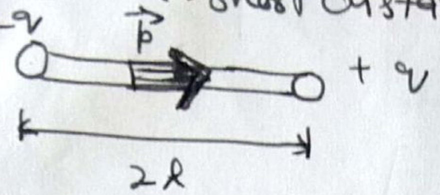
lectric dipole → [Electric dipole Moment]   PHYS-102   Unit-01

n electric dipole is a pair of equal and opposite point charges placed at a short distance apart.



The Product of one charge and the distance between the charges is called the magnitude of the electric dipole moment p

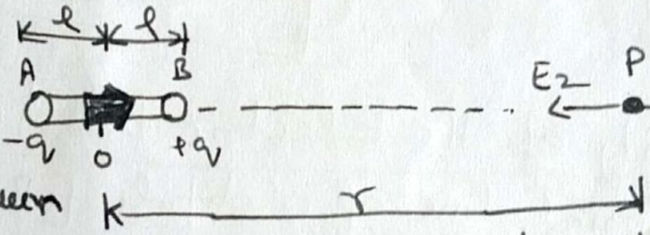
$$\boxed{P = q \times 2l = 2ql}$$

Unit — Coulom-meter   Quantity - Vector

Direction → Along the line Joining the two charges pointing from the negative charge to the positive charge.

Electric field due to Electric dipole →

① Intensity of the electric field at a point on the axis of dipole [endon position]



Suppose an electric dipole AB made by two charges -q and +q having a small distance 2l meter between them is situated in vaccum. Let P is a point on the axis at a distance r meter from the mid point O of the dipole. we have to determine the intensity of the electric field at P

Let E_1 and E_2 be the magnitudes of the intensities of the electric field at p due to charges +q and -q of the dipole respectively. The distance of the point P from the charge +q is $(r-l)$ and from the charge -q is $(r+l)$ so

$E_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{(r-l)^2}$ (along the dipole axis)

$E_2 = \frac{1}{4\pi\epsilon_0}\frac{q}{(r+l)^2}$ (Opposite to the dipole)

The resultant intensity E at the point P will be equal to their difference and in the direction of dipole axis (since $E_1 > E_2$). That is

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$$E = E_1 - E_2 = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(r-l)^2} - \frac{1}{(r+l)^2}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{(r+l)^2-(r-l)^2}{(r^2-l^2)^2}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{4lr}{(r^2-l^2)^2}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{2(2ql)r}{(r^2-l^2)^2}\right]$$

But $2ql = p$ (electric dipole) so

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\frac{2pr}{(r^2-l^2)^2}}$$

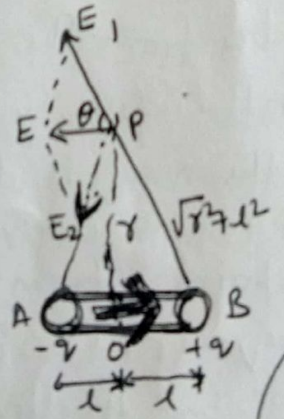
The direction of E is along the dipole axis (from the negative charge towards the positive charge).
if r is very large compare to 2l ($r \gg 2l$) then l^2 may be neglected.

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\frac{2p}{r^3} \text{ Newton/Coulomb}}$$

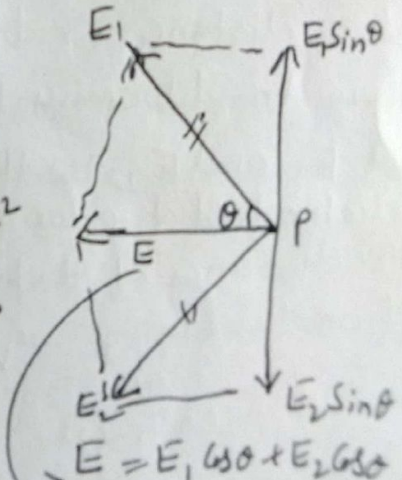
in vector notation $$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\vec{p}}{r^3}} \quad (1)$$

(2) Intensity of the Electric field at a point on the Equatorial line of a dipole [Broad-side on position]

from fig E_1 and E_2 is the magnitudes of the intensities of the electric field at P due the charge +q and -q of dipole respectively. The distance of P from each charge is $\sqrt{r^2+l^2}$. Therefore



①



②

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+l^2)}, \text{ (away from +q)}$$

and

$$E_2 = \frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+l^2)}, \text{ (towards -q)}$$

The magnitudes of E_1 and E_2 are equal but direction are different.

