Angular momentum and tensor of inertia

Angular Momentum of a Rotating Rigid Body

In rotational motion $\vec{\tau} = \frac{d\vec{L}}{dt}$ resemblances to the translational motion $\vec{F} = \frac{d\vec{p}}{dt}$

linear momentum $\vec{P} = M\vec{V}$ with $\vec{L} = I\vec{\omega}$

An arbitrary displacement of a rigid body can be resolved into a displacement of the centre of mass plus a rotation about some instantaneous axis through the centre of mass.

the angular momentum

$$\vec{L} = \vec{R} \times M\vec{V} + \Sigma r'_j \times m_j \dot{r}'_j \quad -①$$

the torque

$$\vec{\tau} = \vec{R} \times \vec{F} + \Sigma r'_j \times f_j \quad -②$$

where r'_j = position vector of m_j relative to the centre of mass, $\vec{\tau} = \frac{d\vec{L}}{dt}$

we have

$$\vec{R} \times \vec{F} + \Sigma r'_j \times f_j = \frac{d}{dt}(\vec{R} \times M\vec{V}) + \frac{d}{dt}(\Sigma r'_j \times m_j \dot{r}'_j)$$

$$= \vec{R} \times M\vec{A} + \frac{d}{dt}(\Sigma r'_j \times m_j \dot{r}'_j) \quad -③$$

Here $\vec{F} = M\vec{A}$

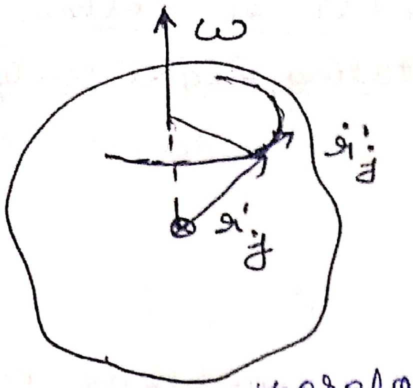
the terms involving R cancel.

and eq. becomes.

$$\Sigma r'_j \times f_j = \frac{d}{dt}(\Sigma r'_j \times m_j \dot{r}'_j) \quad -④$$

the rotational motion can be found by taking torque and angular momentum about the centre of mass which is independent of the centre of mass motion. the angular momentum $\vec{L}_0$ about the centre of mass is

$$L_0 = \Sigma r'_j \times m_j \dot{r}'_j \quad -⑤$$



Now we express L_o in terms of the instantaneous angular velocity ω.

r_j is a rotating vector

$$\dot{r}'_j = \omega \times r'_j$$

therefore,

$$L_o = \Sigma r'_j \times m_j (\omega \times r'_j)$$

then simplify the notation

Put L for L_o and r_j for r'_j

$$L = \Sigma r_j \times m_j (\omega \times r_j) \quad \text{—(6)}$$

Since $\omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

we have $\omega \times r = (z\omega_y - y\omega_z)\hat{i} + (x\omega_z - z\omega_x)\hat{j}$
$+ (y\omega_x - x\omega_y)\hat{k}$ —(7)

Let us compute one component of L say L_x and dropping subscript j we have.

$$[r \times (\omega \times r)]_x = y(\omega \times r)_z - z(\omega \times r)_y \quad \text{—(8)}$$

If we substitute the results of eq. (7) into eq. (8) the result is.

$$[r \times (\omega \times r)]_x = y(y\omega_x - x\omega_y) - z(x\omega_z - z\omega_x)$$
$$= y^2\omega_x + z^2\omega_x - xy\,\omega_y - xz\,\omega_z$$
$$= (y^2 + z^2)\omega_x - xy\,\omega_y - xz\,\omega_z \quad \text{—(9)}$$

Hence eq. (6) becomes

$$L_x = \Sigma m_j (y_j^2 + z_j^2)\omega_x - \sum_i m_j x_j y_j \omega_y - \Sigma m_j x_j z_j \omega_z \quad \text{—(10)}$$

Now let us introduce the following symbols.

