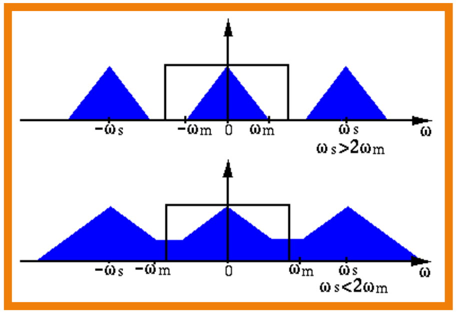
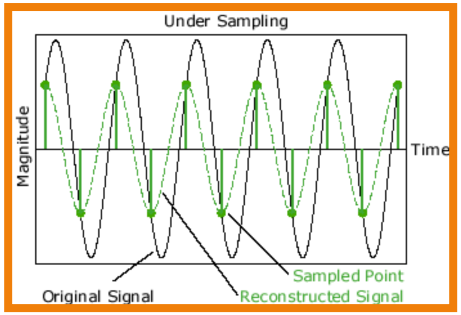
SAMPLING THEOREM

- **Objectives:**

  Representation Using Impulses
  FT of a Sampled Signal
  Signal Reconstruction
  Signal Interpolation
  Aliasing
  Multirate Signal Processin



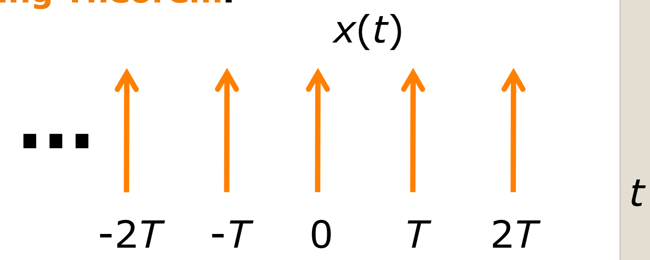


Representation of a CT Signal Using Impulse Functions

- **The goal of this lecture is to convince you that bandlimited CT signals, when sampled properly, can be represented as discrete-time signals with NO loss of information. This remarkable result is known as the Sampling Theorem.**

- **Recall our expression for a pulse train:**

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



- **A sampled version of a CT signal, $x(t)$, is:**

$$x_s(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta\big(t - nT\big) = \sum_{n=-\infty}^{\infty} x(nT)\delta\big(t - nT\big)$$

This is known as idealized sampling.

- **We can derive the complex Fourier series of a pulse train:**

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \text{where} \quad \omega_0 = 2\pi / T$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\Big[e^{-jk\omega_0 t}\Big]_{t=0} = \frac{1}{T}$$

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\omega_0 t}$$

Fourier Transform of a Sampled Signal

- The Fourier series of our sampled signal, $x_s(t)$ is:

$$x_s(t) = p(t)x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} x(t) e^{jk\omega_0 t}$$

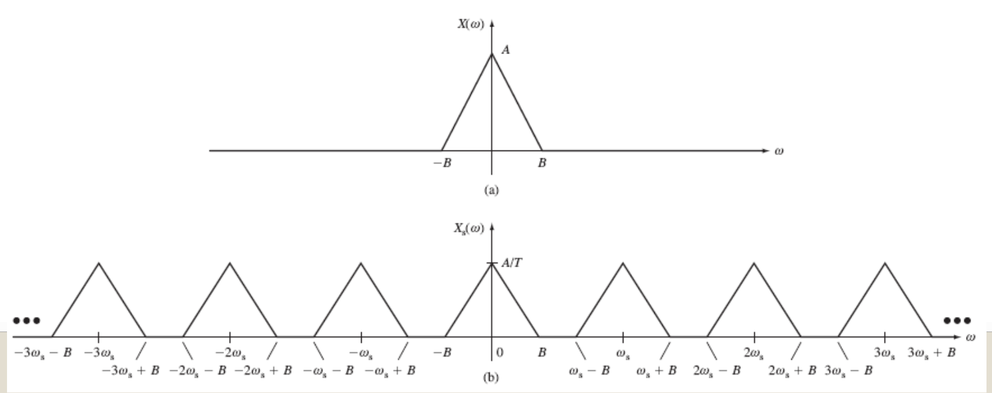
- **Recalling the Fourier transform properties of linearity (the transform of a sum is the sum of the transforms) and modulation (multiplication by a complex exponential produces a shift in the frequency domain), we can write an expression for the Fourier transform of our sampled signal:**

$$X_s\left(e^{j\omega}\right) = \mathcal{F}\{p(t)x(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \frac{1}{T} x(t) e^{jk\omega_0 t}\right\} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \mathcal{F}\left\{x(t) e^{jk\omega_0 t}\right\}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(e^{j(\omega - k\omega_0)})$$

