Angular momentum of a particle -

The angular momentum $\vec{L}$ of a particle which has momentum $\vec{p}$ and position vector $\vec{r}$ with respect to a given coordinate system.

$$\vec{L} = \vec{r} \times \vec{p}$$

The unit of angular momentum is $kg \times m^2/sec$ in the SI system or $g.cm^2/sec$ in cgs units.

Calculation of angular momentum -

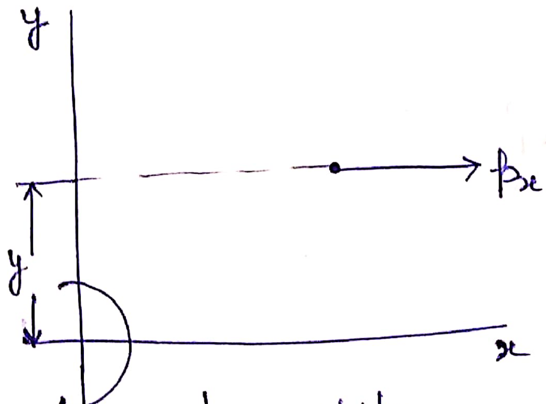
Consider motion in the xy plane, first in the x direction and then in the y direction

Using $\vec{r} = (x, y, 0)$

and $\vec{p} = (p_x, p_y, 0)$

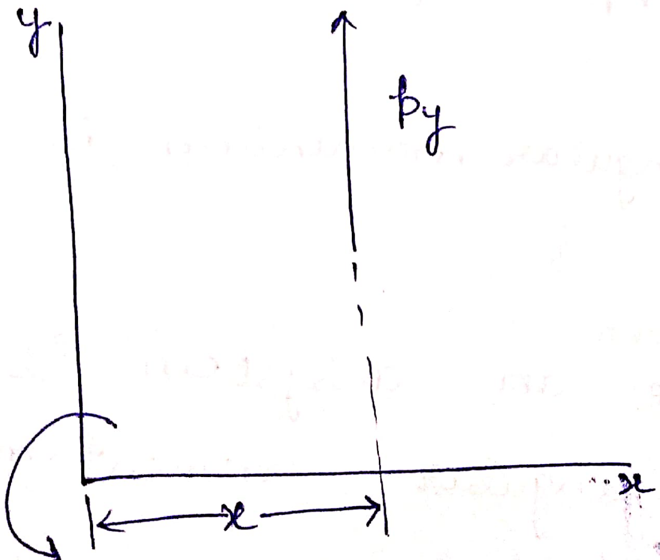
then we have $\vec{L} = \vec{r} \times \vec{p}$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ p_x & p_y & 0 \end{vmatrix}$

$$\vec{L} = (x p_y - y p_x)\hat{k}$$

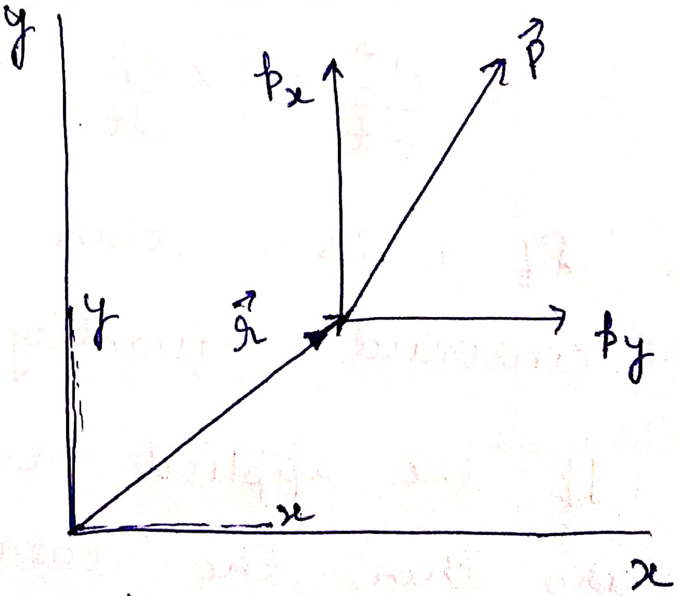


(a)

$L_z = -y p_x$



$L_z = x p_y$ (b)



$L_z = x p_y - y p_x$

(c)

fig.

Torque →

the torque due to force $\vec{F}$ which acts on a particle at position $\vec{r}$ is defined by $\vec{\tau} = \vec{r} \times \vec{F}$

$$\tau = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

the torque and force are entirely different physical quantities as Torque depends on the origin we choose but force does not. $\vec{\tau}$ and $\vec{F}$ are always mutually perpendicular.