As $I_{xx} = \sum m_j (y_j^2 + z_j^2)$
$I_{xy} = -\sum m_j x_j y_j$ — (11)
$I_{xz} = -\sum m_j x_j z_j$

I_{xx}, I_{xy}, I_{xz} are the moment of inertia.

Put values from eq. (11) to eq. (10).

$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$ (12a) in the same we can derivate for L_y and L_z.

$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$ (12b)

$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$ (12c)

If we fixed axis rotation about the z-direction $\omega = \omega\hat{k}$ then eq. (12c) becomes.

$$L_z = I_{zz}\omega$$

$$\boxed{L_z = \sum m_j (x_j^2 + y_j^2)\omega} \quad (12)$$

Eq. (12) shows that angular velocity in the z-direction can produce angular momentum about any of the three coordinate axes. these equations (12a), (12b), (12c) are the equations of rotational motion.

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the tensor of inertia -

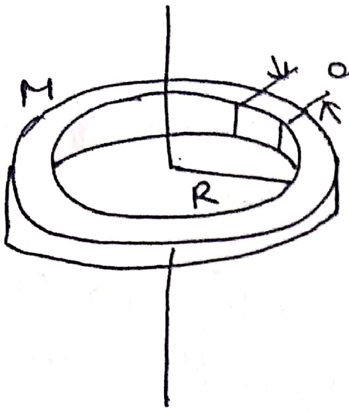
$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

$\left[\begin{array}{l} I_{xy} = I_{yx} \\ I_{yz} = I_{zy} \\ I_{xz} = I_{zx} \end{array}\right]$ symmetrical products of inertia

I_{xx}, I_{yy}, I_{zz} these are the moment of inertia]

Moments of inertia of some objects

① Uniform thin hoop of mass M and radius R, axis through the centre and perpendicular to the plane of the hoop



$$I = \int R^2\, dm$$

$$I = \int \frac{R^2 M}{2\pi R}\, ds \qquad dm = \lambda\, ds = \frac{M}{2\pi R}\, ds$$

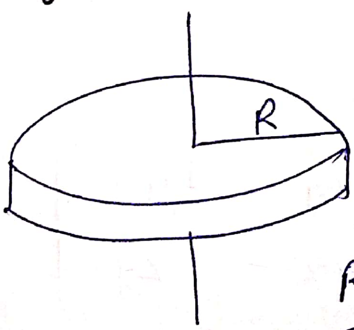
$$I = \int_0^{2\pi R} R\left(\frac{M}{2\pi}\right) ds$$

$$= \left(R\frac{M}{2\pi}\right) s\Big|_0^{2\pi R}$$

$$I = \frac{RM}{\cancel{2\pi}}\, \cancel{2\pi} R$$

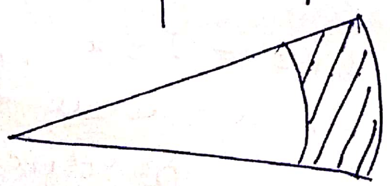
$$\boxed{I = MR^2}$$

② Uniform disk of mass M, radius R, axis through the centre and perpendicular to the plane of the disk.



$$I = \int R^2\, dm$$
$$= \int R^2 \sigma\, dS$$

$$I = \int R^{\cancel{2}} \frac{M}{\pi R^{\cancel{2}}}\, dS \qquad \sigma = \frac{M}{\pi R^2}$$



$$I = \int_0^{2\pi}\int_0^{R} \frac{M}{\pi}\, R\, dR\, d\theta$$

We have taken $dS = R\,dR\,d\theta$ in the plane polar coordinates.

$$I = \frac{M}{\cancel{\pi}}\, \frac{R^2}{2}\, (2\cancel{\pi})$$

$$I = \frac{M}{\pi R^2}\iint R^2 \; R\,dR\,d\theta$$

$$= \frac{2\cancel{\pi} M}{\cancel{\pi} R^2} \; \frac{R^4}{4}\Big|_0^R$$

$$\boxed{I = \frac{\cancel{2}M}{R^2}\,\frac{R^4}{\cancel{4}\,2} = \frac{1}{2}MR^2}$$