SIGNALS AND SYSTEMS

Z -Transform

Introduction

- The z-transform is the discrete-time counterpart of the Laplace transform.

- It can be used to assess the characteristic of discrete-time systems in terms of its impulse response and frequency response.

- The z-transform can be used determine the solution to the difference equation.

Definition of *Z*-Transform

- For a given sequence $x[n]$, its z-transform $X(z)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Inverse Z Transform

- Recall the definition of the inverse Laplace transform via contour integration:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st}\, ds = \frac{1}{2\pi j}\oint_C X(s) e^{st}\, ds$$

- The inverse Z-transform follows from this:

$$x[n] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1}\, dz$$

Evaluation of this integral is beyond the scope of this course. Instead, as with the Laplace transform, we will restrict our interest in the inverse transform to rational forms (ratio of polynomials). We will see shortly that this is convenient since linear constant-coefficient difference equations can be converted to polynomials using the Z-transform.

- As with the Laplace transform, there are two common approaches:
  - Long Division
  - Partial Fractions Expansion
- Expansion by long division Is also known as the power series expansion approach and can be easily demonstrated by an example.

Long Division

- **Consider:**

$$X(z) = \frac{z^2 - 1}{z^3 + 2z + 4}$$

Solution:

$$\begin{array}{r|l}
 & z^{-1} \\
\hline
z^3 + 2z + 4 & z^2 - 1 \\
 & \underline{z^2 + 2 + 4z^{-1}} \\
 & \quad -3 - 4z^{-1}
\end{array}$$

$$\begin{array}{r|l}
 & z^{-1} + 0z^{-2} - 3z^{-3} \\
\hline
z^3 + 2z + 4 & z^2 - 1 \\
 & \underline{z^2 + 2 + 4z^{-1}} \\
 & \quad -3 - 4z^{-1} \\
 & \quad \underline{-3 \qquad\quad - 6z^{-2} - 12z^{-3}} \\
 & \qquad -4z^{-1} + 6z^{-2} + 12z^{-3}
\end{array}$$

$$\begin{array}{r|l}
 & z^{-1} + 0z^{-2} - 3z^{-3} - 4z^{-4} \\
\hline
z^3 + 2z + 4 & z^2 - 1 \\
 & \underline{z^2 + 2 + 4z^{-1}} \\
 & \quad -3 - 4z^{-1} \\
 & \quad \underline{-3 \qquad\quad - 6z^{-2} - 12z^{-3}} \\
 & \qquad -4z^{-1} + 6z^{-2} + 12z^{-3} \\
 & \qquad \underline{-4z^{-1} \qquad\quad - 8z^{-3} - 16z^{-4}} \\
 & \qquad\qquad 6z^{-2} + 20z^{-3} + 16z^{-4}
\end{array}$$

$$\therefore \quad X(z) = z^{-1} + 0z^{-2} - 3z^{-3} - 4z^{-4} + \ldots$$

$$\Rightarrow \quad x[n] = 0\delta[n] + 1\delta[n-1] - 3\delta[n-3] - 4\delta[n-4] + \ldots$$

Inverse *Z*-Transform Using Partial Fractions

- The partial fractions approach is preferred if we want a closed-form solution rather than the numerical solution long division provides.
- **Example:**

$$X(z) = \frac{z^3 + 1}{z^3 - z^2 - z - 2}$$

In this example, the order of the numerator and denominator are the same. For this case, we can use a trick of factoring $X(z)/z$:

$$A(z) = z^3 - z^2 - z - 2 = (z-2)(z + 0.5 + j0.866)(z + 0.5 - j0.866)$$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z + 0.5 + j0.866} + \frac{\overline{c_1}}{z + 0.5 - j0.866} + \frac{c_3}{z-2}$$

$$c_0 = \left[\frac{X(z)}{z}(z)\right]_{z=0} = \frac{1}{-2} = -0.5$$

$$c_1 = \left[\frac{X(z)}{z}\left(z + 0.5 + j0.866\right)\right]_{z=-0.5-j0.866} = 0.429 + j0.0825$$

$$c_3 = \left[\frac{X(z)}{z}\left(z - 2\right)\right]_{z=2} = 0.643$$

Inverse Z-Transform (Cont.