On resolving E_1 and E_2 into two component parallel and perpendicular to AB, the components perpendicular to AB ($E_1 \sin\theta$ and $E_2 \sin\theta$) cancel each other, while the components parallel to AB [$E_1 \cos\theta$ and $E_2 \cos\theta$] being in the same direction. So the resultant intensity of electric field at the point P is

$$E = E_1 \cos\theta + E_2 \cos\theta = \frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+l^2)}\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+l^2)}\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+l^2)} 2\cos\theta$$

From fig (1) $\cos\theta = \frac{BO}{BP} = \frac{l}{(r^2+l^2)^{1/2}}$

$$\text{So } E = \frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+l^2)}\frac{2l}{(r^2+l^2)^{1/2}} = \frac{1}{4\pi\epsilon_0}\frac{2ql}{(r^2+l^2)^{3/2}}$$

But $2ql = P$ (moment of electric dipole)

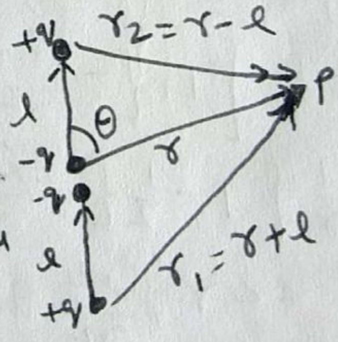
$$\boxed{\therefore E = \frac{1}{4\pi\epsilon_0}\frac{P}{(r^2+l^2)^{3/2}}}$$

The direction of electric field E is anti parallel to dipole axis.

If $r >> 2l$ the l^2 may be neglected in comparison to r^2

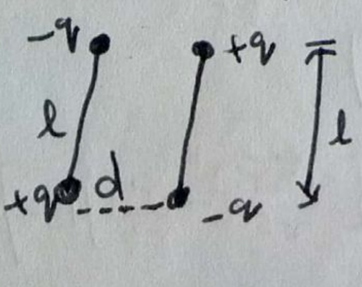
then $\boxed{\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{P}}{r^3} \text{ Newton/Coulomb}}$ (In vector notation)

Minus sign is put because $\vec{E}$ is anti parallel to $\vec{P}$

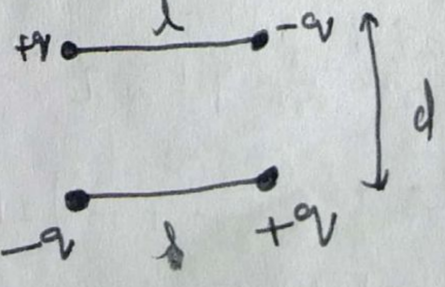


<u>Electric Quadrupole</u> → It is a system of four charges each of magnitude $|q|$ with two of positive sign and two of negative sign. The separation between the charges is assumed to be very small.

So Quadrupole is an arrangement of two dipole



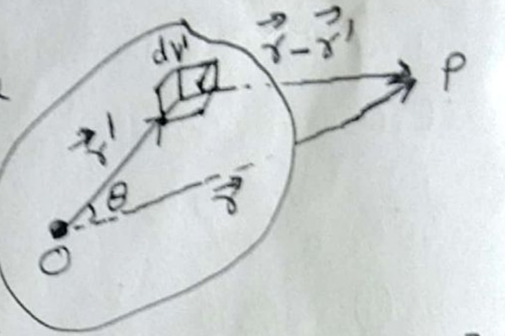
or



Multipole Expansion of electric potential →

or

Electric potential of an arbitrary charge distribution in terms of multipoles →



Let us consider volume V' occupied by the charge distribution. Let dV' be an element of volume at position $\vec{r}'$ from an arbitrary point O within the charge distribution and $\rho(\vec{r}')$ the charge density at point $\vec{r}'$.
Then the potential at a distance point P, whose position vector is $\vec{r}$, due to the volume element dV' is given by,

$$d\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dV'$$

Here $|\vec{r}-\vec{r}'|$ is the distance of the point P from the volume element dV'.
The potential at P due to the entire charge distribution is therefore

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' \quad \text{——— (1)}$$

Since $r' << r$ the quantity $|\vec{r}-\vec{r}'|^{-1}$ can be expanded in a series of ascending powers of r'/r.

