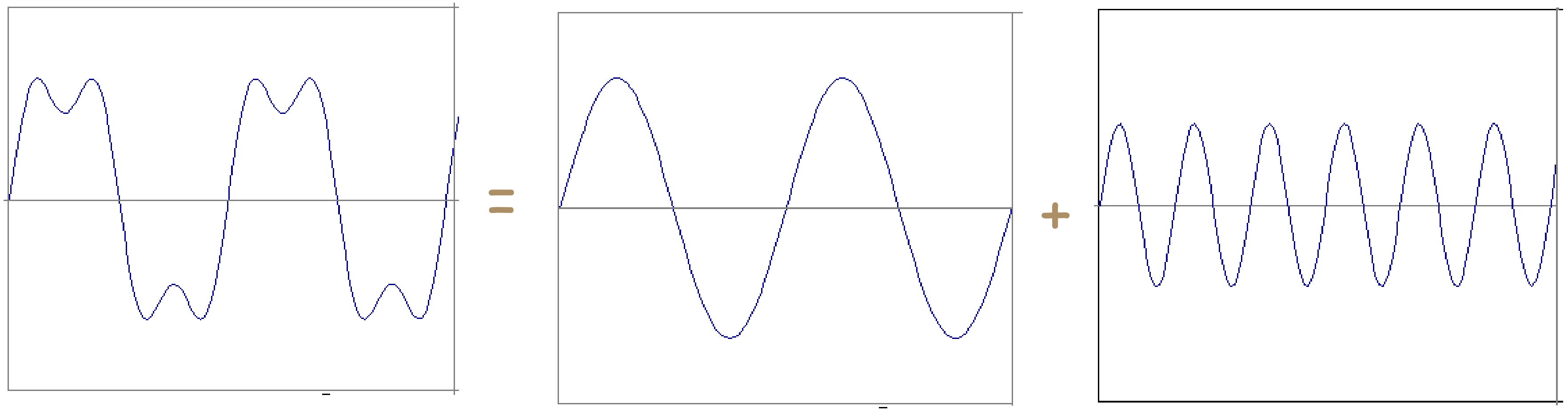
$$F(\omega) = R(\omega) + iI(\omega)$$

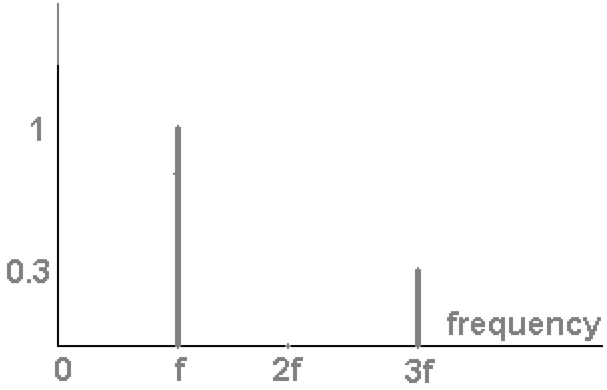
$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

$$F(\omega) \longrightarrow \boxed{\text{Inverse Fourier Transform}} \longrightarrow f(x)$$

