Simplex Algorithm (Maximization case)

1) Express mathematical model of LPP in standard form by adding slack variables in the L.H.S. of constraints and assign a zero cofficient to these in the objective function.

* If any b_i is negative, the multiply inequality by (−1), so as to make b_i positive *

2) An initial BFS is obtained by setting: $x_1 = x_2 = \cdots = x_n = 0$. Hence, we have $s_1 = b_1$, $s_2 = b_2$, ... $s_m = b_m$.

basis matrix $\therefore B = [s_1\ s_2 \ldots s_m]$. $X_B = B^{-1}b \rightarrow$ basic soln.

3) Set up simplex tableau.

Identity matrix

cost coff. of Basic variables C_B	Basic Variables B.V	c_j / X_B	c_1	c_2	...	c_n	0	0	...	0	min ratio x_{Bi}/a_{ij}
			x_1	x_2	...	x_n	s_1	s_2	...	s_m	
			a_{11}	a_{12}		a_{1n}	1	0		0	
			a_{21}	a_{22}		⋮	0	1		⋮	
			⋮	⋮		⋮	⋮	0		0	
			a_{n1}	a_{nk}		a_{nn}	0	0		1	
$z_j = \sum C_B x_j$											
		$c_j - z_j$	$c_1 - z_1$	$c_2 - z_2$		$c_n - z_n$	0	0		0	

4) Compute $c_j - z_j$ $(j = 1, 2, \ldots n)$ where $z_j = C_B x_j$

Examine sign of $c_j - z_j$:

(1) If $c_j - z_j \le 0$ $\forall\ 1 \le j \le n$, then current solution is optimum.

(2)(a) If atleast one $(c_j - z_j) > 0$, then choose the one with largest positive value. The corresponding column is called key or pivot column. The variable corresponding to this column enters the basis (↑) (i.e, non-basic variable becomes a basic variable)

(b) Determine departing variable: Compute

$$\min \left\{ \frac{x_{Bi}}{a_{ij}} : a_{ij} > 0,\ i = 1, \ldots m \right\}$$

where a_{ij} are entries in j^{th} column (step 2(a)).

Let it be x_k. Then x_k leaves.

Pivot element: a_{kj} (i.e, element at intersection of x_j^{th} column & k^{th} row)

5) Find new BFS using information in step 4.
~~Make~~ (a) Divide k^{th} row by a_{kj}
(b) Make all other entries in j^{th} column zero by row-operations (except k^{th} row element which is 1) i.e, $a_{kj} = 1$
(c) Compute $c_j - z_j$

6) If any $c_j - z_j > 0$ repeat steps 4(2), 5 again until all $c_j - z_j \le 0$ (i.e, optimum solution is reached)

Remarks! (1) When determining leaving variable, if all entries in j^{th} column are negative or zero, then solution is unbounded

(2) If there is a tie while selecting leaving variable choose the one which is near to top!

Note: (1) $c_j - z_j$ → represents the net contribution to the objective function that results by introducing one unit of each of the respective column variables ~~i.e.~~ So, if $c_k - z_k > 0$, that means we can increase the profit by $c_k - z_k$ units by introducing one unit of x_k.

(2) For basic variables $c_j - z_j = 0$.
For non-basic variables $c_j - z_j > 0$ or < 0.

Ex $\max z = 10x_1 + 6x_2 + 4x_3$

subject to
$$x_1 + x_2 + x_3 \le 100$$
$$10x_1 + 4x_2 + 5x_3 \le 600$$
$$2x_1 + 2x_2 + 6x_3 \le 300$$
$$x_1, x_2, x_3 \ge 0$$

Solve the above LPP using simplex method.