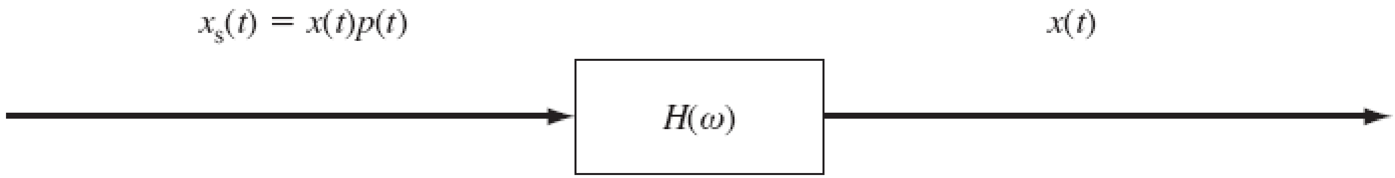
- **If our original signal, $x(t)$, is bandlimited:**

$$\left|X(e^{j\omega})\right| = 0 \quad \text{for} \quad \omega > B$$

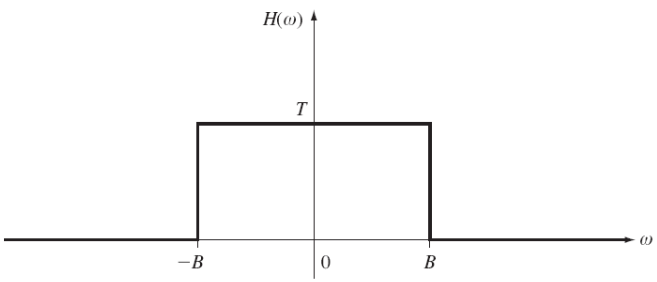


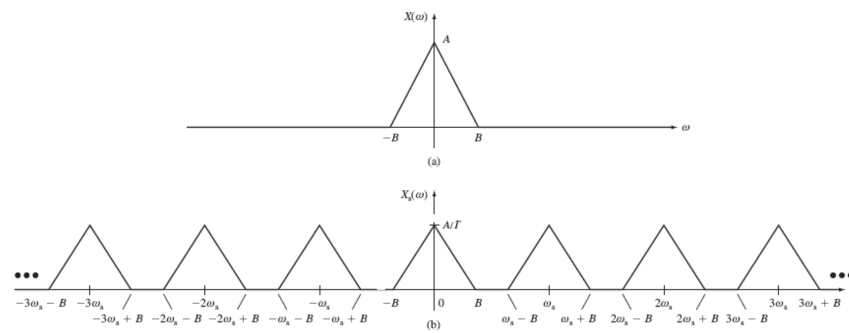
Signal Reconstruction

- Note that if $\omega_s \geq 2B$, the replicas of $X\left(e^{j\omega}\right)$ do not overlap in the frequency domain. We can recover the original signal exactly.



Lowpass filter





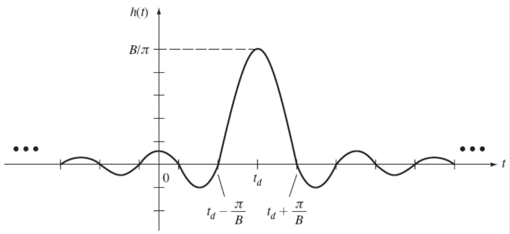
- The sampling frequency $\omega_s = 2B$, is referred to as the **Nyquist sampling frequency**.
- There are two practical problems associated with this approach:
  - The lowpass filter is not physically realizable. Why?
  - The input signal is typically not bandlimited. Explain.