We can compute the inverse using our table of common transforms:

$$X(z) = c_0 + \frac{c_1 z}{z + 0.5 + j0.866} + \frac{\overline{c}_1 z}{z + 0.5 - j0.866} + \frac{c_3 z}{z - 2}$$

$$= c_0 + \frac{c_1}{1 + 0.5 + j0.866\ z^{-1}} + \frac{\overline{c}_1}{1 + 0.5 - j0.866\ z^{-1}} + \frac{c_3}{1 - 2z^{-1}}$$

$$x[n] = c_0\delta[n] + c_1(-0.5 - j0.866)^n u[n] + \overline{c}_1(-0.5 + j0.866)^n u[n] + c_3 2^n u[n]$$

The exponential terms can be converted to a single cosine using a magnitude/phase conversion:

$$|p_1| = \sqrt{(0.5)^2 + (0.866)^2} = 1$$

$$\angle p_1 = \pi + \tan^{-1}\frac{0.866}{0.5} = \frac{4\pi}{3}\ \text{rad}$$

$$|c_1| = \sqrt{(0.429)^2 + (0.0825)^2} = 0.437$$

$$\angle c_1 = \tan^{-1}\frac{0.0825}{0.429} = 0.19\ \text{rad} \quad (10.89\ ^\circ)$$

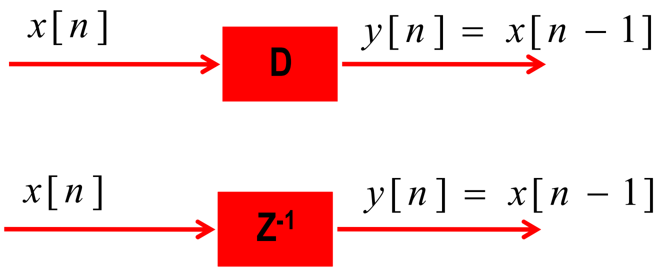
$$x[n] = c_0\delta[n] + c_1(-0.5 - j0.866)^n u[n] + \overline{c}_1(-0.5 + j0.866)^n u[n] + c_3 2^n u[n]$$

$$= c_0\delta[n] + 2|c_1||p_1|\cos(\angle p_1 n + \angle c_1) + c_3(2)^n u[n]$$

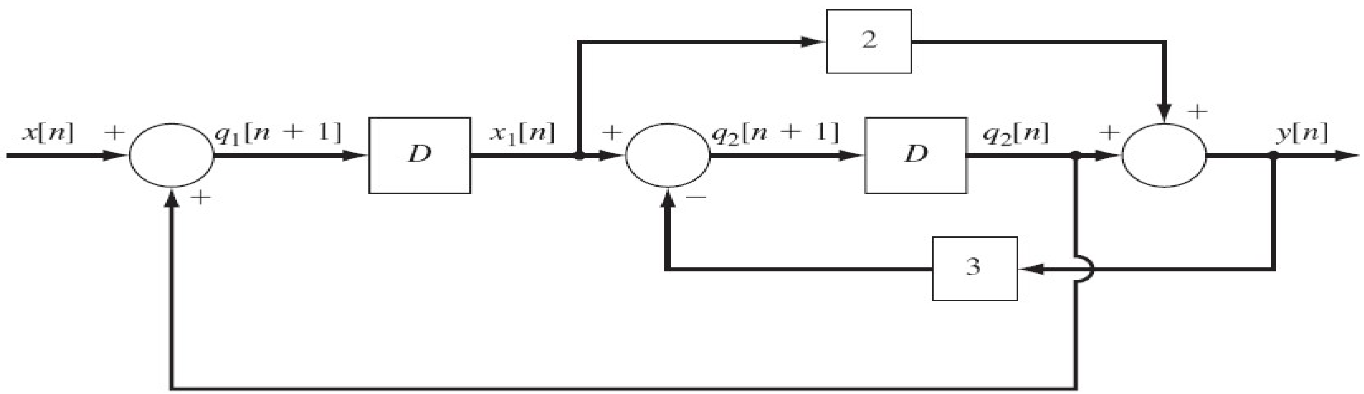
$$= -0.5\delta[n] + 0.874\ \cos(\frac{4\pi}{3}n + 0.19) + 0.643\ (2)^n u[n]$$

