Simplex Method:

It is an iterative procedure and is one of the most powerful method for solving LPP. This can be applied for solving LPP in 2 or more variables (without any restriction on number of constraints).

If we apply simplex method to LPP with 2 variables we will see that value of variables at each iteration are extreme points of feasible region (in case feasible solution to LPP exists). Hence, we move from one extreme point to another in such a way that the value of objective function is improved. Since, number of extreme points in feasible region are finite, optimal solution is ~~reached~~ acheived in finite number of steps.

Feasible solution: A solution to LPP is feasible if all variables in it assume non-negative values i.e, $x_i \geq 0 \quad 1 \leq i \leq n$.

Consider LPP : $\max/\min \ z = CX$
subject to $AX (\leq \text{ or } \geq) b$
$X \geq 0$.
→ Cannonical Form

Standard form of LPP: $\max/\min \ z = CX$
subject to $AX = b$
$X \geq 0$.

$X_{n \times 1}, A_{m \times n}, C_{1 \times n}, b_{m \times 1}$ are matrices of given order (see subscript).

To apply analytical methods for solving LPP we need to convert it first into standard form. For this we need slack / surplus variables. Lets see the definition of these two:

Slack variable: Consider the i^{th} constraint:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \quad 1 \leq i \leq m$$

$\because$ L.H.S. $\leq$ R.H.S, so we add a variable s_i (≥ 0) such that

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s_i = b_i$$

$s_i \rightarrow$ is called slack variable. It represents ~~unsed~~ unused capacity in LPP.

Surplus variable: Consider the i^{th} constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i \quad 1 \leq i \leq m$$

$\because$ L.H.S. $\geq$ R.H.S, so we substract a variable s_i (≥ 0) such that

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - s_i = b_i$$

$s_i \rightarrow$ is called surplus variable. As the name suggests, it represents surplus capacity in LPP

Important: Cost / Profit associated with slack/surplus variables is zero [as unused or surplus capacity do not contribute ~~towards~~ to profit].

Example: Convert the given LPP in standard form

max $z = 2x + 3y$

subject to

$$x + 5y \leq 6$$
$$2x - y \geq 3$$
$$x, y \geq 0.$$

Soln Add slack variable to 1^{st} constraint & a surplus variable to 2^{nd} constraint

$\therefore$

$$\left.\begin{array}{l} \max z = 2x+3y \\ \text{subject to} \\ \quad x+5y+s_1=6 \\ \quad 2x-y-s_2=3 \\ \quad x, y, s_1, s_2 \geq 0 \end{array}\right] \text{standard form}$$

Consider $Ax=b$ $(m<n)$ where A is $m\times n$ matrix

Let $B_{m\times m}$ be submatrix of A with linearly independent columns of A. Then solution obtained by setting $(n-m)$-variables not associated with columns of B, equal to zero, and solving the resultant system, is called Basic solution to given system

i.e, $X_B = B^{-1}b$. $B \to$ Basis matrix

[Refer to case where $\text{rank}(Ab) = \text{rank}(A) <$ no. of unknowns in case of system of equations]

Basic feasible solution: (BFS) A solution which is basic as well as feasible is called BFS.

Note: If less then m-variables are positive then the solution is called degenerate BFS. (i.e, if, say, $x_i > 0$ ~~for~~ $i=1,2,\ldots k; k+2, \ldots m$ and $x_{k+1}=0$ then $(x_1, x_2, \ldots x_m, \underbrace{0,0,\ldots 0}_{n-m})$ form degenrate BFS).

Optimal Solution: Feasible solution which maximizes or minimizes, the objective function of LPP.

Unbounded Solution: A solution is called unbounded if the value of objective function increase or decrease indefinitely.

Ex Obtain all BFS of the following system of linear equations

$$2x_1 + 6x_2 + 2x_3 = 3$$
$$6x_1 + 4x_2 + 4x_3 = 2$$

Solution Given system of equations $Ax = b$ is

$$\underset{A}{\begin{pmatrix} 2 & 6 & 2 \\ 6 & 4 & 4 \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{b}{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

Since $\rho(A) = 2$, the maximum number of l.I columns of A is 2. Thus we can take any of the following 2x2 basis matrix B

$$\begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix}, \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}$$

Consider $B = \begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}$ Then $x_3 = 0$. $X_{B_1} = B^{-1}b = -\frac{1}{28}\begin{pmatrix} 4 & -6 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

$\therefore$ one BFS is $(0, 1/2, 0)$

Now Let $B = \begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix}$ $\therefore x_2 = 0$ $X_{B_2} = B^{-1}b = -\frac{1}{4}\begin{pmatrix} 4 & -2 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 7/2 \end{pmatrix}$

$\therefore (-2, 0, 7/2) \rightarrow$ Not a BFS (as $x_1 = -2 < 0$)

Now, let $B = \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}$ $\therefore x_1 = 0$ $X_{B_3} = B^{-1}b = \frac{1}{16}\begin{pmatrix} 4 & -2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

$\therefore$ BFS is $(0, 1/2, 0)$

Hence, this system has only one BFS $(0, 1/2, 0)$.

