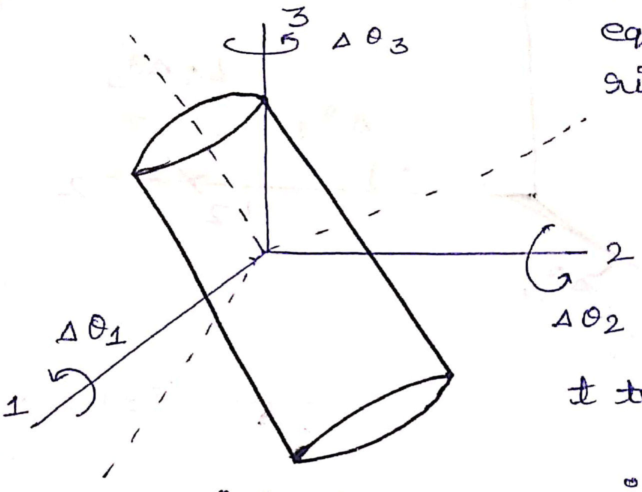
Euler's Equations

→ Now Here we derive the exact equations of motion for a rigid body.



to calculate $\frac{d\vec{L}}{dt}$ the change in the components of $\vec{L}$ in the time interval from t to $t+\Delta t$.

Let us introduce an inertial coordinate system which coincides with the instantaneous position of the body's principle axes at time t. the components of the angular velocity ω at time t $\omega_1, \omega_2, \omega_3$.

At that time the components of L are

$L_1 = I_1\omega_1$, $L_2 = I_2\omega_2$, $L_3 = I_3\omega_3$

In time interval Δt, the principal axes rotate away from the 1, 2, 3 axes. the rotation angle about the 1 axis $\Delta\theta_1 = \omega_1\Delta t$, $\Delta\theta_2 = \omega_2\Delta t$ and $\Delta\theta_3 = \omega_3\Delta t$. the corresponding change $\Delta L_1 = L_1(t+\Delta t) - L_1(t)$ can be found to first order by treating the three rotations one by one.

Here are two ways L_1 can change. If ω_1 varies, $I_1\omega_1$ will change.

In addition, rotations about the other two axes cause L_2 and L_3 to change direction which contribute to the angular momentum along the first axis.

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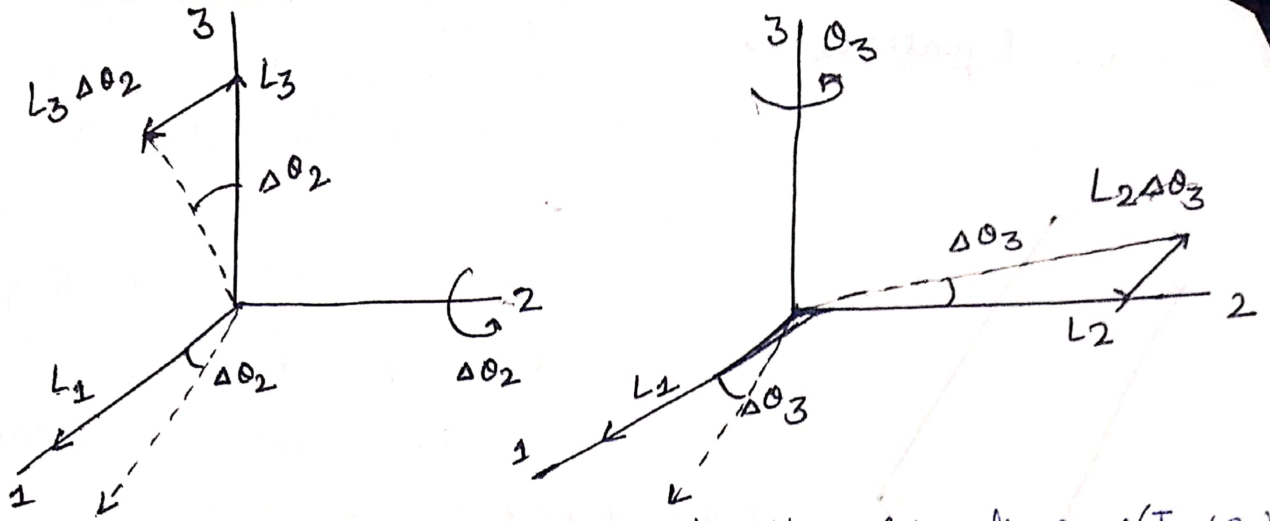

the first contribution to ΔL_1 is from $\Delta(I_1\omega_1)$.
Since the moments of inertia are constant to the first order for small angular displacements about the principle axes.

$$\Delta(I_1\omega_1) = I_1 \Delta\omega_1$$

to find the remaining contributions to ΔL_1, consider first rotation about the 2 axis through angle $\Delta\theta_2$.

the rotation of L_1 causes no change along the 1 axis to first order.
the rotation of L_3 contributes $L_3\Delta\theta_2 = I_3\omega_3\Delta\theta_2$ along the 1 axis.
Rotation about the 3 axis contributes $-L_2\Delta\theta_3 = -I_2\omega_2\Delta\theta_3$ to ΔL_1.

Adding all the contributions gives.

$$\Delta L_1 = I_1\Delta\omega_1 + I_3\omega_3\Delta\theta_2 - I_2\omega_2\Delta\theta_3$$

dividing by Δt and taking the limit $\Delta t \to 0$ yields →

$$\frac{dL_1}{dt} = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\,\omega_3\omega_2$$

the other components can be treated in same manner.

$$\frac{dL_2}{dt} = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\,\omega_1\omega_3$$

$$\frac{dL_3}{dt} = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\,\omega_2\omega_1$$

Since $\vec{\tau} = \frac{d\vec{L}}{dt}$

$$\tau_1 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\,\omega_3\omega_2$$

$$\tau_2 = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\,\omega_1\omega_3$$

$$\tau_3 = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\,\omega_2\omega_1$$

where τ_1, τ_2, τ_3 are the components of τ along the axes of the inertial system 1, 2, 3.

these equations were derived by Euler and are known as Euler's equations of rigid body motion.

At some time t we set up the 1, 2, 3 inertial system to coincide with the body's instantaneous direction of the

principal axes. T_1, T_2, T_3 are the components of torque along the 1, 2, 3 axes at time t.
In same manner $\omega_1, \omega_2, \omega_3$ are the components of ω along the 1, 2, 3 axes at time t and $\frac{d\omega_1}{dt}$, $\frac{d\omega_2}{dt}$, $\frac{d\omega_3}{dt}$ are the instantaneous rates of change of these components.

Difficulties in Euler's Equations –

the Euler's equations do not show us how to find the orientation of these coordinate systems in space.

In the 1, 2, 3 system, the components of $\overset{\frown}{I}$ are constant, but we do not know the orientation of the axes. Euler's equations can not be integrated directly to give angles specifying the orientation of the body relative to the x, y, z laboratary system.

Euler overcome this difficulty by expressing $\omega_1, \omega_2, \omega_3$ in terms of a set of angles relating the principle axes to the axes of the x, y, z coordinate system.