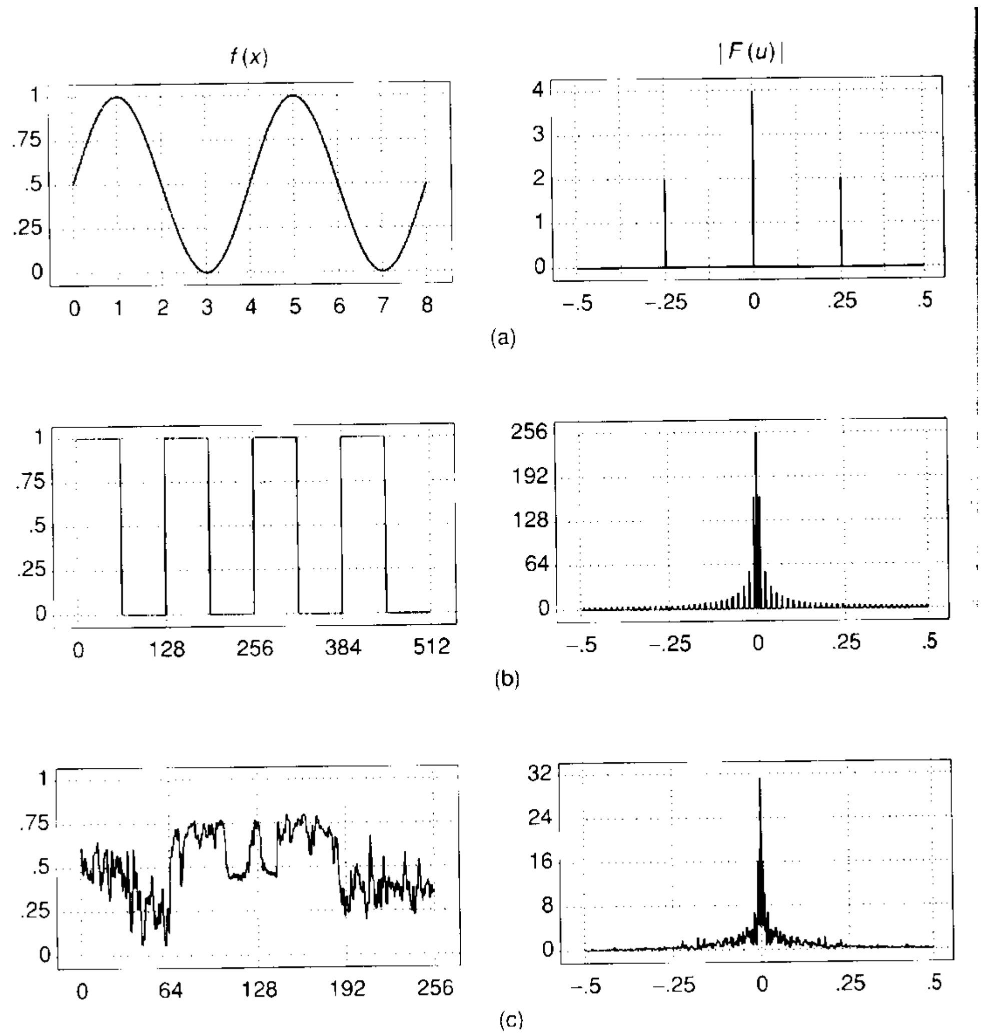
Frequency Spectra

- example : $g(t) = \sin(2pi\ f\ t) + (1/3)\sin(2pi\ (3f)\ t)$





Frequency Spectra



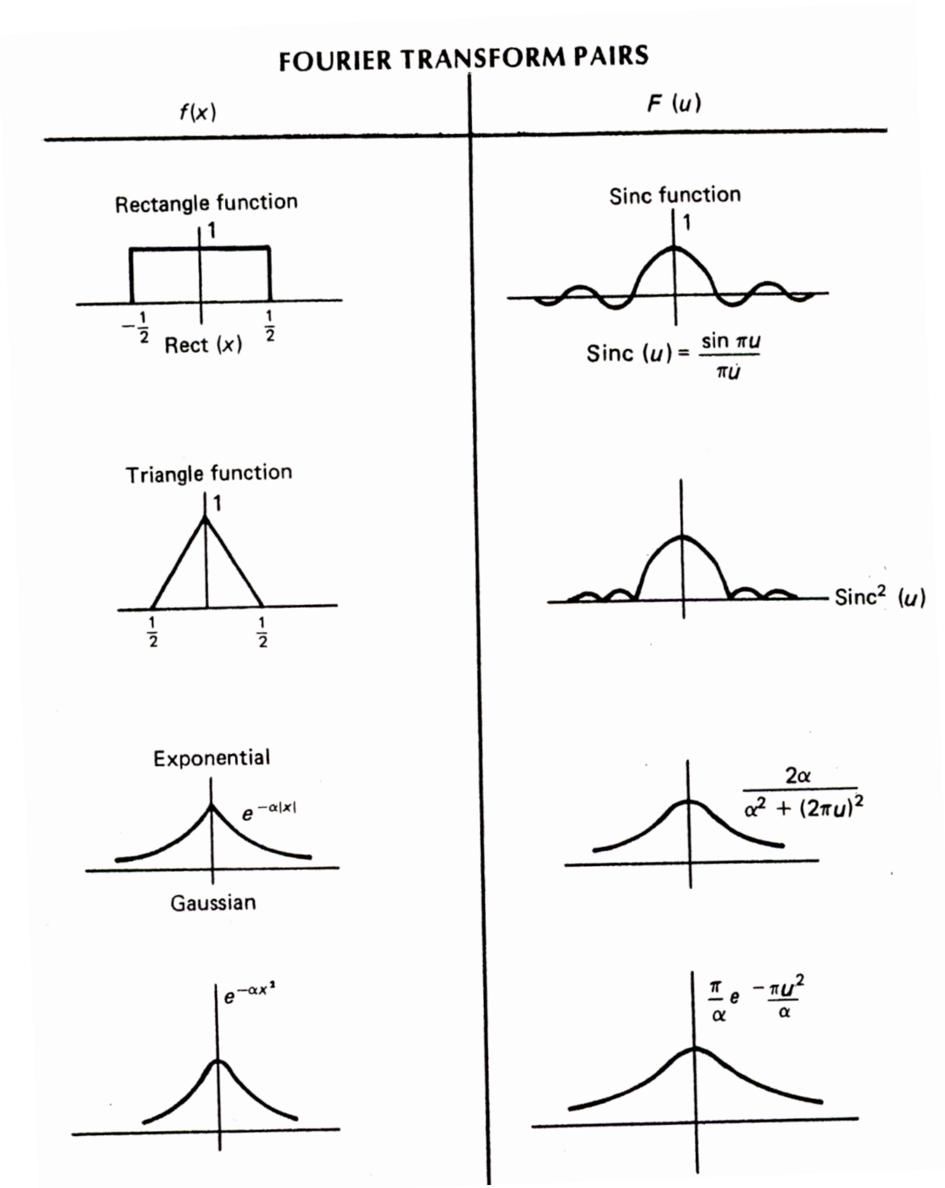
