Degeneracy: This ~~may~~ occurs when the one of the constraints on the right hand side is zero.

Degeneracy may occur at:

I **Initial stage:** When one or more basic variables assume value zero in I.BFS (i.e, one or more b_i's are zero)

II **Subsequent Iterations:** When min-ratio are equal for 2 or more rows for selecting outgoing variables. In such ~~a~~ case one or more ~~rows~~ variables (b_i's) become zero in next iteration and the problem becomes degenerate.

Consequences: The subsequent iterations may not produce improvements in the value of objective fn. It is also possible that same sequence of simplex iterations may repeat endlessly. This is called cycling.

Method to Resolve Degeneracy (Tie):

Step I: Divide each element in the tied rows by the corresponding positive cofficient of the key column (entering variable column) in that row

Step II: Compare the resulting ratio (from left to right) first in unit matrix and then in body matrix, column by column

Step III: The row which first contains the smallest ratio will be outgoing variable.

After this proceed with usual simplex iterations for finding optimal solution.

Or

① $\min_i \left\{ \frac{s_{i1}}{x_{ij}} : x_{ij} > 0 \right\} \rightarrow$ 1st column of unit matrix $\rightarrow$ Tie does not get resolved $\rightarrow$ 2nd column of unit matrix

② $\min_i \left\{ \frac{s_{i2}}{x_{ij}} : x_{ij} > 0 \right\}$

Example! Solve LPP max $z = 3x_1 + 8x_2$

subject to $x_1 + 4x_2 \le 8$

$x_1 + 3x_2 \le 6$

$x_1, x_2 \geq 0$.

Soln Standard LPP max $z = 2x_1 + 8x_2 + 0s_1 + 0s_2$

subject to $x_1 + 4x_2 + s_1 = 8$

$x_1 + 3x_2 + s_2 = 6$

$x_1, x_2, s_1, s_2 \geq 0$

Suppose we take s_1 as leaving variable

		C_j	3	8	0	0	
C_B	BV	X_B	x_1	x_2	s_1	s_2	min Ratio
0	s_1	8	1	$\boxed{4}$	1	0	$8/4 = 2$ →
0	s_2	6	1	3 ↑	0	1	$6/3 = 2$
		$C_j - Z_j$	3	8	0	0	
8	x_2	2	$1/4$	1	$1/4$	0	8
0	s_2	0	$\boxed{1/4}$ ↑	0	$-3/4$	1	0 →
		$C_j - Z_j$	1	0	-2	0	
8	x_2	2	0	1	$\boxed{1}$	-1	2 →
3	x_1	0	1	0	-3 ↑	4	—
		$C_j - Z_j$	0	0	1	-4	
0	s_1	2	0	1	1	-1	
3	x_1	6	1	3	0	1	
		$C_j - Z_j$	0	-1	0	-3	

∴ optimal soln is $x_1 = 6$, $x_2 = 0$, $z_{max} = 18$

Note! (1) In 2nd and 3rd iteration value z does not change

(2) In 4th iteration degeneracy is removed automatically. (This happens in certain cases only, not in all cases)

Now, suppose we apply above rules of removing degeneracy:

(51)

(i) Take ratio of s_1 (1st column of unit matrix) to x_2 (entering variable)

$$\min\left\{\frac{s_{i1}}{x_{i2}} : x_{i2} > 0\right\} = \left\{\frac{1}{4}, \frac{0}{3}\right\} = 0 \quad \therefore\ s_2 \text{ leaves.}$$

		c_j	3	8	0	0	
CB	BV	x_B	x_1	x_2	s_1	s_2	min ratio
0	s_1	8	1	4	1	0	2
0	s_2	6	1	$\boxed{3}$	0	1	2 →
		$c_j - z_j$	3	8 ↑	0	0	
0	s_1	0	$-1/3$	0	1	$-4/3$	–
8	x_2	2	$\boxed{1/3}$	1	0	$1/3$	6 →
		$c_j - z_j$	$1/3$ ↑	0	0	$-8/3$	
0	s_1	2	0	1	1	-1	
3	x_1	6	1	3	0	1	
		$c_j - z_j$	0	-1	0	-3	

optimal soln is $x_1 = 6$, $x_2 = 0$, $z_{max} = 18$

Here you can see no. of iterations are less than previous.

