Principle of virtual Work -

An infinitesimal virtual displacement of ith particle of a system of N particles is denoted by $\delta \vec{r}_i$. Here the displacement of position coordinates only there is no involvement of variation of time. it means.

$$\delta \vec{r}_i = \delta \vec{r}_i (q_1, q_2 \ldots, q_n) \quad -①$$

Suppose system is in equilibrium. So the total force applying on any particle is zero i.e.

$$\vec{F}_i = 0 \qquad i = 1, 2, \ldots N \quad -②$$

the virtual work of the force $\vec{F}_i$ in the virtual displacement $\delta \vec{r}_i$ also 0

$$\delta W_i = 0$$

the Sum of virtual work.

$$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{r}_i = 0 \quad -③$$

the statement of principle of virtual work
" the work done is 0 in the case of an arbitrary virtual displacement of a system from a position of equilibrium.

the total force on the ith particle.

$$\vec{F}_i = \underbrace{\vec{F}_i^{a}}_{\text{applied force}} + \underbrace{f_i}_{\text{force of constraint}} \quad -④$$

eq. (3) becomes.

$$\sum_{i=1}^{N} \vec{F}_i^{a} \cdot \delta\vec{r}_i + \underbrace{\sum_{i=1}^{N} \vec{f}_i \cdot \delta\vec{r}_i}_{} = 0$$

workless constraints.

$$\sum_{i=1}^{N} \vec{F}_i^{a} \cdot \delta\vec{r}_i = 0$$

i.e., the virtual work of applied force is 0 for the equilibrium of a system.

D' Alembert's Principle -

the force acting on the ith particle.

$$\vec{F}_i = \frac{d\vec{p}_i}{dt} = \dot{\vec{p}}_i$$

$$\vec{F}_i - \dot{\vec{p}}_i = 0 \qquad i = 1, 2, \ldots N$$

It means that any particle in the system is in equilibrium under force equals to the actual force $\vec{F}_i$ plus reversed effective force $\dot{\vec{p}}_i$.

So for virtual displacement $\delta\vec{r}_i$.

$$\sum_{i=1}^{N} (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$$

$$\vec{F}_i = \vec{F}_i^{a} + \vec{f}_i$$

$$\sum_{i=1}^{N} (\vec{F}_i^{a} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i + \sum_{i=1}^{N} \vec{f}_i \cdot \delta\vec{r}_i = 0$$

the virtual work of the constraints is zero i.e. $\sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$

Date ___ /___ /________

So $\sum_{i=1}^{N} (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$ —①

this is the D'Alembert's principle.
the forces of constraints are not appearing in the equations.

therefore, the D'Alembert's principle may be written as

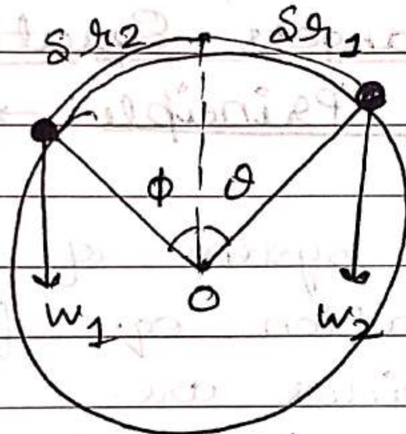
$$\boxed{\sum_{i=1}^{N} (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta r_i = 0}$$

VIRTUAL WORK

Date ___/___/______

Ex- two heavy particles of weights W_1 and W_2 are connected ~~to~~ by a light inextensible string and hang over a fixed smooth circular cylinder of radius R, the axis of which is horizontal [Fig.1]. Find the condition of equilibrium of the system by applying the principle of virtual work.

Sol.



By the principle of virtual work.

$$\sum_i^N F_i \cdot \delta r_i = 0$$

$$W_1 \sin\theta\, \delta r_1 + W_2 \sin\phi\, \delta r_2 = 0$$

$$\delta r_1 = R\,d\theta$$

$$d\theta = \frac{\delta r_1}{R} \quad \left[\because \text{Angle} = \frac{\text{arc}}{\text{Radius}}\right]$$

$$\left.\begin{array}{l} f_1 = W_1 \sin\theta \\ f_2 = W_2 \sin\phi \end{array}\right]$$