Soln standard form of LPP:

$$\max z = 10x_1 + 6x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to
$$x_1 + x_2 + x_3 + s_1 = 100 \quad \text{—} (*)$$
$$10x_1 + 4x_2 + 5x_3 + s_2 = 600$$
$$2x_1 + 2x_2 + 6x_3 + s_3 = 300$$
$$x_i, s_i \ge 0 \quad 1 \le i \le 3$$

If x_1, x_2, x_3 are not produced, then the unused capacity will be s_1, s_2, s_3 whose value will be 100, 600, 300 respectively (see (*))

$\therefore$ Initial BFS is: $x_1 = x_2 = x_3 = 0$, $s_1 = 100$, $s_2 = 600$, $s_3 = 300$.

		C_j	10	6	4	0	0	0	min ratio
C_B	BV(B)	X_B	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	100	1	1	1	1	0	0	100/1 = 100
0	s_2	600	$\boxed{10}$	4	5	0	1	0	600/10 = 60 →
0	s_3	300	2	2	6	0	0	1	300/2 = 150
		$C_j - Z_j$	10 ↑	6	4	0	0	0	

$$Z_j = \sum_{i=1}^{m} C_{B_i} a_{ij}$$

$\therefore$ ~~$Z_1 = \sum C_{B_i} a_{i1} = 0 \cdot 1 + 0 \cdot 10 + 0 \cdot 2 = 0$~~

~~$Z_2 = \sum C_{B_i} a_{12} = 0 \cdot 1 + 0 \cdot 4 + 0 \cdot 2 = 0$~~

~~$Z_3 = \sum C_{B_1} a_{31} = 0 \cdot 1 = 0$.~~

$Z_1 = C_{B_1}a_{11} + C_{B_2}a_{21} + C_{B_3}a_{31} = 0 \qquad c_1 = 10$

$Z_2 = C_{B_1}a_{12} + C_{B_2}a_{22} + C_{B_3}a_{32} = 0 \qquad c_2 = 6$

$Z_3 = C_{B_1}a_{13} + C_{B_2}a_{23} + C_{B_3}a_{33} = 0 \qquad c_3 = 4$

$\therefore C_1 - Z_1 = 10, \quad C_2 - Z_2 = 6, \quad C_3 - Z_3 = 4$

$\max(\cancel{z_j}\ c_j - z_j) = \max\{10, 6, 4, 0, 0, 0\} = 10$

$\Rightarrow \boxed{x_1}$ is the entering ~~set~~ variable

$\min\left\{\frac{x_{Bi}}{a_{i1}} : a_{i1} > 0\right\} = \min\left\{\frac{100}{1}, \frac{600}{10}, \frac{300}{2}\right\} = 60$

$\Rightarrow \boxed{s_2}$ is leaving vaiible.

pivot element $= 10$. (a_{21})

* Now, divide 2^{nd} row by $a_{21}(=10)$ so as to make pivot element 1
* Using this make other entries in ~~th~~ column x_1 zero. using row operation.

	c_j		10	6	4	0	0	0	
C_B	B.V	X_B	x_1	x_2	x_3	s_1	s_2	s_3	min ratio x_B/x_2
0	s_1	40	0	$\boxed{6/10}$	5/10	1	$-1/10$	0	400/6
10	x_1	60	1	4/10	5/10	0	1/10	0	600/4
0	s_3	180	0	12/10	5	0	$-4/10$	1	1800/12
		$c_j - z_j$	0	2	-1	0	-1	0	

$z_1 = c_{B1}a_{11} + c_{B2}a_{21} + c_{B3}a_{31} = 0 + 10 + 0 = 10$

$z_2 = 0 \cdot \frac{6}{10} + 10\,\frac{4}{10} + 0\,\cancel{12}/10 = 0 + 4 + 0 = 4$

$z_3 = 0\,\frac{5}{10} + 10\,\frac{5}{10} + 0 \cdot 5 = 0 + 5 + 0 = 5$

$\therefore c_1 - z_1 = 10 - 10 = 0$. $c_2 - z_2 = 6 - 4 = 2$, $c_3 - z_3 = 4 - 5 = -1$

Similarly compute for columns s_1, s_2 & s_3

$\max_j c_j - z_j = 2 \rightarrow \therefore$ x_2 enters

$\min\left\{\frac{x_B}{x_{j2}}, x_{j2} > 0\right\} = \min\left\{\frac{40}{(6/10)}, \frac{60}{(4/10)}, \frac{180}{(12/10)}\right\} = \frac{400}{6}$

(s_1 leaves)

pivot element: $6/10$.

