SIGNALS & SYSTEMS

Z-Transform

Comparison of Laplace ad Z-Transform

- **Fourier transform plays a key role in analyzing and representing discrete-time signals and systems, but** it is **not** applicable **for all signals.**
- **Continuous systems: Laplace transform is a generalization of the Fourier transform.**
- **Discrete systems : Z-transform, generalization of DTFT, converges for a broader class of signals.**
- **In Laplace Transform we evaluate the complex sinusoidal representation of a continuous signal.**
- **In the Z-Transform, it is on the complex sinusoidal representation of a discrete-time signal.**

Relation between DTFT and Z-Transform

- **The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems**
- **Because of the convergence condition, in many cases, the DTFT of a sequence may not exist, thereby making it impossible to make use of such frequency-domain characterization in these cases**

- **A generalization of the DTFT defined by** $X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ **leads to the *Z*-transform**
- ***Z*-transform may exist for many sequences for which the DTFT does not exist.**
- **Use of *Z*-transform permits simple algebraic manipulations**

Introduction

- The z-transform is the discrete-time counterpart of the Laplace transform.

- It can be used to assess the characteristic of discrete-time systems in terms of its impulse response and frequency response.

- The z-transform can be used determine the solution to the difference equation.

Definition of *Z*-Transform

- For a given sequence $x[n]$, its z-transform $X(z)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Z-Transform Example

A signal is defined as

$$x(n) = a^n \qquad n \geq 0$$
$$= 0 \qquad \text{elsewhere}$$

$$X(z) = \sum_{n=0}^{\infty} (a)^n z^{-n} = 1 + a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + \ldots..$$

(Open form)

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

(Close form)

From the close form solution, there is a pole where $z = a$ and a zero.

Solution for Difference Equation and Transfer Function

- The z-transform can be used to determine the solution to difference equation. Given that the input-output relationship of a linear time-invariant system is as follows

$$y(n) = a(1)y(n\text{-}1) + a(2)y(n\text{-}2) = b(0)x(n) + b(1)x(n\text{-}1) + b(2)x(n\text{-}2)$$

- The z-transform is

$$Y(z) + a(1)Y(z)\, z^{-1} + a(2)Y(z)\, z^{-2} = b(0)X(z) + b(1)X(z)\, z^{-1} + b(2)X(z)\, z^{-2}$$

$$H(z) = \frac{Y(z)}{X(z)} = \left[\frac{b(0) + b(1)z^{-1} + b(2)z^{-2}}{1 + a(1))z^{-1} + a(2)z^{-2}}\right]$$

where $H(z)$ is the transfer function.

General Form of the Transfer Function

- For more general case, the transfer function is in the form

$$H(z) = \frac{\sum_{n=0}^{N} b(n) z^{-n}}{\sum_{n=0}^{N} a(n) z^{-n}}$$

where N is the polynomial order. The transfer function when factorized in term of the roots is

$$H(z) = \frac{\prod_{n=0}^{N} (1 - \beta(n) z^{-n})}{\prod_{n=0}^{N} (1 - \alpha(n) z^{-n})}$$

Inverse Z-Transform

- The system impulse response $h(n)$ is obtained from $H(z)$ by taking the inverse z-transform. If the following transfer function is used as example

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + a(1)z^{-1} + a(2)z^{-2}}$$

$$= \frac{1}{(1 - \alpha(1)z^{-1})(1 - \alpha(2)z^{-1})}$$

- The application of the partial fraction expansion results in

$$H(z) = \frac{1}{(1-\alpha(1)z^{-1})(1-\alpha(2)z^{-1})} = \frac{A_0}{1-\alpha(1)z^{-1}} + \frac{A_1}{1-\alpha(2)z^{-1}}$$

Elementary signals

Unit step function $x(n) = 1 \qquad n \geq 0$

$$X(z) = \sum_{n=0}^{\infty} 1 z^{-n} = 1 + z^{-1} + z^{-2} + \ldots\ldots\ldots\ldots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}; \quad |z| > 1$$

Power function $x(n) = a^n \qquad n \geq 0$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = 1 + az^{-1} + a^2 z^{-2} + \ldots\ldots\ldots\ldots = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}; \quad |z| > |a|$$

Ramp function $x(n) = n \qquad n \geq 0$

$$X(z) = \sum_{n=0}^{\infty} nz^{-n} = 0 + z^{-1} + 2z^{-2} + 3z^{-3} \ldots\ldots\ldots = \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z-1)^2}; \quad |z| > 1$$

The values of z for which $X(z)$ is finite are known as region of convergence (**ROC**)

Try other signals: impulse function, ….

Significance of ROC

For causal sequence $x(n) = a^n \quad for \quad n \geq 0 \quad and \quad 0 \quad for \quad n < 0$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}; \qquad |z| > |a|$$

For Anti causal sequence $x(n) = 0 \quad for \quad n \geq 0 \quad and \quad -a^n \quad for \quad n < 0$

$$X(z) = \sum_{n=-\infty}^{-1} - a^n z^{-n} = -\sum_{n=-\infty}^{-1} (a^{-1}z)^{-n} = \frac{z}{z-a}; \quad |z| < |a|$$

The two sequences have same *X(z)* but their ROC is different. Without ROC we can not uniquely determine the sequence *x(n)*. Generally, for causal sequence, the ROC is exterior of the circle having radius *a* and for anti causal sequence it is interior of the circle.