Example max $z = x_1 + 2x_2 + x_3$

subject to

$$2x_1 + x_2 - x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6 \Rightarrow 2x_1 - x_2 + 5x_3 \le 6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

standard LPP: max $z = x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + 0s_3$

subject to

$$2x_1 + x_2 - x_3 + s_1 = 2$$
$$2x_1 - x_2 + 5x_3 + s_2 = 6$$
$$4x_1 + x_2 + x_3 + s_3 = 6$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

	c_j		1	2	1	0	0	0	
CB	BV	XB	x_1	x_2	x_3	s_1	s_2	s_3	min Ratio
0	s_1	2	2	$\boxed{1}$	−1	1	0	0	2/1 = 2 →
0	s_2	6	+2	−1	5	0	1	0	—
0	s_3	6	4	1	1	0	0	1	6/1 = 6
		$c_j - z_j$	1	2 ↑	1	0	0	0	
2	x_2	2	2	1	−1	1	0	0	—
0	s_2	8	4	0	4	1	1	0	8/4 = 2
0	s_3	4	2	0	$\boxed{2}$	−1	0	1	4/2 = 2 →
		$c_j - z_j$	−3	0	3 ↑	−2	0	0	

Now tie b/w 2nd & 3rd row

$\therefore \min_i \left\{ \frac{x_{i2}}{x_{i3}} : x_{i3} > 0 \right\}$ ($\because$ $x_2 \to$ 1st column of unit matrix)

$= \left\{ -\infty, \frac{0}{4}, \frac{0}{2} \right\} \to$ Tie not solved

Now, $\min_i \left\{ \frac{s_{i2}}{x_{i3}} : x_{i3} > 0 \right\} = \left\{ \frac{1}{4}, \frac{0}{2} \right\} = 0 \therefore s_3$ leave

$\to$ (2nd column of unit matrix)

			1	2	1	0	0	0	
CB	BV	XB	x_1	x_2	x_3	s_1	s_2	s_3	min Ratio
2	x_2	4	3	1	0	1/2	0	1/2	
0	s_2	0	0	0	0	3	1	−2	
1	x_3	2	1	0	1	−1/2	0	1/2	
		$c_j - z_j$	−6	0	0	−1/2	0	−3/2	

optimality is reached. Optimal soln is

$\boxed{x_1 = 0 \quad x_2 = 4 \quad x_3 = 2}$ $\boxed{z_{max} = 10}$

Ex (Tie b/w artificial & slack (surplus variable) (52)