Signal Interpolation

- **The frequency response of the lowpass, or interpolation, filter is:**

$$H(e^{j\omega}) = \begin{cases} T, & -B \le \omega \le B \\ 0, & \text{elsewhere} \end{cases}$$

- **The impulse response of this filter is given by:**



$$h(t) = \frac{BT}{\pi}\frac{\sin(Bt/\pi)}{(Bt/\pi)} \equiv \frac{BT}{\pi}\text{sinc}\,(Bt/\pi B \quad -\infty \le t \le \infty$$

- **The output of the interpolating filter is given by the convolution integral:**

$$y(t) = h(t) * x_s(t) = \int_{-\infty}^{\infty} x_s(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right]h(t-\tau)d\tau = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)h(t-\tau)d\tau$$

$$= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x(nT)\delta(t-nT)h(t-\tau)d\tau$$

- **Using the sifting property of the impulse:**

$$y(t) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x(nT)\delta(t-nT)h(t-\tau)d\tau$$

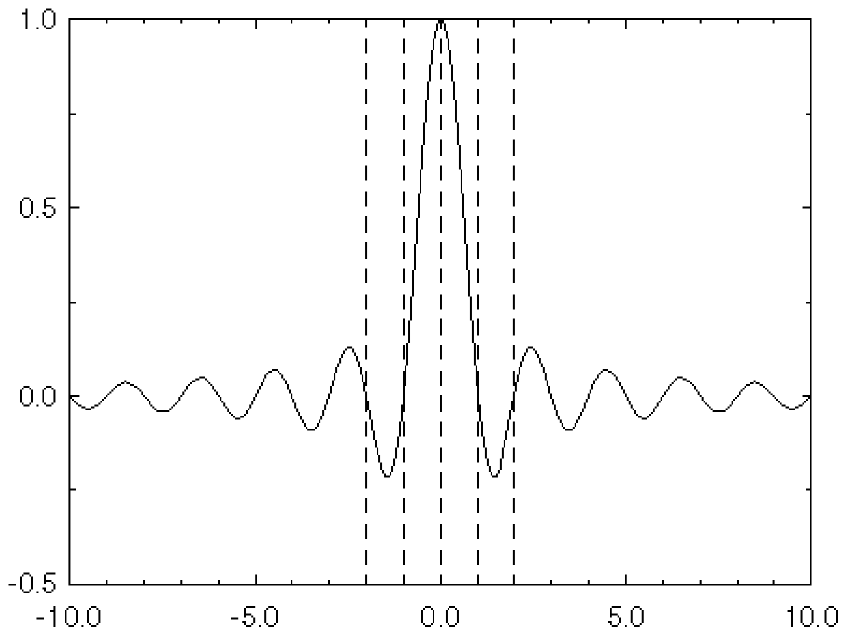
$$= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT)$$

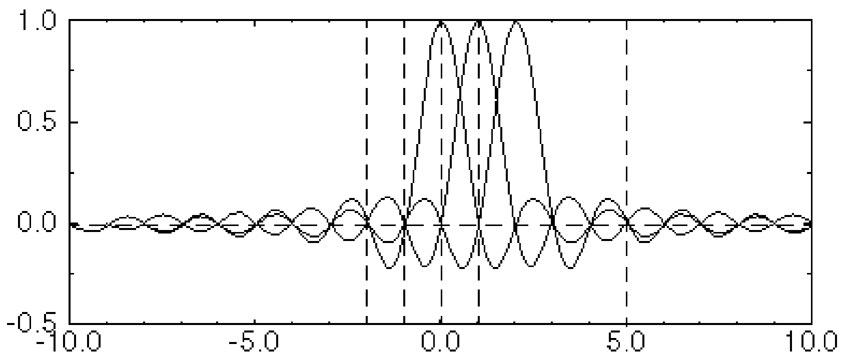
Signal Interpolation (Cont.)