$$|\vec{r}-\vec{r}'|^{-1} = (r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{-1/2} = \frac{1}{r}\left[1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right]^{-1/2}$$

$$= \frac{1}{r}\left[1 + \left\{-\frac{2\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right\}\right]^{-1/2}$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left\{-\frac{2\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right\} + \frac{3}{8}\left\{-\frac{2\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right\} + \cdots\right]$$

Since $r' << r$, $\frac{r'^2}{r^2}$ is negligible compared to $\frac{2\vec{r}\cdot\vec{r}'}{r^2}$, but it may not be dropped in the first set of brackets because it is of the same order as the dominant term in the second set of bracket. Omitting terms involving cube and higher powers of $\vec{r}'$, the last expression gives

$$|\vec{r}-\vec{r}'|^{-1} = \frac{1}{r}\left[1 - \frac{1}{2}\left\{-\frac{2\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right\} + \frac{3}{8}\left\{\frac{-2\vec{r}\cdot\vec{r}'}{r^2}\right\}^2 + \cdots\right]$$

$$= \left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} - \frac{r'^2}{2r^3} + \frac{3}{2}\frac{(\vec{r}\cdot\vec{r}')^2}{r^5} + \cdots\right] = \left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \frac{1}{2}\left\{\frac{3(\vec{r}\cdot\vec{r}')^2}{r^5} - \frac{r'^2}{r^3}\right\} + \cdots\right]$$

Making this substitution in eq (1) we get

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{v'}\left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \frac{1}{2}\left\{\frac{3(\vec{r}\cdot\vec{r}')^2}{r^5} - \frac{r'^2}{r^3}\right\} + \cdots\right]\rho(\vec{r}')\,dv' \quad \text{—(2)}$$

The third term on the right side of this equation can be expressed as

$$\frac{1}{2}\left\{\frac{3(\vec{r}\cdot\vec{r}')^2}{r^5} - \frac{r'^2}{r^3}\right\} = \frac{1}{2r^5}\left\{3(\vec{r}\cdot\vec{r}')^2 - r'^2 r^2\right\}$$

$$= \frac{1}{2r^5}\left\{3\left(\sum_{i=1}^{3} x_i x_i'\right)^2 - r'^2\left(\sum_{J=1}^{3} x_J^2\right)\right\}$$

Here x_i an x_J are Cartesian components of $\vec{r}$ and x_i', x_J' are the Cartesian component of $\vec{r}'$ therefore

$$\frac{1}{2}\left\{\frac{3(\vec{r}\cdot\vec{r}')^2}{r^5} - \frac{r'^2}{r^3}\right\} = \frac{1}{2r^5}\left\{3\left(\sum_{i=1}^{3} x_i x_i' \sum_{J=1}^{3} x_J x_J'\right) - r'^2\left(\sum_{J=1}^{3} x_J^2\right)\right\}$$

$$= \frac{1}{2r^5}\left\{3\left(\sum_{i=1}^{3} x_i x_i' \sum_{J=1}^{3} x_J x_J'\right) - r'^2 \sum_{i=1}^{3}\sum_{J=1}^{3} x_i x_J \delta_{iJ}\right\}$$

Here $\delta_{iJ} = \begin{cases} 0 \text{ for } i \neq J \\ 1 \text{ for } i = J \end{cases}$

$$\therefore \frac{1}{2}\left\{\frac{3(\vec{r}\cdot\vec{r}')^2}{r^5} - \frac{r^2}{r^3}\right\} = \frac{1}{2r^5}\sum_{i=1}^{3}\sum_{J=1}^{3} x_i x_J\left\{3x_i' x_J' - r'^2\delta_{iJ}\right\}$$

Putting this value in eq (2)

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int_{v'}\rho(\vec{r}')\,dv' + \frac{\vec{r}}{r^3}\int_{v'}\vec{r}'\rho(\vec{r}')\,dv' + \sum_{i=1}^{3}\sum_{J=1}^{3}\frac{x_i x_J}{2r^5}\int_{v'}(3x_i' x_J' - r'^2\delta_{iJ})\rho(\vec{r}')\,dv' + \cdots\right] \quad \text{—(3)}$$

Eq (3) shows that the potential due to a charge distribution may be expressed as a summation of an infinite number of terms, $\phi(\vec{r}) = \phi_1(\vec{r}) + \phi_2(\vec{r}) + \phi_3(\vec{r}) + \cdots$