Torque is a important physical quantity because it is intimately related to the rate of change of angular momentum.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

$u \times mu$
$= 0$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

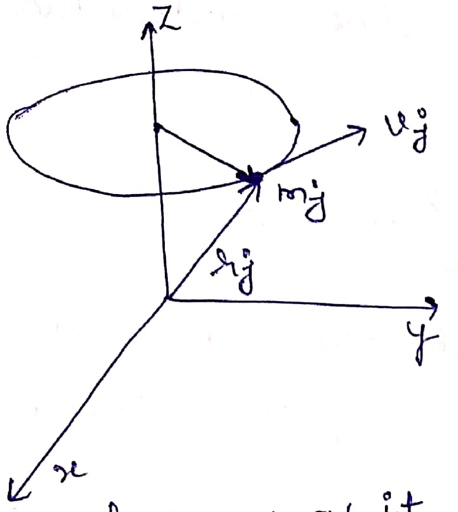
If τ is 0 then angular momentum is a conserved quantity.

If the applied torque on a system is zero then the total angular momentum of this system should be a conserved / constant quantity.

↓

this is the principle of conservation of angular momentum.

Fixed Axis Rotation and the angular momentum-



Here we can choose the axis of rotation in the z-direction the rotating object can be a wheel the only restriction being that it is rigid structure. which means its shape does not change as it rotates.

When a rigid body rotates about an axis, every particle in the body remains at a fixed distance from the axis. For each particle in the body

$|\vec{r}| =$ Constant

For a body rotating about the z-axis.

$$|\vec{v}_j| = |\dot{r}_j| = \omega \rho_j \quad \text{—(1)}$$

ρ_j = perpendicular distance from the axis of rotation to particle m_j of the rigid body and $\vec{\rho}_j$ is the corresponding vector.

ω = Angular velocity / Rate of rotation

the angular momentum of the jth particle of the body $\vec{L}(j)$ is

$$\vec{L}(j) = r_j \times m_j v_j$$

Here we are concerned only with L_z. (the component of angular momentum along the axis of rotation.

$$\vec{L}_z(j) = m_j v_j \times (\text{distance to z-axis})$$

$$\vec{L}_z(j) = m_j v_j \rho_j$$

$$\vec{L}_z(j) = m_j \rho_j^2 \omega$$

the z- component of the total angular momentum of the body L_z is the sum of the individual

z- components

$$\vec{L}_z = \sum_j \vec{L}_{z(j)} = \sum m_j \rho_j^2 \omega \quad (2)$$

here we have taken that the angular velocity ω to be constant throughout the body.

So $L_z = I\omega$

where $I = \sum_j m_j \rho_j^2$

↓

Moment of inertia

I depends on both the distribution of mass in the body and the location of the axis of rotation.

Now for continuous distribution.

$$\sum_j m_j \rho_j^2 \rightarrow \int \rho^2 \, dm$$

and $I = \int \rho^2 \, dm = \int (x^2 + y^2) \, dm$

we can replace $dm = \omega \, dV$

↓ the density at the position of dm

↓ the volume

ynamics of Pure Rotation about an Axis -

Here we consider fixed axis rotation with no translation of the axis. Motion like this, where there is an axis of rotation at rest is called pure rotation.

Consider a body rotating with angular velocity ω about the z-axis.

$$L_z = I\omega$$

$$\tau = \frac{dL}{dt} \qquad \tau_z = \frac{d}{dt}(I\omega)$$

$$\tau_z = I\frac{d\omega}{dt} = I\alpha$$

$\alpha = \frac{d\omega}{dt}$ is the angular acceleration.

$$\vec{\tau} = \vec{I}\vec{\alpha} \quad \text{—(1)}$$

eq. (1) is analogous to $\vec{F} = m\vec{a}$ and in fact there is a close analogy b/w linear and rotational motion. We can develop this further by evaluating the kinetic energy of a body undergoing pure rotation.

$$K = \Sigma \frac{1}{2} m_j v_j^2$$

$$= \Sigma \frac{1}{2} m_j \rho_j^2 \omega^2$$

$$= \frac{1}{2} I\omega^2$$

Angular momentum and tensor of inertia

Angular Momentum of a Rotating Rigid Body

In rotational motion $\vec{\tau} = \frac{d\vec{L}}{dt}$ resemblances to the translational motion

$$\vec{F} = \frac{d\vec{p}}{dt}$$

linear momentum $\vec{P} = M\vec{V}$ with $\vec{L} = I\vec{\omega}$

An arbitrary displacement of a rigid body can be resolved into a displacement of the centre of mass plus a rotation about some instantaneous axis through the centre of mass.