Transfer Functions

- In addition to our normal transfer function components, such as summation and multiplication, we use one important additional component: delay.
- This is often denoted by its Z-transform equivalent.
- We can illustrate this with an example (assume initial conditions are zero):

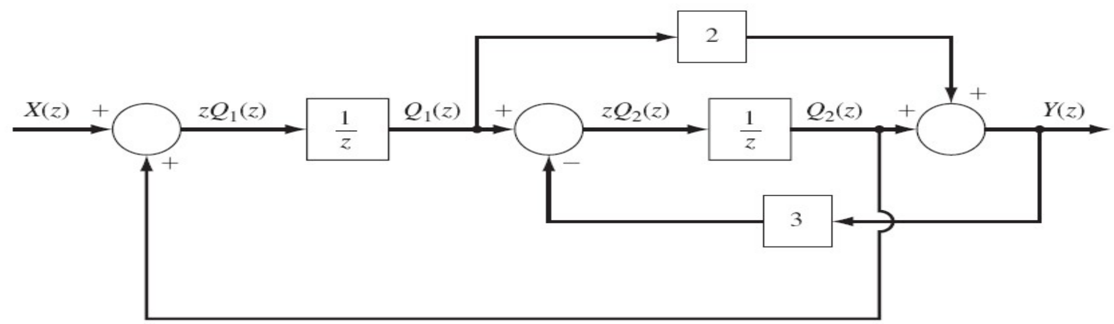


$$Y(z) = z^{-1} X(z)$$



Transfer Function Example

- **Redraw using *Z*-transform:**



- **Write equations for the behavior at each of the summation nodes:**

$$zQ_1(z) = Q_2(z) + X(z)$$
$$zQ_2(z) = Q_1(z) - 3Y(z)$$
$$Y(z) = 2Q_1(z) + Q_2(z)$$

- **Three equations and three unknowns: solve the first for $Q_1(z)$ and substitute into the other two equations.**

$$Q_1(z) = z^{-1}Q_2(z) + z^{-1}X(z)$$
$$zQ_2(z) = \left[z^{-1}Q_2(z) + z^{-1}X(z)\right] - 3Y(z)$$
$$Q_2(z) = z^{-2}Q_2(z) + z^{-2}X(z) - 3z^{-1}Y(z)$$
$$Q_2(z) = \frac{1}{1 - z^{-2}}\left[z^{-2}X(z) - 3z^{-1}Y(z)\right]$$
$$Y(z) = 2z^{-1}\left[\frac{1}{1 - z^{-2}}\left[z^{-2}X(z) - 3z^{-1}Y(z)\right]\right] + 2z^{-1}X(z) + \frac{1}{1 - z^{-2}}\left[z^{-2}X(z) - 3z^{-1}Y(z)\right]$$

Simplify.. .

