Subsequences: Let $\langle x_n \rangle$ be sequence of real numbers and $\langle m_k \rangle_{k=1}^{\infty}$ be strictly ↑ sequence of natural number i.e, $m_1 < m_2 < \dots < m_k < \dots$ then the seq $\langle x_{m_k} \rangle$ is called subsequence of $\langle x_n \rangle$.

Note: (1) If $\langle y_n \rangle$ is subsequence of $\langle x_n \rangle$ then each $y_n = x_{m_n}$ for some $m_n \geq n$.
(2) Every sequence is a subsequence of itself.

Example: $\langle n^2 \rangle$ be a sequence.
$\langle 1^2, 3^2, 5^2, \dots \rangle$, $\langle 2^2, 4^2, 6^2, \dots \rangle$ are its subsequences

Theorem: If $\langle y_n \rangle$ is a subsequence of $\langle x_n \rangle$ then
(1) $\langle y_n \rangle$ is bdd if $\langle x_n \rangle$ is bdd.
(2) $\langle y_n \rangle$ is monotonic if $\langle x_n \rangle$ is monotonic
(3) $\langle y_n \rangle$ is convergent if $\langle x_n \rangle$ is convergent. Moreover, if $x_n \to l$ then $y_n \to l$.

Proof: (1) $\langle y_n \rangle$ is subseq of $\langle x_n \rangle \Rightarrow y_n = x_{m_n}$ for some $m_n \geq n$.
Since $\langle x_n \rangle$ is bdd $\Rightarrow |x_n| < M \quad \forall n \in \mathbb{N}$.
$\Rightarrow |y_n| = |x_{m_n}| < M \quad (\because m_n \in \mathbb{N})$
$\Rightarrow \langle y_n \rangle$ is bdd.
(2) $\langle x_n \rangle$ is monotonic ↑. $\Rightarrow x_{m_n} \leq x_{m_{n+1}}$ for $m_{n+1} > m_n$
$\Rightarrow y_n \leq y_{n+1} \quad \forall n \in \mathbb{N}$.

$\Rightarrow \langle y_n \rangle$ is monotonic ↑.
Similarly prove ~~for~~ if $\langle x_n \rangle$ is monotonic ↓.

(3) Let $\epsilon > 0$ be given. Then $\exists\, n_0 \in \mathbb{N}$ s.t ~~xxx~~
$|x_n - l| < \epsilon \quad \forall n \geq n_0$. Take $m_n \geq n$, then
$|x_{m_n} - l| < \epsilon \Rightarrow |y_n - l| < \epsilon$
$\Rightarrow \langle y_n \rangle \to l$ //

<u>Theorem</u>: Prove that $\langle x_n \rangle \to l$ iff $\lim_{n\to\infty} x_{2n} = l = \lim_{n\to\infty} x_{2n+1}$

<u>Proof</u>: If $\langle x_n \rangle \to l$ then $\langle x_{2n} \rangle \to l$ & $\langle x_{2n+1} \rangle \to l$ from above theorem.

<u>Converse</u>: Let $\langle x_{2n} \rangle \to l$ & $\langle x_{2n+1} \rangle \to l$.
$\Rightarrow \exists\, p_0$ & q_0 s.t. $|x_{2n} - l| < \epsilon \quad \forall n \geq p_0$
& $|x_{2n+1} - l| < \epsilon \quad \forall n \geq q_0$

Let $n_0 = \max(p_0, q_0)$. $\Rightarrow |x_n - l| < \epsilon \quad \forall n \geq n_0$ //

<u>Ex</u>: $\langle (-1)^{n-1} \rangle$ is not convergent.
<u>Sol</u>: Let $x_n = (-1)^{n-1}$ Then $x_{2n} = (-1)^{2n-1}$ & $x_{2n+1} = (-1)^{2n}$
clearly $\langle x_{2n} \rangle$ & $\langle x_{2n+1} \rangle$ are subsequences of $\langle x_n \rangle$.
But $\langle x_{2n} \rangle \to (-1)$ & $\langle x_{2n+1} \rangle \to 1$
$\Rightarrow \lim_{n\to\infty} x_{2n} \neq \lim_{n\to\infty} x_{2n+1} \Rightarrow \langle x_n \rangle$ not convergent (above thm).

<u>Ex</u> $\langle \sin \frac{n\pi}{2} \rangle$ is not convergent.
$x_n = \sin \frac{n\pi}{2}$ $\quad x_{2n} = \sin n\pi = 0$
$x_{4n-3} = \sin(4n-3)\frac{\pi}{2} = 1$ $\quad x_{4n-1} = \sin(4n-1)\frac{\pi}{2} = -1$
$\therefore$ Three subsequences $\langle x_{2n} \rangle$, $\langle x_{4n-3} \rangle$ and $\langle x_{4n-1} \rangle$ of $\langle x_n \rangle$ converge to different limits.
Hence, $\langle x_n \rangle$ is not convergent (from above thm).

Bolzano-Weierstrass theorem for sequences: Every bounded sequence has convergent subsequence.