Ex max $z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - Ma_1$

subject to

$$2x_1 + 2x_2 - x_3 - s_1 + a_1 \geq 2$$

$$3x_1 - 4x_2 + s_2 \leq 3$$

$$x_2 + 3x_3 + s_3 \leq 5$$

$$x_1, x_2, x_3 \geq 0, \; s_1, s_2, s_3, a_1 \geq 0$$

		c_j	5	-2	3	0	0	0	-M	
C_B	BV	X_B	x_1	x_2	x_3	s_1	s_2	s_3	a_1	min Ratio
-M	a_1	2	$\boxed{2}$	2	-1	-1	0	0	1	2/2=1
0	s_2	3	3	-4	0	0	1	0	0	3/3=1
0	s_3	5	0	1	4	0	0	1	0	—
		$c_j - z_j$	5+2M	-2+2M	3-M	-M	0	0	0	
5	x_1	1	1	1	-1/2	-1/2	0	0	-	—
0	s_2	0	0	-7	$\boxed{3/2}$	3/2	1	0	-	0
0	s_3	5	0	1	3	0	0	1	-	5/3
		$c_j - z_j$	0	-7	11/2	5/2	0	0		
5	x_1	1	1	-4/3	0	0	1/3	0	-	—
3	x_3	0	0	-14/3	1	1	2/3	0	-	—
0	s_3	5	0	$\boxed{15}$	0	-3	-2	1	-	5/15
		$c_j - z_j$	0	56/3	0	-3	-11/3	0		
5	x_1	13/9	1	0	0	-4/45	7/45	4/45	-	—
3	x_3	14/9	0	0	0	$\boxed{+1/15}$	2/45	14/45	-	$\frac{14}{9} \times \frac{15}{3}$
-2	x_2	1/3	0	1	0	-1/5	-2/15	1/15	-	[illegible]
		$c_j - z_j$	0	0	0	11/15	-53/45	-56/45		
5	x_1	23/3	1	0	4	0	1/3	4/3		[illegible]
0	s_1	70/3	0	0	15	1	2/3	14/3		—
-2	x_2	5	0	1	3	0	0	1		[illegible]
		$c_j - z_j$	0	0	-11	0	-5/3	-14/3		

∴ optimal soln $x_1 = 23/3 \quad x_2 = 5 \quad x_3 = 0 \quad z_{max} = \frac{85}{3}$

→ If there is a tie b/w artificial & slack/surplus, artificial should leave

Two Phase Method: Another method to solve LPP involving artificial variables. Procedure is seperated into 2 phases:

Phase I: (1) Convert problem in standard form.

(2) Assign -1 to each ~~aiftifi~~ artificial variable in case of maximization problem (or 1 in case of minimization problem). All other variables are assigned zero value in objective fn. (x_j's, slack & surplus).

(3) Write down auxillary problem.

(4) Solve it using simplex method

(a) If artificial variable is present in basis with +ive value, then the problem has no feasible solution. Hence stop.

(b) If optimality is reached and artificial variable is either absent in basis or is present with zero value then proceed to phase II. (as BFS to original problem has been found)

Phase II: * Use the optimum BFS of phase I as starting solution for the original LPP i.e, the final ~~&~~ simplex table of phase I is taken as initial simplex table of phase II and * in place of objective fn of phase I, take the original objective function * Delete the entries in the column headed by artificial variables $A_1, A_2, \ldots$ ~~Am~~. * Apply usual simplex method to get optimum solution.

Ex Use 2-phase simplex method to solve

min $z = x_1 + x_2$

subject to

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Standard LPP

min $z = x_1 + x_2 + 0s_1 + 0s_2 + A_1 + A_2$

subject to

$2x_1 + x_2 - s_1 + A_1 = 4$

$x_1 + 7x_2 - s_2 + A_2 = 7$

$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

Soln Phase I: Auxillary LPP.

min $z = 0x_1 + 0x_2 + 0s_1 + 0s_2 + A_1 + A_2$

subject to

$2x_1 + x_2 - s_1 + A_1 = 4$

$x_1 + 7x_2 - s_2 + A_2 = 7$

$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

		c_j	0	0	0	0	1	1	
CB	BV	XB	x_1	x_2	s_1	s_2	A_1	A_2	min Ratio
1	A_1	4	2	1	-1	0	1	0	4
1	A_2	7	1	$\boxed{7}$	0	-1	0	1	1 →
		$c_j - z_j$	-3	-8 ↑	1	1	0	0	
1	A_1	3	$\boxed{13/7}$	0	-1	1/7	1	-1/7	21/13 →
0	x_2	1	1/7	1	0	-1/7	0	1/7	7
		$c_j - z_j$	-13/7 ↑	0	1	-1/7	0	8/7	
0	x_1	21/13	1	0	-7/13	1/13	7/13	-1/13	
0	x_2	10/13	0	1	1/13	-2/13	-1/13	2/13	
		$c_j - z_j$	0	0	0	0	1	1	

Since all $c_j - z_j \geq 0$ and artificial variable is not present in basis, so this table gives optimal BFS. Hence we go to phase II.