Note! This is a degenerate BFS as one of the basic variable is zero ($x_1 = 0$ in X_{B_1} & $x_3 = 0$ is X_{B_3})

Ex Obtain all ~~BFS~~ Basic solutions to the system

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5.$$

Solution $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $X_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $x_3 = 0$

$B = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$ $X_B = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ $x_2 = 0$

$B = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ $X_B = \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix}$ $x_1 = 0$.

Note: All above three basic solutions are non-degenerate.

Theorem: (Reduction of FS to BFS) If an LPP has a feasible solution, then it also has a basic feasible solution

Proof: Consider the LPP max $z = CX$
subject to $AX = b$, $X \geq 0$

$C_{1\times n}$, $X_{n\times 1}$, $b_{m\times 1}$, $A_{m\times n}$ real matrices.

Let $\rho(A) = m$. Since, feasible solution exists, so
$\rho(A,b) = \rho(A)$, $m < n$.

Let $X = (x_1, x_2, \ldots x_n)$ be feasible soln. with $x_i \geq 0$ $\forall i$

Let X have p-positive components and remaining $(n-p)$ components are zero. Relabel the components so that positive components are first 'p' and accordingly relabel columns of A. Then

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_p x_p = b$$

where $a_1, a_2, \ldots a_p$ are first p columns of A.

Now, 2 cases arise:

Case I: Vectors $a_1, a_2, \ldots a_p$ form l.i set. Then $p \leq m$

If $p = m$, the given solution is non-degenerate BFS with $x_1, x_2, \ldots x_p$ as basic variables.

If $p < m$, then the set $\{a_1, a_2, \ldots a_p\}$ can be extended to $\{a_1, \ldots a_p, a_{p+1}, \ldots a_m\}$ to form basis for columns of A. Then we have

$$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = b.$$

where $x_j = 0$, $j = p+1, p+2, \ldots m$.

Thus, we have a degenerate BFS with $m-p$ of the basic variables zero.

Case II: Set $\{a_1, a_2, \ldots a_p\}$ is l.d. Obviously $p>m$

$\Rightarrow \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_p a_p = 0$ α_i's are constant, (defn of l.d. vectors)

$\Rightarrow \exists r$, s.t. $\alpha_r \neq 0$. Then $a_r = -\sum_{j \neq r}^{p} \frac{\alpha_j}{\alpha_r} a_j$

$\therefore \sum_{j \neq r}^{p} a_j x_j + \left(-\sum_{j \neq r}^{p} \frac{\alpha_j}{\alpha_r} a_j\right) x_r = b.$

or, $\sum_{j \neq r}^{p} \left(x_j - x_r \frac{\alpha_j}{\alpha_r}\right) a_j = b$

This is a solution in which not more than $(p-1)$ non-zero components are there.

To ~~ensure~~ ensure that these are positive, choose a_r, in such a way that $x_j - x_r \frac{\alpha_j}{\alpha_r} \geq 0 \ \forall j \neq r$.

$\Rightarrow$ either $\alpha_j = 0$ or $\frac{x_j}{\alpha_j} > \frac{x_r}{\alpha_r}$, if $\alpha_j > 0$.

$\frac{x_j}{\alpha_j} < \frac{x_r}{\alpha_r}$ if $\alpha_j < 0$.

$\therefore$ Select a_r such that $\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j}, \alpha_j > 0 \right\}$

Then all $(p-1)$ variables are $\left(x_j - x_r \frac{\alpha_j}{\alpha_r}\right)$ are non-negative.

So, we have a F.S. with not more than $(p-1)$ non-zero components.

If the corresponding set of $(p-1)$ columns of A is l.i, we have BFS. If not, case II applies and we again have l.d set. So repeat the process till we arrive at F.S. with associated set of columns of A l.i. Hence, from case I, we have BFS

Corollary: There exists only finite number of BFS. to a LPP.