- Inserting our expression for the impulse response:

$$y(t) = \frac{BT}{\pi} \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}\,(\frac{B}{\pi}(t-nT))$$

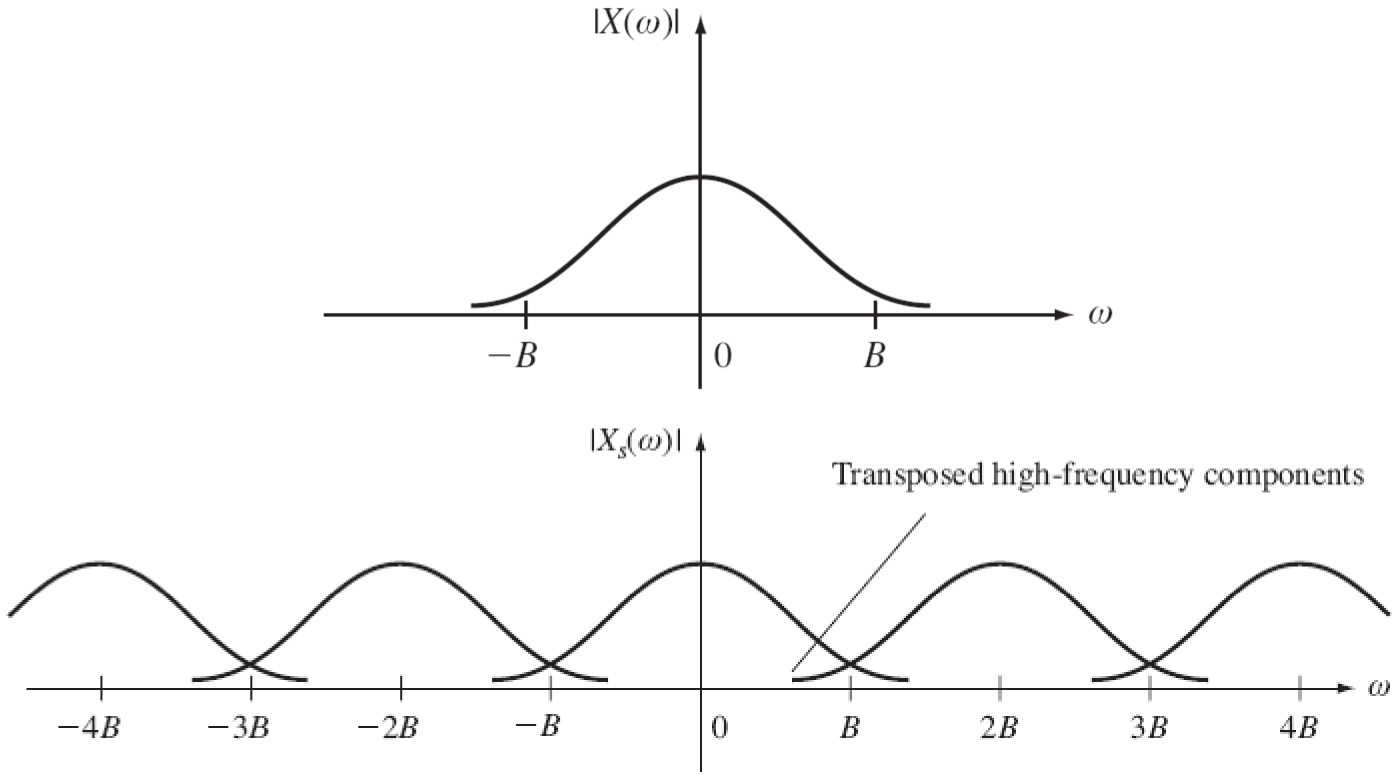
- This has an interesting graphical interpretation shown to the right.
- This formula describes a way to perfectly reconstruct a signal from its samples.
- Applications include digital to analog conversion, and changing the sample frequency (or period) from one value to another, a process we call resampling (up/down).
- But remember that this is still a noncausal system so in practical systems we must approximate this equation. Such implementations
are studied more extensively in an introductory DSP class.