* Repeat the above process

		C_j	10	6	4	0	0	0
C_B	B.V	X_B	x_1	x_2	x_3	s_1	s_2	s_3
6	x_2	$\frac{400}{6}$	0	1	5/6	10/6	-1/6	0
10	x_1	100/3	1	0	1/6	-2/3	1/6	0
0	s_3	100	0	0	4	-2	0	1
		$C_j - Z_j$	0	0	-8/3	-10/3	-2/3	0

Since all $C_j - Z_j \le 0$, hence optimum solution is obtained.

$$x_1 = \frac{100}{3}, \quad x_2 = \frac{400}{6} = \frac{200}{3}, \quad x_3 = 0.$$

$$Z_{max} = 10\left(\frac{100}{3}\right) + 6\left(\frac{200}{3}\right) + 4 \cdot 0 = \frac{2200}{3}$$ — Ans

<u>Example 2</u>: maximize $z = 4000x_1 + 2000x_2 + 5000x_3$

subject to
$$12x_1 + 7x_2 + 9x_3 \le 1260$$
$$22x_1 + 18x_2 + 16x_3 \le 19008$$
$$2x_1 + 4x_2 + 3x_3 \le 396$$
$$x_1, x_2, x_3 \ge 0.$$

Solve the LPP using simplex method.

<u>Solution 2</u>: max $z = 4000x_1 + 2000x_2 + 5000x_3 + 0s_1 + 0s_2 + 0s_3$

subject to
$$12x_1 + 7x_2 + 9x_3 + s_1 = 1260$$
$$22x_1 + 18x_2 + 16x_3 + s_2 = 19008$$
$$2x_1 + 4x_2 + 3x_3 + s_3 = 396$$
$$x_1, x_2, x_3 \ge 0$$

		C_j	4000	2000	5000	0	0	0	X_B/x_3
C_B	BV	X_B	x_1	x_2	x_3	s_1	s_2	s_3	min Ratio
0	s_1	1260	12	7	9	1	0	0	1260/9
0	s_2	19008	22	18	16	0	1	0	19008/16
0	s_3	396	2	4	$\boxed{3}$	0	0	1	396/3 →
		$C_j - Z_j$	4000	2000	5000 ↑	0	0	0	

		C_j	4000	2000	5000	0	0	0	x_B/x_1
C_B	BV	X_B	x_1	x_2	x_3	s_1	s_2	s_3	min ratio
0	s_2	72	$\boxed{6}$	-5	0	1	0	-3	$72/6=12$ →
0	s_1	16896	$34/3$	$-10/3$	0	0	1	$-16/3$	1490.8
5000	x_3	132	$2/3$	$4/3$	1	0	0	$1/3$	198
		$C_j - Z_j$	$\frac{2000}{3}$ ↑	$\frac{-14000}{3}$	0	0	0	$\frac{-5000}{3}$	

		C_j	4000	2000	5000	0	0	0	min Ratio
CB	BV	X_B	x_1	x_2	x_3	s_1	s_2	s_3	
4000	x_1	12	1	$-5/6$	0	$1/6$	0	$-1/2$	
0	s_2	16760	0	$55/9$	0	$-17/9$	1	$1/3$	
5000	x_3	124	0	$17/9$	1	$-1/9$	0	$2/3$	
		$C_j - Z_j$	0	$-\frac{37000}{9}$	0	$\frac{-1000}{9}$	0	$-\frac{4000}{3}$	

Since all $C_j - Z_j \le 0$, so optimum solution is obtained

$x_1 = 12$, $x_2 = 0$ $x_3 = 124$ $Z_{max} = 668000$.

Ans