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2z + 1}{z^2 + 3z + 5}$$

The Properties of ROC

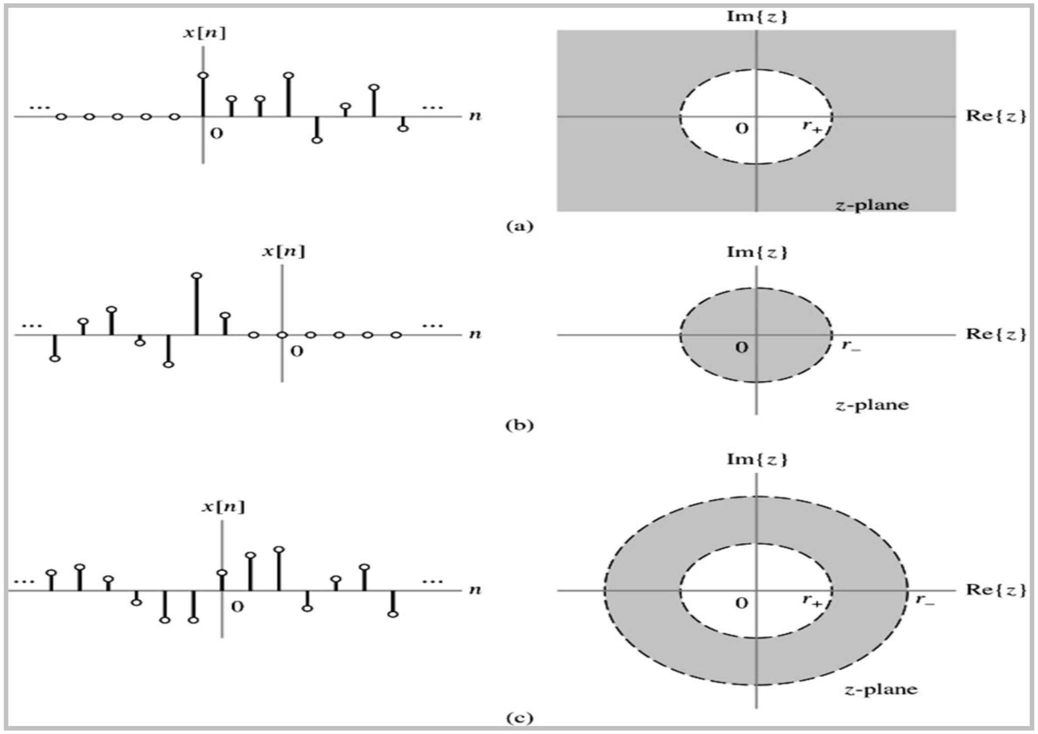


Figure : The relationship between the ROC and the time extent of a signal.

(a) A right-sided signal has an ROC of the form $|z| > r_+$.

(b) A left-sided signal has an ROC of the form $|z| < r_-$.

(c) A two-sided signal has an ROC of the form $r_+ < |z| < r_-$.