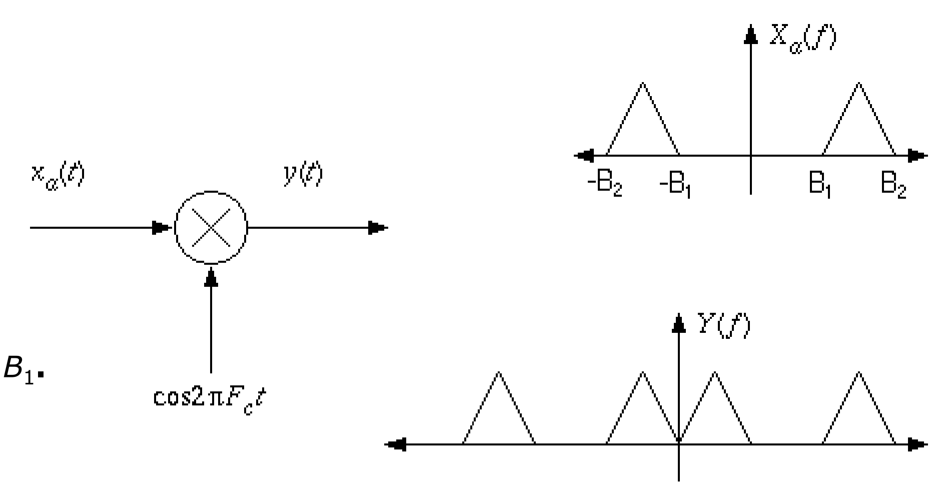
Aliasing

- **Recall that a time-limited signal cannot be bandlimited. Since all signals are more or less time-limited, they cannot be bandlimited. Therefore, we must lowpass filter most signals before sampling. This is called an anti-aliasing filter and are typically built into an analog to digital (A/D) converter.**
- **If the signal is not bandlimited distortion will occur when the signal is sampled. We refer to this distortion as aliasing**



Sampling of Narrowband Signals

- **What is the lowest sample frequency we can use for the narrowband signal shown to the right?**
- **Recalling that the process of sampling shifts the spectrum of the signal, we can derive a generalization of the Sampling Theorem in terms of the physical bandwidth occupied by the signal.**
- **A general guideline is**   **, where $B = B_2 - B_1$.**
- **A more rigorous equation depends on B_1 and B_2:**



$$2B \le f_s \le 4B$$

$$f_s = 2B\frac{r'}{r} \quad \text{where} \quad r' = \frac{f_c + B/2}{B}$$

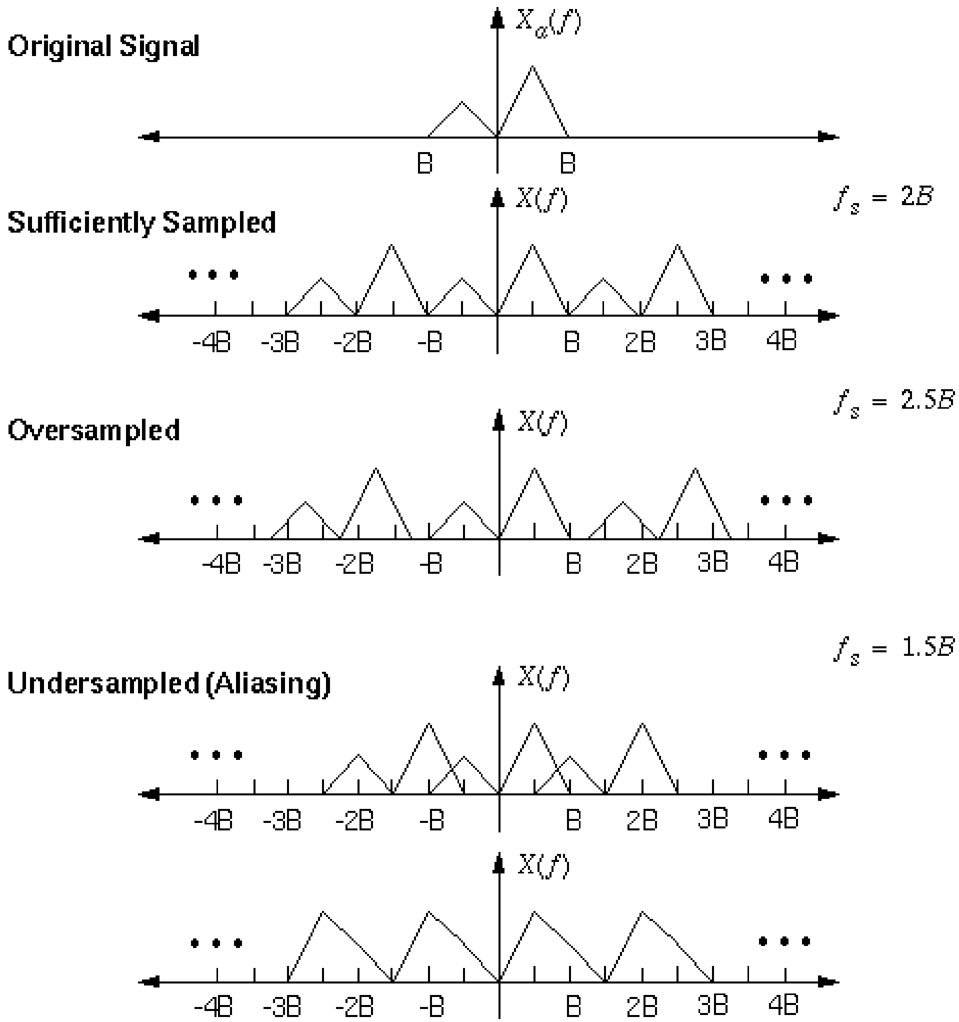
- **Sampling can also be thought of as a modulation operation, since it shifts a signal's spectrum in frequency.**