$$W_1 \sin\theta\, R\,\delta\theta + W_2 \sin\phi\, R\,\delta\phi = 0$$

$$W_1 \sin\theta\, \delta\theta + W_2 \sin\phi\, \delta\phi = 0$$

But $\theta + \phi =$ Constant

Date ___ / ___ / ______

$$\delta\theta + \delta\phi = \text{Const}$$

$$\delta\phi = -\delta\theta$$

$$W_1 \sin\theta\, \delta\theta - W_2 \sin\phi\, \delta\theta = 0$$

$$(W_1 \sin\theta - W_2 \sin\phi)\, \delta\theta = 0$$

$$W_1 \sin\theta - W_2 \sin\phi = 0$$

or $\boxed{\dfrac{W_1}{W_2} = \dfrac{\sin\phi}{\sin\theta}}$ Ans.

Derive Lagrange's Equations from D'Alembert's Principle →

Consider a system of N particles. the transformation eq. for position vect of the particles are-

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_k \ldots q_n, t) \quad \text{(1)}$$

$$q_k \, (k = 1, 2, \ldots, n)$$

Derivate eq. (1) w.r. to t

$$\frac{d\vec{r}_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial r_i}{\partial q_k}\frac{dq_k}{dt} + \cdots \frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

$$v_i = \dot{r}_i = \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \quad \text{(2)}$$

where $\dot{q}_k$ are the generalized coordinates
q_k are the generalized velocities.

the virtual displacement is given by -

$$\delta \vec{r}_i = \frac{\partial r_i}{\partial q_1} \delta q_1 + \frac{\partial r_i}{\partial q_2} \delta q_2 + --- + \frac{\partial r_i}{\partial q_k} \delta q_k + -- +$$

$$\frac{\partial r_i}{\partial q_n} \delta q_n$$

$$\delta r_i = \sum_{i=1}^{n} \frac{\partial r_i}{\partial q_k} \delta q_k \quad -③$$

the virtual displacements do not depend on time.

According to D'Alembert's principle.

$$\sum_{i=1}^{N} (F_i - \dot{p}_i) \cdot \delta r_i = 0 \quad -④$$

$$\sum_{i=1}^{N} F_i \cdot \delta r_i = \sum_{i=1}^{N} \vec{F}_i \cdot \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k} \delta q_k$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{N} \left[\vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right] \delta q_k$$

$$= \sum_{k=1}^{n} G_k \delta q_k \quad -⑤$$

$$G_k = \sum_{i=1}^{N} \left[\vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right]$$

$$G_k = \sum_{i=1}^{N} \left[F_{xi} \frac{\partial x_i}{\partial q_k} + F_{yi} \cdot \frac{\partial y_i}{\partial q_k} + F_{zi} \frac{\partial z_i}{\partial q_k} \right] \quad -⑥$$

are called the components of generalized force associated with the generalized coordinates q_k.

Further

$$\sum_{i=1}^{N} \vec{\dot{p}}_i \cdot \delta \vec{r}_i = \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k$$

$$= \sum_{k=1}^{n} \left[\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right] \delta q_k \quad \text{--- (7)}$$

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

$$= \sum_{i=1}^{N} \left[\frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_k} \right) \right] \quad \text{--- (8)}$$

$$\frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_k} \right) = \frac{\partial}{\partial q_k} \left(\frac{d \vec{r}_i}{dt} \right) = \frac{\partial \vec{v}_i}{\partial q_k} \quad \text{--- (9)}$$

therefore, eq. (7) is and $\frac{\partial \dot{r}_i}{\partial \dot{q}_k} = \frac{\partial v_i}{\partial \dot{q}_k}$ — (10)

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \sum_{i=1}^{N} \left[\frac{d}{dt} \left[m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_k} \right] - m_i \vec{v}_i \cdot \frac{\partial v_i}{\partial q_k} \right] \quad \text{--- (11)}$$

Substituting in eq. (7) we get-

$$\sum_{i=1}^{N} \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{k=1}^{n} \sum_{i=1}^{N} \left[\frac{d}{dt} \left(m_i \vec{v}_i \cdot \frac{\partial v_i}{\partial \dot{q}_k} \right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_k} \right] \delta q_k$$

$$= \sum_{k=1}^{n} \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_k} \left(\sum_{i=1}^{N} \frac{1}{2} m_i (v_i \cdot v_i) \right) \right\} - \frac{\partial}{\partial q_k} \left\{ \sum_{i=1}^{N} \frac{1}{2} m_i (v_i \cdot v_i) \right\} \right] \delta q_k$$

$$= \sum_{k=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right] \delta q_k \quad \text{--- (12)}$$

$\sum_i \frac{1}{2} m_i (u_i \cdot u_i)$

$= \sum_i \frac{1}{2} m_i v_i^2 = T =$ K.E. of the system.