Poles: -1/2,1/4

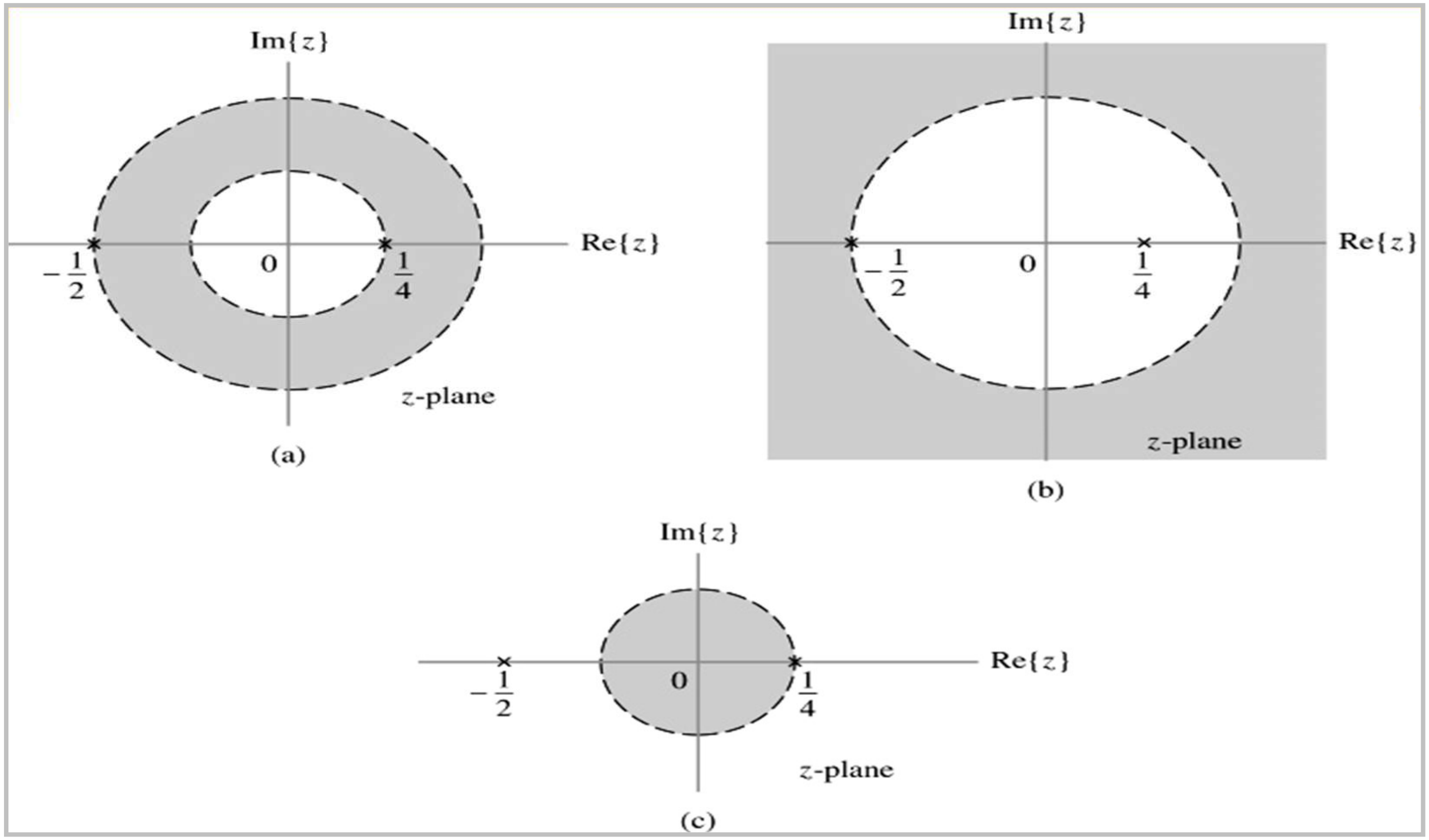


Figure : ROCs .

(a) Two-sided signal $x[n]$ has ROC in between the poles.

(b) Right-sided signal $y[n]$ has ROC outside of the circle containing the pole of largest magnitude.

(c) Left-sided signal $w[n]$ has ROC inside the circle containing the pole of smallest magnitude.

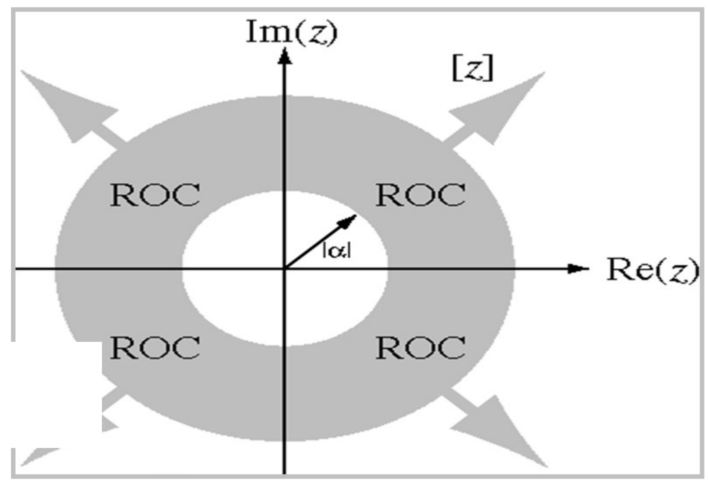
☑Taking a path analogous to that used the development of the

☑ Laplace transform, the z transform of the **causal** DT signal is

$$A\,\alpha^{n}\,\mathrm{u}[n]\,,\ |\alpha| > 0$$

$$\mathrm{X}(z) = A\sum_{n=-\infty}^{\infty}\alpha^{n}\,\mathrm{u}[n]z^{-n} = A\sum_{n=0}^{\infty}\alpha^{n}z^{-n} = A\sum_{n=0}^{\infty}\left(\frac{\alpha}{z}\right)^{n}$$

and the series converges if $|z| > |\alpha|$. This defines the ROC as the **exterior of a circle** in the z plane centered at the origin, of radius $|\alpha|$.