Fourier Transform

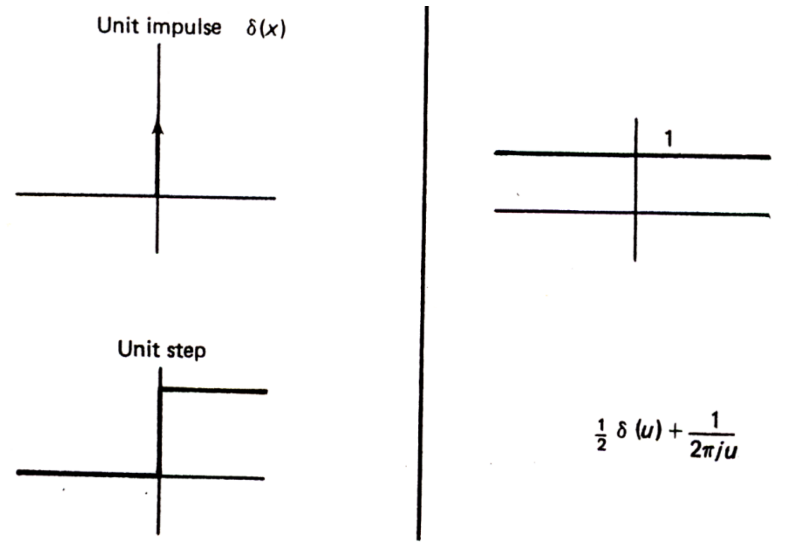
One of the most useful features of the Fourier transform (and Fourier series) is the simple "inverse" Fourier transform.