Substitute $\sum_i F_i \cdot \delta r_i$ from eq. (5) and $\sum_i \dot{p}_i \cdot \delta r_i$ from eq. (12) to eq. (8) then the D' Alembert's principle becomes.

$$\sum_{k=1}^{n} \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} \right\} - G_k \right] \delta q_k = 0 \quad (13)$$

constraints are holonomic, it means any virtual displacement δq_k is independent of δq_j. therefore, the coefficient in square bracket for each δq_k must be zero

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - G_k = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k} \quad (14)$$

this is the general form of lagrange's equations.

For conservative system

$$F_i = -\nabla_i V = -\hat{i}\frac{\partial V}{\partial x_i} - \hat{j}\frac{\partial V}{\partial y_i} - \hat{k}\frac{\partial V}{\partial z_i}$$

From eq. (6) the generalized force components are

$$G_k = -\sum_{i=1}^{N} \left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k} \right] \quad (15)$$

So it can be written as-

$$G_k = \frac{-\partial V}{\partial q_k} \quad -(16)$$

then eq. (14) becomes-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \frac{-\partial V}{\partial q_k} \quad -(17)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial (T-V)}{\partial q_k} = 0 \quad -(18)$$

V is scalar potential function which depends on q_k not generalized velociti

$$\text{So} \quad \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_k}\right) - \frac{\partial (T-V)}{\partial q_k} = 0$$

Now define new function

Laglangian $\boxed{L = T - V}$ —(19)

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0} \quad -(20)$$

this eq. known as laglange's equation for conservative system.

Laglange's Equations Formation →

the laglangian function L is given by-

$$L = T - V \quad (21)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad (22)$$

T = K.E; V=P.E

V = P.E.

to obtain the equations of motion of the system.

Find partial derivatives of L i.e. $\frac{\partial L}{\partial \dot{q}_k}$ and $\frac{\partial L}{\partial q_k}$ and substitute in eq. (22).

Kinetic energy in generalized coordinates.

$$T = \sum_i \frac{1}{2} m_i v_i^2$$

$$= \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right)^2$$

$$T = M_0 + \sum_k M_k \dot{q}_k + \frac{1}{2} \sum_{k,l} M_{kl} \dot{q}_k \dot{q}_l \quad (23)$$

$$M_0 = \sum_k \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t} \right)^2, \; M_k = \sum_i m_i \frac{\partial r_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

and $M_{kl} = \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_k} \cdot \frac{\partial \vec{r}_i}{\partial q_l}$

eq (23) is the expression for the kinetic energy.

First term in eq. (23) is independent of generalized velocities, while second and third are linear and quadratic.

For scleronomic systems.

So $v_i = \dot{r}_i = \sum_k \frac{\partial r_i}{\partial q_k} \dot{q}_k$

therefore $T = \sum_i \frac{1}{2} m_i v_i^2$

$$= \frac{1}{2} \sum_{k,l} M_{kl} \dot{q}_k \dot{q}_l \quad (24)$$

In this case, T is a homogeneous quadratic form in generalized velocities.

Note →

Prove $\frac{\partial v_i}{\partial \dot{q}_k} = \frac{\partial r_i}{\partial q_k}$

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \sum_{j=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_k} \dot{q}_j + \frac{\partial^2 r_i}{\partial t \partial q_k} \quad (1)$$

Here $\frac{\partial r_i}{\partial q_k}$ is a single quantity being the function of the generalized coordinates q_j and time t.

$$v_i = \frac{dr_i}{dt} = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t}$$

$$\frac{\partial v_i}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{dr_i}{dt}\right)$$

$$= \sum_{j=1}^{n} \frac{\partial^2 r_i}{\partial q_k \partial q_j} \dot{q}_j + \frac{\partial^2 r_i}{\partial q_k \partial t} \quad (2)$$

From eq. (1) and (2)

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \frac{\partial v_i}{\partial q_k}$$

$$\frac{\partial v_i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t}\right]$$

$$= \frac{\partial r_i}{\partial q_j} \delta_{jk}$$

$$\frac{\partial v_i}{\partial \dot{q}_k} = \frac{\partial r_i}{\partial q_k}$$