Phase II

		C_j	1	1	0	0
C_B	BV	X_B	x_1	x_2	s_1	s_2
1	x_1	21/13	1	0	-7/13	1/13
1	x_2	10/13	0	1	1/13	-2/13
		$C_j - Z_j$	0	0	6/13	1/13

Since all $C_j - Z_j \geq 0$ ∴ optimum soln.

$x_1 = \frac{21}{13}$ $x_2 = \frac{10}{13}$ $Z_{max} = \frac{31}{13}$ — Ans

Ex Solve the LPP using 2-Phase method.

max $z = 6x_1 + 8x_2$

subject to

$2x_1 + x_2 \leq 1$

$x_1 + 4x_2 \geq 6$

$x_1, x_2 \geq 0$

Standard Form

max $z = 6x_1 + 8x_2 + 0s_1 + 0s_2 - A_1$

subject to

$2x_1 + x_2 + s_1 = 1$

$x_1 + 4x_2 - s_2 + A_1 = 6$

$x_1, x_2, s_1, s_2, A_1 \geq 0$

Phase I Auxillary LPP

		C_j	0	0	0	0	-1	
C_B	BV	X_B	x_1	x_2	s_1	s_2	A_1	min ratio
0	s_1	1	2	$\boxed{1}$	1	0	0	1 →
-1	A_1	6	1	4	0	-1	1	6
		$C_j - Z_j$	1	4 ↑	0	-1	0	
0	x_2	1	2	1	1	0	0	
-1	A_1	2	-7	0	-4	-1	1	
		$C_j - Z_j$	-7	0	-5	-1	0	

∴ All $C_j - Z_j \leq 0$, so optimality is reached. But artificial variable A_1 is present in basis at +ive value ($A_1 = 2$). Hence, feasible solution to given LPP does not exist.

Ex. Using 2 Phase method solve

max $z = 2x_1 + x_2 + 3x_3 + 0s_1 - A_1$

subject to

$x_1 + x_2 + 2x_3 + s_1 \leq 5$

$2x_1 + 3x_2 + 4x_3 + A_1 = 12$

Phase I: Auxillary LPP max $z = -A$

subject to $x_1 + x_2 + 2x_3 + s_1 = 5$

$2x_1 + 3x_2 + 4x_3 + A_1 = 12$

		c_j	0	0	0	0	-1	
CB	BV	X_B	x_1	x_2	x_3	s_1	A_1	Ratio
0	s_1	5	1	1	$\boxed{2}$	1	0	5/2
-1	A_1	12	2	3	4	0	1	3
		$c_j - z_j$	2	3	4	0	0	
0	x_3	5/2	1/2	1/2	1	1/2	0	5
-1	A_1	2	0	1	0	-2	1	2
		$c_j - z_j$	0	1	0	-2	0	
0	x_3	3/2	1/2	0	1	3/2	-1/2	
0	x_2	2	0	1	0	-2	1	
		$c_j - z_j$	0	0	0	0	-1	

∵ All $c_j - z_j \leq 0$, so optimality is reached. No artifical variable is present in basic solution. Hence, we get initial BFS. So, we go to Phase II.

Phase II:

		c_j	2	1	3	0	
CB	BV	X_B	x_1	x_2	x_3	s_1	
3	x_3	3/2	1/2	0	1	3/2	3
1	x_2	2	0	1	0	-2	—
		$c_j - z_j$	1/2	0	0	-5/2	
2	x_1	3	1	0	2	3	
1	x_2	2	0	1	0	-2	
		$c_j - z_j$	0	0	-1	-4	

∵ All $c_j - z_j \leq 0$ optimality is reached

$\boxed{x_1 = 3 \quad x_2 = 2 \quad z_{max} = 8}$ — Ans