Proof: Let S be set of all distinct points of bounded sequence $\langle x_n \rangle$. Then S is also bdd.

Case I: S is finite.

$\Rightarrow \exists$ an $s \in S$ which is infinitely many times repeated in $\langle x_n \rangle$

Let $\langle m_n \rangle_{n=1}^{\infty}$ be strictly $\uparrow$ sequence of +ive numbers

s.t. $x_{m_n} = s \ \forall n \in \mathbb{N}$.

$\Rightarrow \langle x_{m_n} \rangle$ is subseq of $\langle x_n \rangle$ and $\langle x_{m_n} \rangle \to s$ //.

Case II: S is infinite

From Bolzano-Weierstrass theorem for sets ("Every bdd infinite ~~set~~ subset of $\mathbb{R}$ has atleast one limit point")

$\Rightarrow S$ has a limit point, say x, in $\mathbb{R}$

Now, we construct a subseq. of $\langle x_n \rangle$ which converges to x.

For each $k \in \mathbb{N}$, let $I_k = \left(x - \frac{1}{k}, x + \frac{1}{k}\right)$

$\Rightarrow I_k$ contains infinitely many elements of S ($\because x$ is limit pt of S and I_k is $\left(\frac{1}{k}\right)$-nbd of x)

Choose $x_{m_1} \in I_{m_1}$, $x_{m_2} \in I_{m_2}$, -- $x_{m_n} \in I_{m_n}$ where $m_{k+1} > m_k$

$\therefore$ We have subseq $\langle x_{m_n} \rangle$ of $\langle x_n \rangle$ s.t. $x_{m_n} \in I_{m_n}$

$\Rightarrow x_{m_n} \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ or, $|x_{m_n} - x| < \frac{1}{n} \ \forall n \in \mathbb{N}$

$\Rightarrow \langle x_{m_n} \rangle \to x$ //.

Cauchy Sequence: A sequence $\langle x_n \rangle$ is said to cauchy if for each $\epsilon > 0$, however small, $\exists\, n_0 \in \mathbb{N}$ such that

$n, m \geq n_0 \Rightarrow |x_n - x_m| < \epsilon$.

Cauchy sequence is also called Fundamental sequence

Theorem 1 Every convergent sequence is cauchy sequence.

Proof: Let $\langle x_n \rangle \to l$.

$\Rightarrow$ Given $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t. $|x_n - l| < \frac{\epsilon}{2}$ $\forall n \geq n_0$.

& $|x_m - l| < \frac{\epsilon}{2}$ $\forall m \geq n_0$

Thus, for, $n, m \geq n_0$, we have

$|x_n - x_m| \leq |x_n - l| + |x_m - l| < \epsilon$

$\Rightarrow \langle x_n \rangle$ is cauchy sequence. //

Theorem 2: Every cauchy sequence is bounded.

Proof: Let $\langle x_n \rangle$ be a cauchy sequence. Choose $\epsilon = 1$. Then

$\exists\, n_0 \in \mathbb{N}$ s.t. for $n, m \geq n_0 \Rightarrow |x_n - x_m| < 1$

In particular take $m = n_0 + 1$. (i.e, fixing value of m)

$\Rightarrow |x_n - x_{n_0+1}| < 1$ $\forall n \geq n_0$

$\Rightarrow |x_n| - |x_{n_0+1}| < |x_n - x_{n_0+1}| < 1$ (triangle inequality)

$\Rightarrow |x_n| < 1 + |x_{n_0+1}| = \lambda$ (say) $\forall n \geq n_0$

Let $M = \max\{|x_1|, \ldots |x_{n_0}|, \lambda\}$. ~~Then $|x_n| \leq M$ $\forall n \geq n_0$.~~

$\Rightarrow |x_n| \leq M$ $\forall n \in \mathbb{N}$

$\Rightarrow \langle x_n \rangle$ is bounded.

Converse of Theorem 1:

Theorem 3: Every cauchy sequence in $\mathbb{R}$, is convergent

Proof (Thm 3): Let $\langle x_n \rangle$ be cauchy sequence

$\Rightarrow$ $\langle x_n \rangle$ is bdd (Theorem 2)

$\Rightarrow$ It has a convergent subsequence. Let it be $\langle y_n \rangle$.
(Bolzano-Weierstrass thm for seq)

Let $\langle y_n \rangle \to l$. To prove $\langle x_n \rangle \to l$.

$\because$ $\langle x_n \rangle$ is cauchy, given $\epsilon > 0$, $\exists$ $m_0 \in \mathbb{N}$ s.t. for $n, m \geq m_0$

$|x_n - x_m| < \frac{\epsilon}{2}$ — (1)

$\langle y_n \rangle \to l \Rightarrow \exists\, p_0 \in \mathbb{N}$ s.t. $|y_n - l| < \frac{\epsilon}{2}$ $\forall n \geq p_0$ — (2)

Let $n_0 = \max(m_0, p_0)$

~~Also~~ Also $y_n = x_{m_n}$ for some $m_n \geq n_0$ [$\because$ $\langle y_n \rangle$ is subseq of $\langle x_n \rangle$)
— (3)

$$\Rightarrow |x_n - l| = |x_n - x_{m_n} + x_{m_n} - l| \quad \text{(from (3))}$$
$$\leq |x_n - x_{m_n}| + |y_n - l|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\Rightarrow$ $|x_n - l| < \epsilon$ $\forall n \geq n_0$ $\Rightarrow$ $\langle x_n \rangle \to l$ //.