and

$$f_c = (B_1 + B_2)/2$$

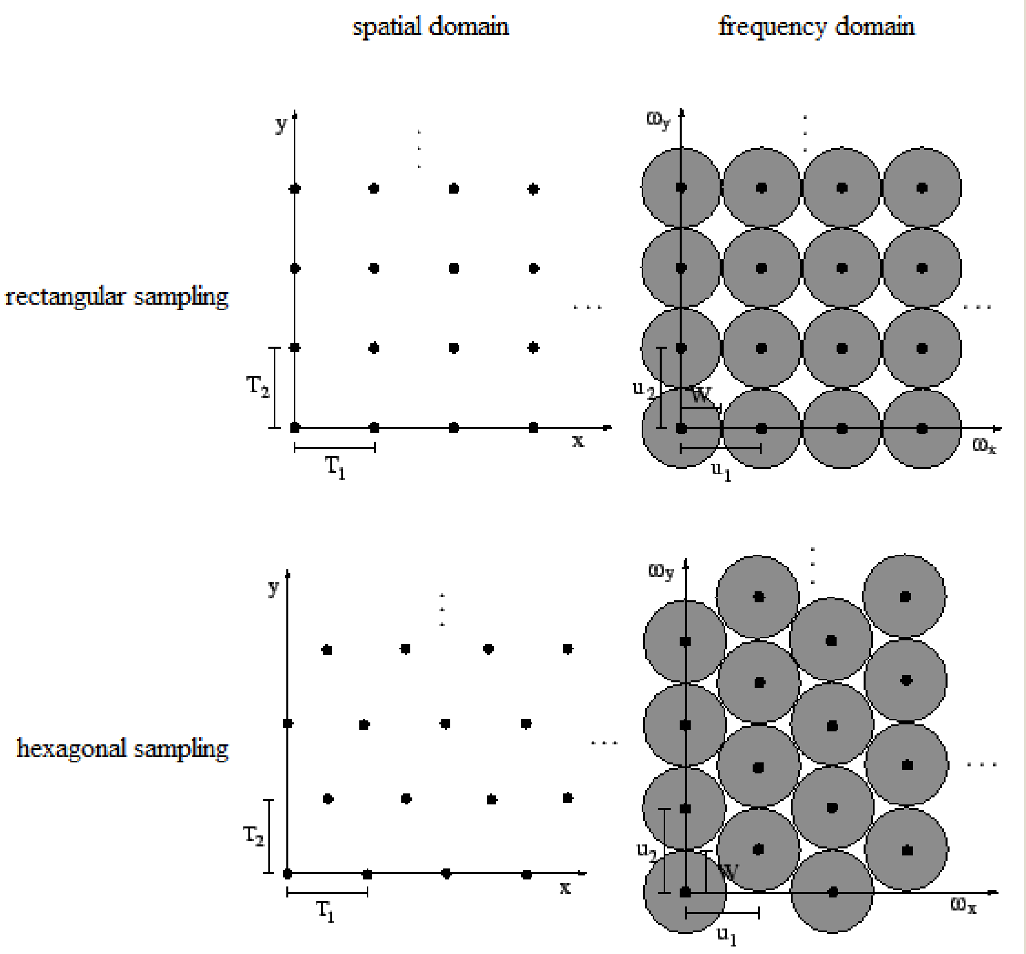
$$r = \lfloor r' \rfloor \quad \text{(greatest integer greater than or equal to } r\text{)}$$