the angular momentum

$$\vec{L} = \vec{R} \times M\vec{V} + \Sigma r'_j \times m_j \dot{r}'_j \quad -①$$

the torque

$$\vec{\tau} = \vec{R} \times \vec{F} + \Sigma r'_j \times f_j \quad -②$$

where r'_j = position vector of m_j relative to the centre of mass, $\vec{\tau} = \frac{d\vec{L}}{dt}$

we have

$$\vec{R} \times \vec{F} + \Sigma r'_j \times f_j = \frac{d}{dt}(\vec{R} \times M\vec{V}) + \frac{d}{dt}(\Sigma r'_j \times m_j \dot{r}'_j)$$

$$= \vec{R} \times M\vec{A} + \frac{d}{dt}(\Sigma r'_j \times m_j \dot{r}'_j) \quad -③$$

Here $\vec{F} = M\vec{A}$

the terms involving R cancel.

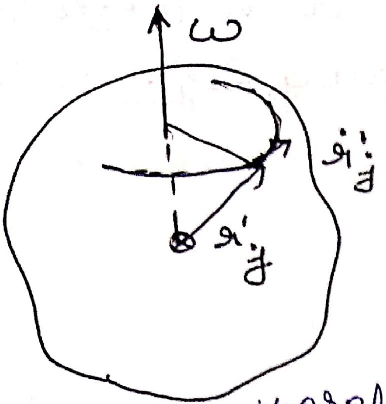
and eq. becomes.

$$\Sigma r'_j \times f_j = \frac{d}{dt}(\Sigma r'_j \times m_j \dot{r}'_j) \quad -④$$

the rotational motion can be found by taking torque and angular momentum about the centre of mass independent of the centre of mass motion.

the angular momentum $\vec{L}_o$ about the centre of mass is

$$L_o = \Sigma r'_j \times m_j \dot{r}'_j \quad -⑤$$



Now we express L_0 in terms of the instantaneous angular velocity ω.

r_j is a rotating vector

$\dot{r}'_j = \omega \times r'_j$

therefore,

$$L_0 = \Sigma r'_j \times m_j (\omega \times r'_j)$$

then simplify the notation

Put L for L_0 and r_j for r'_j

$$L = \Sigma r_j \times m_j (\omega \times r_j) \quad -(6)$$

Since $\omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

we have $\omega \times r = (z\omega_y - y\omega_z)\hat{i} + (x\omega_z - z\omega_x)\hat{j} + (y\omega_x - x\omega_y)\hat{k} \quad -(7)$

Let us compute one component of L say L_x and dropping subscript j we have.

$$[r \times (\omega \times r)]_x = y(\omega \times r)_z - z(\omega \times r)_y \quad -(8)$$

If we substitute the results of eq. (7) into eq. (8) the result is.

$$[r \times (\omega \times r)]_x = y(y\omega_x - x\omega_y) - z(x\omega_z - z\omega_x)$$
$$= y^2\omega_x + z^2\omega_x - xy\,\omega_y - xz\,\omega_z$$
$$= (y^2 + z^2)\omega_x - xy\,\omega_y - xz\,\omega_z \quad -(9)$$

Hence eq. (6) becomes

$$L_x = \Sigma m_j (y_j^2 + z_j^2)\omega_x - \sum_i m_j x_j y_j \omega_y - \Sigma m_j x_j z_j \omega_z \quad -(10)$$

Now let us introduce the following symbols.

As $I_{xx} = \Sigma m_j (y_j^2 + z_j^2)$
$I_{xy} = -\Sigma m_j x_j y_j$ — (11)
$I_{xz} = -\Sigma m_j x_j z_j$

I_{xx}, I_{xy}, I_{xz} are the moment of inertia.

Put values from eq. (11) to eq. (10).

$L_z = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$ (12c)) in the same
$L_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z$ (12b)) we can derivate for L_y and L_z.

$L_x = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z$ — (12a)

If we fixed axis rotation about the z-direction $\omega = \omega \hat{k}$ then eq. (12c) becomes.

$L_z = I_{zz} \omega$

$$L_z = \Sigma m_j (x_j^2 + y_j^2) \omega \quad \text{— (12)}$$

Eq. (12) shows that angular velocity in the z-direction can produce angular momentum about any of the three coordinate axes.

these equations (12a), (12b), (12c) are the equations of rotational motion.

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the tensor of inertia -

$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

$I_{xy} = I_{yx}$
$I_{yz} = I_{zy}$
$I_{xz} = I_{zx}$

symmetrical products of inertia

I_{xx}, I_{yy}, I_{zz} these are the moment of inertia]