Cauchy's ~~criterion~~ criterion for convergence: This tells us about the convergence of sequence without having the knowledge of its limit.

Theorem: A sequence of real numbers is convergent iff it is cauchy sequence.

Proof: Follows from theorem 1 & 3 (above)

i.e., Seq. of real number is convergent iff for given $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t. $|x_{n+p} - x_n| < \epsilon$ $\forall n \geq n_0$, $p \geq 0$.

(Note) here $m = n+p$ in cauchy seq. defn)

Ex Using Cauchy's general principle of convergence show that the following sequences are convergent:

(1) $\langle \frac{1}{n} \rangle$ (2) $\langle \frac{n}{n+1} \rangle$.

Soln (1) Let $x_n = \frac{1}{n}$ & $x_{n+p} = \frac{1}{n+p}$ Then for $n \geq n_0$, $p \geq 0$.

$$|x_{n+p} - x_n| = \left| \frac{1}{n+p} - \frac{1}{n} \right| = \frac{p}{n(n+p)} < \frac{p}{n^2} < \epsilon$$

$\Rightarrow n > \sqrt{\frac{p}{\epsilon}}$. Let $n_0 = \left[\sqrt{\frac{p}{\epsilon}}\right]$

$\therefore |x_{n+p} - x_n| < \epsilon$ for $n \geq n_0$, $p \geq 0$.

$\therefore$ From Cauchy's criterion for convergence $\langle \frac{1}{n} \rangle$ converges.

(2) $\langle \frac{n}{n+1} \rangle$ Let $m > n$.

$$|x_m - x_n| = \left| \frac{m}{m+1} - \frac{n}{n+1} \right| = \left| 1 - \frac{1}{m+1} - \left(1 - \frac{1}{n+1}\right) \right|$$

$$= \left| \frac{1}{n+1} - \frac{1}{m+1} \right| < \frac{1}{(m+1)} < \epsilon \quad \text{for } m > \frac{1}{\epsilon} - 1.$$

Let $n_0 = \left[\frac{1}{\epsilon} - 1\right]$. Thus, $|x_m - x_n| < \epsilon \;\; \forall\, m, n \geq n_0$

Hence, from Cauchy's general principle of convergence $\langle \frac{n}{n+1} \rangle$ converges

Ex Show that $x_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{(2n-1)}$ cannot converge.

Soln Choose $\epsilon = \frac{1}{8}$ and $n = 2m$. Then

$$|x_{2m} - x_m| = \frac{1}{2m+1} + \cdots + \frac{1}{4m-1} > \frac{1}{4m} + \text{m-times} \cdots + \frac{1}{4m} = \frac{m}{4m}$$

$\Rightarrow |x_{2m} - x_m| > \frac{1}{4} \nless \epsilon \left(= \frac{1}{8}\right)$ contradiction.

$\therefore \langle x_n \rangle$ is not Cauchy & hence does not converge from Cauchy's general principle of convergence.

Practice Problems

Ex Show that $a_{n+1} = 1-\sqrt{1-a_n}$ $\forall n \geq 1$ & $a_1 < 1$, converges to zero

Soln (By induction) Case I: $0 < a_1 < 1$

$a_2 = 1-\sqrt{1-a_1} < 1$ and $a_2 - a_1 = \sqrt{1-a_1}(\sqrt{1-a_1} - 1) < 0$

$\Rightarrow a_2 < a_1$ (as $\sqrt{1-a_1} < 1$)

Assume $a_n < a_{n-1}$

Now, $a_{n+1} - a_n = 1-\sqrt{1-a_n} - (1-\sqrt{1-a_{n-1}}) = \sqrt{1-a_{n-1}} - \sqrt{1-a_n} < 0$

$\left(\because a_n < a_{n-1} \text{ (assumed)} \Rightarrow \sqrt{1-a_n} > \sqrt{1-a_{n-1}} \right)$

$\Rightarrow a_{n+1} < a_n$ $\forall n$ (By induction)

$\Rightarrow \langle a_n \rangle$ is monotonically $\downarrow$ and bdd below by zero. so converges.

Case 2: $a_1 < 0$

$\Rightarrow \sqrt{1-a_1} > 1 \Rightarrow a_2 > a_1$. Similarly it can be proved

$\langle a_n \rangle$ is monotonically $\uparrow$ and bdd above by zero.

($\because$ all a_i's < 0)

Let $\langle a_n \rangle \to l$ $\quad \therefore \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (1-\sqrt{1-a_n})$

$\Rightarrow l = 1-\sqrt{1-l} \Rightarrow \sqrt{1-l} = 1$

or $l = 0$

Hence $\lim_{n\to\infty} a_n = 0$