Undersampling and Oversampling of a Signal



Sampling is a Universal Engineering Concept

- Note that the concept of sampling is applied to many electronic systems:
  - electronics: CD players, switched capacitor filters, power systems
  - biological systems: EKG, EEG, blood pressure
  - information systems: the stock market.
- Sampling can be applied in space (e.g., images) as well as time, as shown to the right.

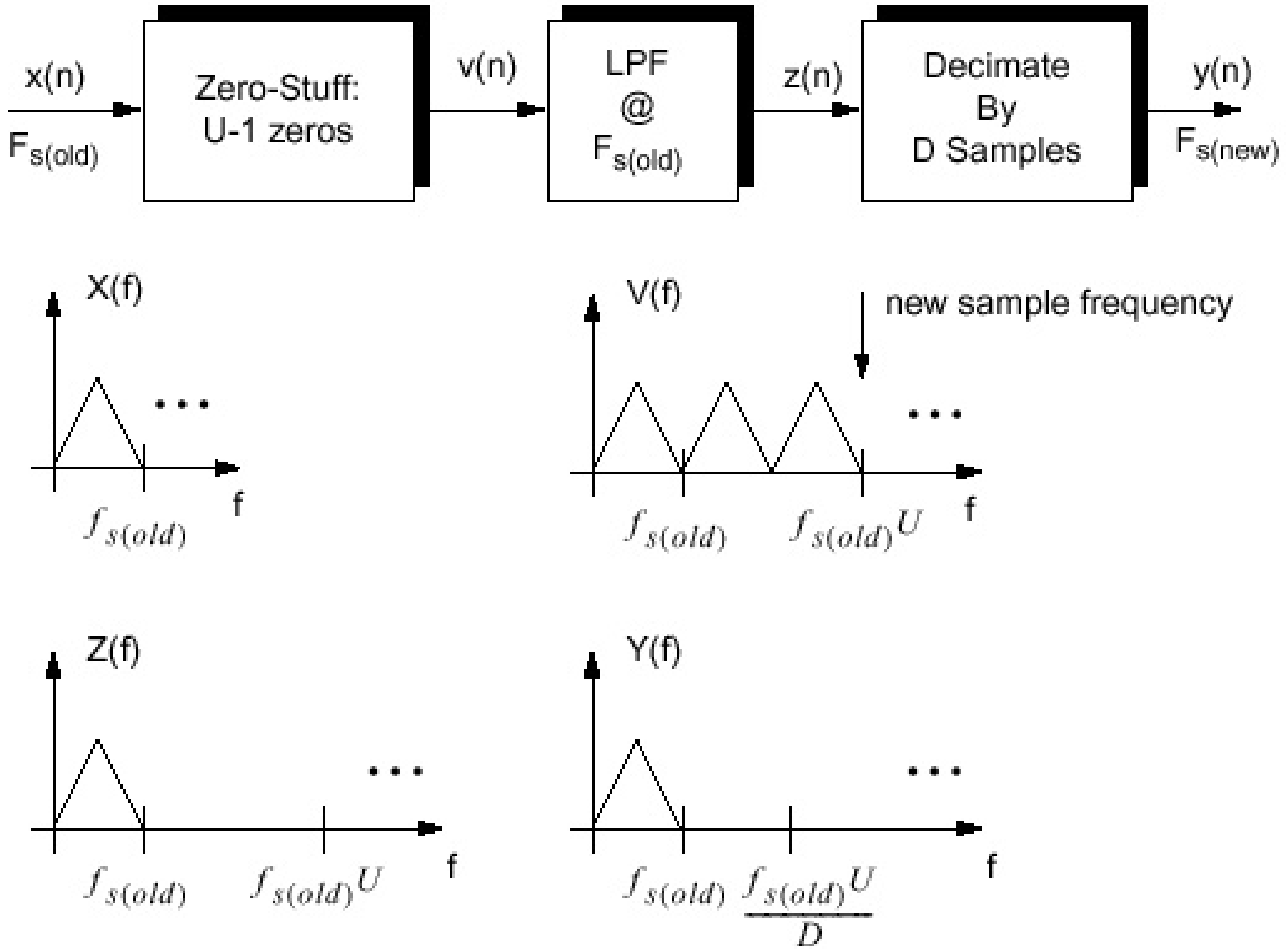


- Full-motion video signals are sampled spatially (e.g., 1280x1024 pixels at 100 pixels/inch) , temporally (e.g., 30 frames/sec), and with respect to color (e.g., RGB at 8 bits/color). How were these settings arrived at?

Downsampling and Upsampling

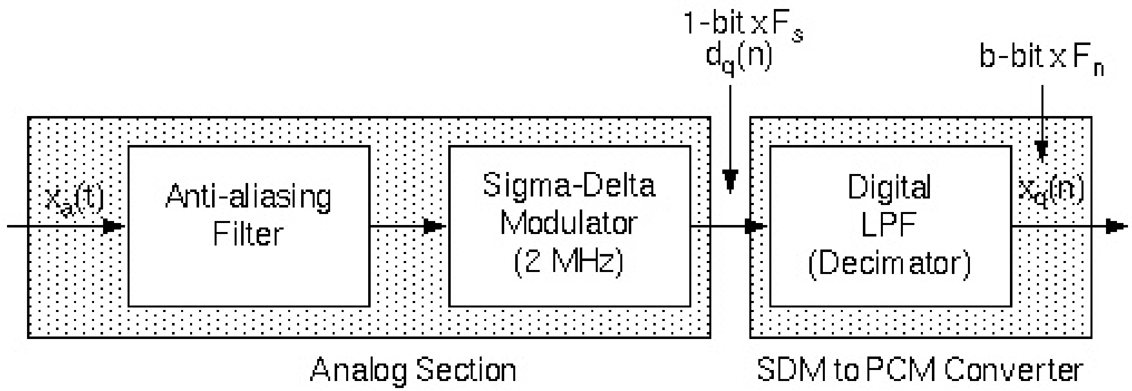
- **Simple sample rate conversions, such as converting from 16 kHz to 8 kHz, can be achieved using digital filters and zero-stuffing:**