Here $\phi_1(\vec{r}) = \frac{1}{4\pi\epsilon_0 r}\int_{v'}\rho(\vec{r}')\,dv'$

$$\phi_2(\vec{r}) = \frac{\vec{r}}{4\pi\epsilon_0 r^3}\int_{v'}(\vec{r}')\rho(\vec{r}')\,dv'$$

$$\phi_3(\vec{r}) = \frac{1}{4\pi\epsilon_0}\cdot\sum_{i=1}^{3}\sum_{J=1}^{3}\frac{x_i x_J}{2r^5}\underbrace{\int_{v'}(3x_i' x_J' - r'^2\delta_{iJ})\rho(\vec{r}')\,dv'}_{Q_{iJ}}$$

and ~~and soon~~

In this third term $\phi_3(\vec{r})$, there are nine components of Q_{ij} corresponding to $i, j = 1,2,3$ of these, six are equal in pairs leaving six distinct component. This set of quantities form the Quadrupole moment tensor and represents an extension of the dipole moment concept. this term $\phi_3(\vec{r})$ is called the potential due to quadrupole moment of the charge distribution

Multipoles - The concept of electric dipole and quadrupole can be extended to a system of a large number of positive and negative charges located at small distances from one another. Such a charge distribution can be looked upon as multipole.

- A single point-charge can be termed as a monopole.
- If a monopole is displaced through a very small distances and the original monopole is replaced by an equal and opposite point charge the system is termed as "dipole".
- Similarly a quadrupole is obtained by displacing a dipole through a very small distance and then replacing the original dipole by one of same magnitude but of opposite sign
- this concept can be continued further to obtain octopole and multipoles of still higher order.
- the dipole potential varies as $\frac{1}{r^2}$ and quadrupole potential varies as $\frac{1}{r^3}$. for the general (2^x)th multipole, the potential at large distances from the system varies as $\frac{1}{r^{x+1}}$ and the field intensity as $\frac{1}{r^{x+2}}$.

Potential energy of the dipole in electric field
or
work done in rotating an electric dipole in an electric field

When an electric dipole of moment $\vec{P}$ is placed in a uniform electric field $\vec{E}$ making an angle θ with the field, a (restoring) torque $\vec{\tau}$ acts upon the dipole which tends to align the dipole in the direction of the field. The magnitude of this torque is given by -

$$\tau = PE \sin\theta \quad -(1)$$

If we rotate the dipole further from this position through an infinitesimally small angle $d\theta$, the work done (torque X angular displacement) would be

$$dW = \tau \, d\theta = PE \sin\theta \, d\theta \quad -(2)$$

Hence the workdone in rotating the dipole through the angle θ from its equilibrium position is.

$$W = \int_0^\theta PE \sin\theta \, d\theta = PE \left[-\cos\theta\right]_0^\theta$$

$$= PE\left[-\cos\theta + \cos 0\right] = PE(1-\cos\theta) \quad -(3)$$

If the dipole rotated through $90°$ from the direction of the field Then the workdone will be $W = PE(1-\cos 90) = PE \quad -(4)$

Similarly, if the dipole be rotated through $180°$ from the direction of the field, then the workdone will be

$$W = PE(1-\cos 180) = PE\left[1-(-1)\right] = 2PE \quad -(5)$$

Poisson and Laplace's equation -

we know, $E = -\text{grad } V$ — (1)

Gauss law in differential form state that the divergence of electric field $\vec{E}$ at any point is equal to $\frac{1}{\epsilon_0}$ times the charge density at that point. So

$$\text{div}\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{— (2)}$$

or

$$\text{Div}(-\text{grad } V) = \frac{\rho}{\epsilon_0} \quad \text{— (3)}$$

Here ρ is volume charge density

$$\text{Div}(+\text{grad} V) = -\frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \nabla V = -\frac{\rho}{\epsilon_0}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \quad \text{— (4)}$$

This equation is known as Poisson equation for the electric potential. In Cartesian co-ordinate this eq. is written as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon_0} \quad \text{— (5)}$$

In a region of space where there are no free charges ($\rho = 0$) then poission equation is,

$$\boxed{\nabla^2 V = 0} \quad \text{— (6)}$$

This is known as Laplace equation.