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{(Fourier transform)}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{("inverse" Fourier transform)}$$

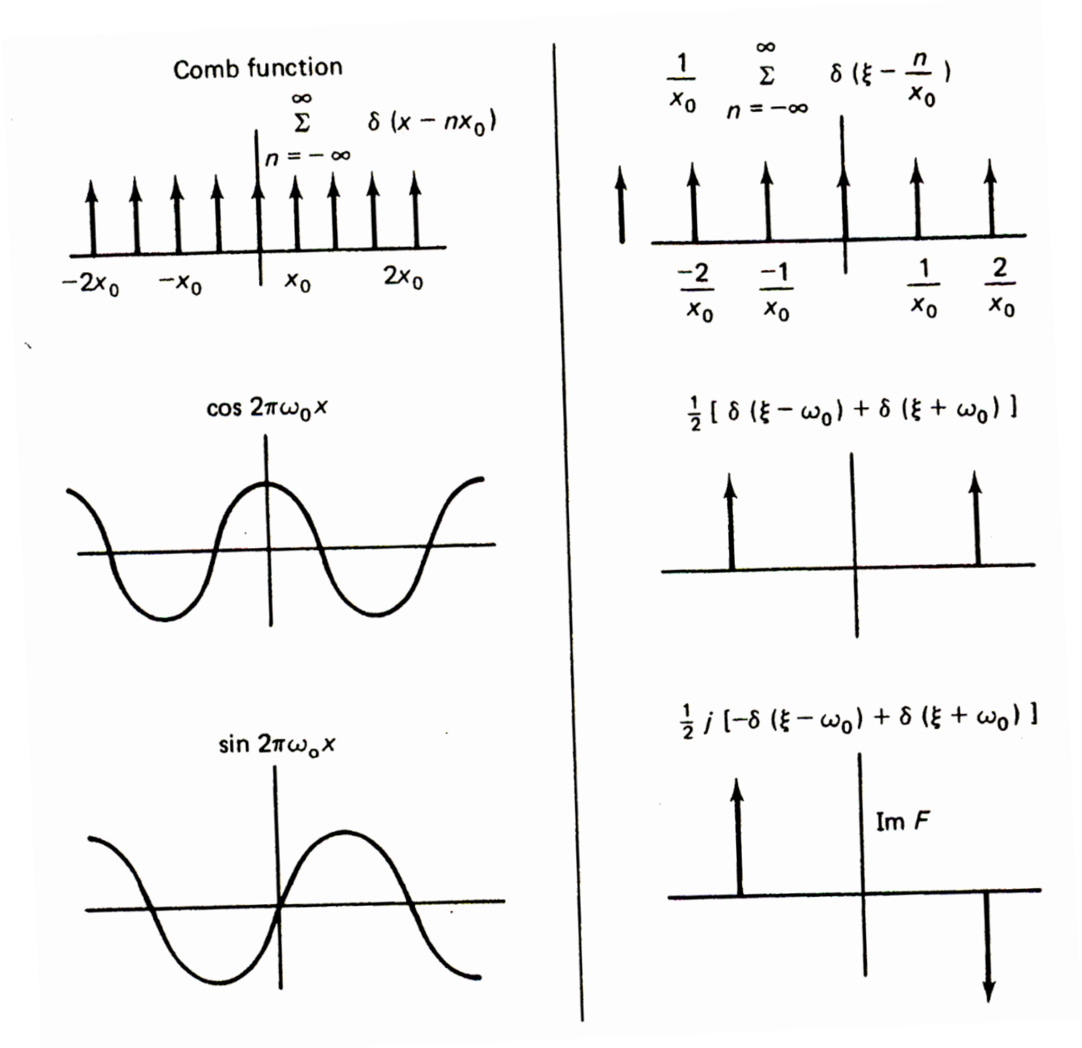
Fourier Transform Pairs (I)





Note that these are derived using angular frequency (e^{-iux})

Fourier Transform Pairs (I)



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Fourier Transform and Convolution

Let $g = f * h$

Then $\boxed{G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx}$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u(x-\tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right]$$

$$\boxed{= F(u) H(u)}$$

Convolution in spatial domain

$\Leftrightarrow$ Multiplication in frequency domain

Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x)+c_2 g(x)$	$c_1 F(u)+c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a\rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x-x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x)*g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi ux}$)

Properties of Fourier Transform

Parseval's theorem:

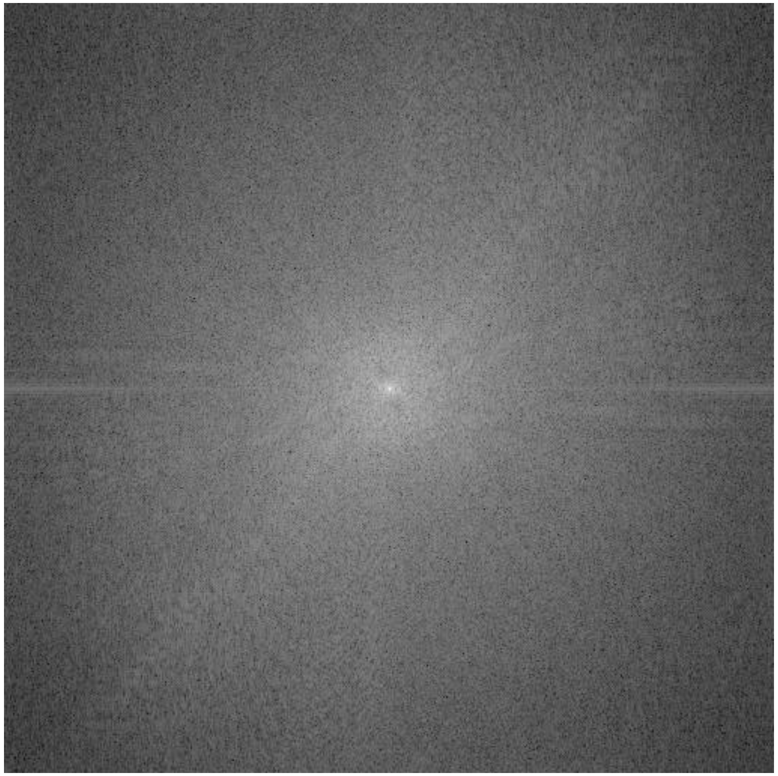
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\xi)|^2 d\xi$$

$$\int_{-\infty}^{\infty} f(x) g^*(x)\, dx = \int_{-\infty}^{\infty} F(\xi) G^*(\xi)\, d\xi$$

$f(x)$	$F(\xi)$
Real(R)	Real part even (RE) Imaginary part odd (IO)
Imaginary (I)	RO,IE
RE,IO	R
RE,IE	I
RE	RE
RO	IO
IE	IE
IO	RO
Complex even (CE)	CE
CO	CO

Image Processing in the Fourier Domain

Magnitude of the FT



Does not look anything like what we have seen

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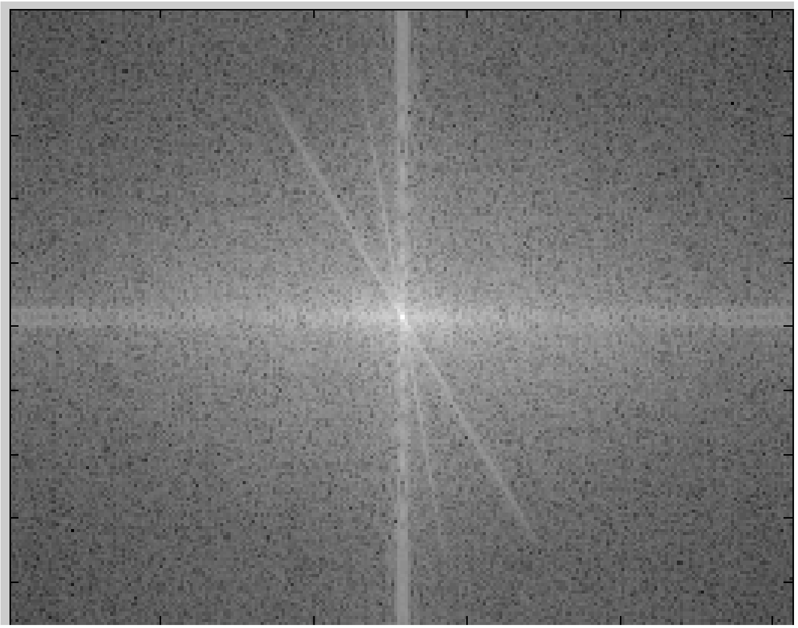

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THANK YOU