Find *X(z)* and ROC for $x(n) = \alpha^n u(n) + \beta^n u(-n-1)$

Answer

$$X(z) = \frac{\beta - \alpha}{\alpha + \beta - z - \alpha\beta z^{-1}} \qquad ROC: |\alpha| < |z| < |\beta|$$

Z-transform pairs

$$\delta[n] \leftrightarrow 1, \quad ROC: all \quad z$$

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}}, \quad ROC: |z| > 1$$

$$-u[-n-1] \leftrightarrow \frac{1}{1-z^{-1}}, \quad ROC: |z| < 1$$

$$\delta[n-m] \leftrightarrow z^{-m},$$

$$ROC: all \quad z \; except \; 0 \; (if \;\; m > 0) \; or \; \infty \; (if \;\; m < 0)$$

Z-transform pairs

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad ROC : |z| > |a|$$

$$-a^n u[-n-1] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad ROC : |z| < |a|$$

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1 - az^{-1})^2}, \quad ROC : |z| > |a|$$

$$-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{(1 - az^{-1})^2}, \quad ROC : |z| < |a|$$

Z-transform pairs

$$[\cos w_0 n]u[n] \leftrightarrow \frac{1-[\cos w_0]z^{-1}}{1-2[\cos w_0]z^{-1}+z^{-2}}, \quad ROC : |z| > 1$$

$$[\cos w_0 n]u[n] = \frac{1}{2}\left(e^{jw_0 n} + e^{-jw_0 n}\right)u[n]$$

$$\frac{1}{2}\left(\frac{1}{1-e^{jw_0}z^{-1}} + \frac{1}{1-e^{-jw_0}z^{-1}}\right)$$

$$[\sin w_0 n]u[n] \leftrightarrow \frac{[\sin w_0]z^{-1}}{1-2[\cos w_0]z^{-1}+z^{-2}}, \quad ROC : |z| > 1$$

Zero and pole

- The Z-transform is most useful when the infinite sum can be expressed in closed form, usually a ratio of polynomials in z (or Z^{-1}).

$$X(z) = \frac{P(z)}{Q(z)}$$

- Zero: The value of z for which $X(z) = 0$
- Pole: The value of z for which $X(z) = \infty$

Convolution

- **Convolution:** $x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k] \quad \Leftrightarrow \quad X(z)H(z)$

Proof:
$$\mathcal{Z}\left[x[n]*h[n]\right]=\mathcal{Z}\left[\sum_{k=-\infty}^{\infty}x[k]h[n-k]\right]=\sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}x[k]h[n-k]\right]z^{-n}$$
$$=\sum_{k=-\infty}^{\infty}x[k]\left[\sum_{n=-\infty}^{\infty}h[n-k]z^{-n}\right]$$

Change of index on the second sum: $m=n-k$

$$\mathcal{Z}\left[x[n]*h[n]\right]=\sum_{k=-\infty}^{\infty}x[k]\left[\sum_{m=-\infty}^{\infty}h[m]z^{-(m+i)}\right]=\left[\sum_{k=-\infty}^{\infty}x[k]z^{-k)}\right]\left[\sum_{m=-\infty}^{\infty}h[m]z^{-m}\right]$$
$$=X(z)H(z)$$

The ROC is at least the intersection of the ROCs of *x*[*n*] and *h*[*n*], but can be a larger region if there is pole/zero cancellation.

- **The system transfer function is completely analogous to the CT case:**

$$h[n] \quad \Leftrightarrow \quad H(z)=\sum_{n=-\infty}^{\infty}h[n]z^{-n}$$

- **Causality:** $h[n]=0 \quad n\leq 0$

Implies the ROC must be the exterior of a circle and include *z* = ∞.

Initial-Value and Final-Value Theorems (One-Sided ZT)

- **Initial Value Theorem:**

$$x[0] = \lim_{z \to \infty} X(z)$$

Proof:

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \sum_{n=0}^{\infty} x[n] z^{-n} = \lim_{z \to \infty} x[0] + x[1] z^{-1} + \ldots = x[0]$$

- **Final Value Theorem:**

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1) X(z)$$

- **Example:**

$$X(z) = \frac{3z^2 - 2z + 4}{z^3 - 2z^2 + 1.5z - 0.5} = \frac{3z^2 - 2z + 4}{(z-1)(z^2 - z + 0.5)}$$

$$\lim_{n \to \infty} x[n] = [(z-1) X(z)]\Big|_{z=1} = \left.\frac{3z^2 - 2z + 4}{z^2 - z + 0.5}\right|_{z=1} = \frac{5}{.5} = 10$$

Table Common *Z*-transform Pairs

SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Properties of *Z*-Transform

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0 \rvert R_x$
$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$