☑ The z transform is

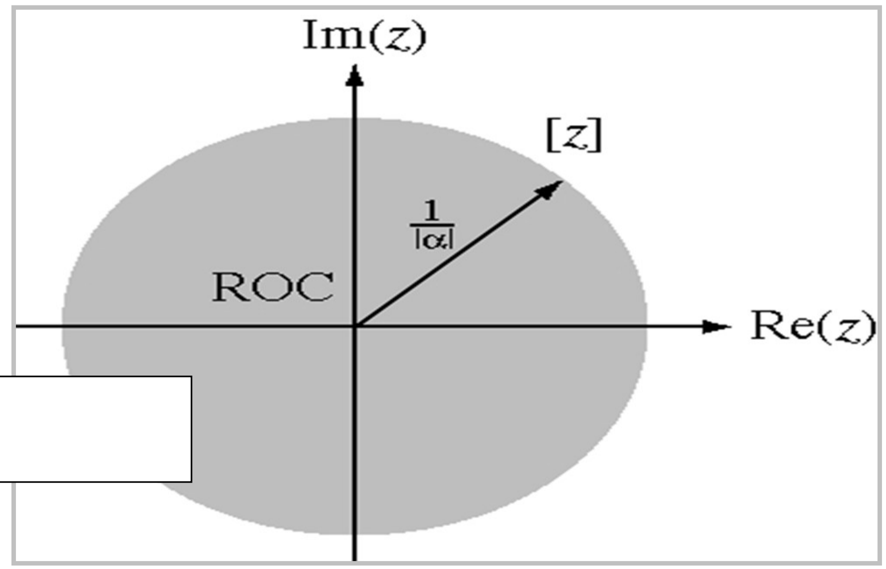
$$\mathrm{X}(z) = A\,\frac{z}{z-\alpha}\ ,\quad |z| > |\alpha|$$

Causal

☑ By similar reasoning, the z transform and region of convergence of the **anti-causal** signal below, are

$$A\alpha^{-n}u[-n], \ |\alpha| > 0$$

$$X(z) = \frac{A}{1-\alpha z} = \frac{Az^{-1}}{z^{-1}-\alpha}, \quad |z| < \frac{1}{|\alpha|}$$



Anti-Causal

Problem and Solution

The z-transform $F(z)$ of the function $f(nT) = a^{nT}$ is

(A) $\dfrac{z}{z-a^T}$ (B) $\dfrac{z}{z+a^T}$

(C) $\dfrac{z}{z-a^{-T}}$ (D) $\dfrac{z}{z+a^{-T}}$

Option (A) is correct.

We have $f(nT) = a^{nT}$

Taking z-transform we get

$$F(z) = \sum_{n=-\infty}^{\infty} a^{nT} z^{-n} = \sum_{n=-\infty}^{\infty} (a^T)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a^T}{z}\right)^n = \frac{z}{z-a^T}$$

Option (C) is correct.

Given z transform

$$C(z) = \frac{z^{-1}(1-z^{-4})}{4(1-z^{-1})^2}$$

Applying final value theorem

$$\lim_{n \to \infty} f(n) = \lim_{z \to 1} (z-1) f(z)$$

$$\lim_{z \to 1} (z-1) F(z) = \lim_{z \to 1} (z-1) \frac{z^{-1}(1-z^{-4})}{4(1-z^{-1})^2} = \lim_{z \to 1} \frac{z^{-1}(1-z^{-4})(z-1)}{4(1-z^{-1})^2}$$

$$= \lim_{z \to 1} \frac{z^{-1} z^{-4}(z^4-1)(z-1)}{4z^{-2}(z-1)^2}$$

$$= \lim_{z \to 1} \frac{z^{-3}}{4} \frac{(z-1)(z+1)(z^2+1)(z-1)}{(z-1)^2}$$

$$= \lim_{z \to 1} \frac{z^{-3}}{4} (z+1)(z^2+1) = 1$$