$$F_{s(new)} = F_{s(old)}\left(\frac{U}{D}\right)$$

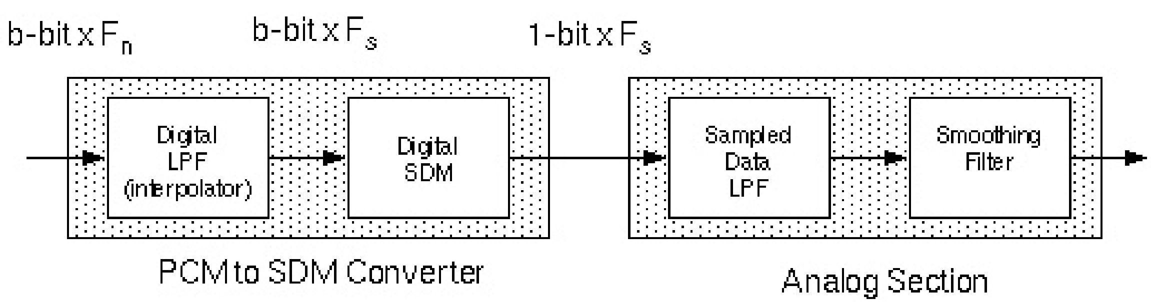


Oversampling

- **Sampling and digital signal processing can be combined to create higher performance samplers ☺**
- **For example, CD players use an oversampling approach that involves sampling the signal at a very high rate and then downsampling it to avoid the need to build high precision converter and filters.**



An Oversampling D/A Converter



Summary

- **Introduced the Sampling Theorem and discussed the conditions under which analog signals can be represented as discrete-time signals with no loss of information.**
- **Discussed the spectrum of a discrete-time signal.**
- **Demonstrated how to reconstruct and interpolate a signal using sinc functions that are a consequence of the Sampling Theorem.**
- **Introduced a variety of applications involving sampling including downsampling and oversampling.**