BPC-401

Unit - I

Mechanics of a Particle -

let us suppose that $\vec{r}$ be the radius vector of a particle from some given origin and $\vec{v}$ is velocity vector.

then we know very well that

$$\vec{v} = \frac{d\vec{r}}{dt} \quad -①$$

Now we define the linear momentum $\vec{p}$ of the particle $\vec{p} = m\vec{v}$ -②

the particle may feel different kind of forces such as gravitational, electrodynamic due to the interactions with external objects and fields.

Now, we recall the Newton's second law of motion which states $\vec{F} = \frac{d\vec{p}}{dt} = \dot{p}$ -③

$$\vec{F} = \frac{d}{dt}(m\vec{v})$$

$$= m\frac{d\vec{v}}{dt} = m\vec{a} \quad -④$$

$$\vec{F} = m\frac{d^2r}{dt^2} \quad -⑤$$

the eq. of motion is thus a differential equation of second order, it is assuming here that $\vec{F}$ does not depend on higher-order derivation In which frame eq. ③ is valid is called the inertial frame.

Conservation Theorem for the linear momentum of a Particle.

If the total force $\vec{F}$ is zero, then $\dot{\vec{p}} = 0$, and the linear momentum $\vec{p}$ is conserved.

The angular momentum of the particle about point O, is defined as-

$$\vec{L} = \vec{r} \times \vec{p} \quad -⑥$$

where $\vec{r}$ is the radius vector from O to the particle.

Now we define the moment of force or torque about O as

$$\vec{N} = \vec{r} \times \vec{F} \quad -⑦$$

$$\vec{r} \times \vec{F} = N = \vec{r} \times \frac{d}{dt}(m\vec{v}) \quad -⑧$$

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times \frac{d}{dt}(m\vec{v})$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \frac{d}{dt}(m\vec{v}) \quad -⑨$$

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{r} \times \frac{d}{dt}(m\vec{v}) \quad -⑩$$

using this expression in eq.(8)

$$\vec{r} \times \vec{F} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = N$$

$$= \frac{d}{dt}(\underbrace{\vec{r} \times \vec{p}}_{\vec{L}})$$

$$N = \frac{d\vec{L}}{dt} \equiv \dot{\vec{L}} \quad -⑪$$

this is the torque equation which yields the conservation of angular momentum.

Conservation theorem for the angular momentum of a Particle - If the total torque $\vec{N}$ is zero then $\dot{L} = 0$, and the angular momentum is conserved.

Now we calculate the work done - the work done by the external force $\vec{F}$ upon the particle in going from point 1 to point 2, this work is

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} \quad -(12)$$

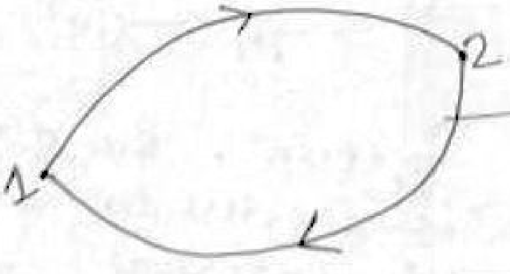
R.H.S. $\int \vec{F} \cdot d\vec{s} = m \int \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = \frac{m}{2} \int_1^2 \frac{d}{dt}(v^2)\, dt$

and therefore $W_{12} = \frac{m}{2}(v_2^2 - v_1^2) \quad -(13)$

the scalar quantity $\frac{mv^2}{2}$ is called the kinetic energy of the particle and this is denoted by T, so the work done is equal to the change in the kinetic energy.

$$W_{12} = T_2 - T_1 \quad -(14)$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force is called to be conservative force system.



→ work done around a closed circuit is zero

$$\boxed{\oint \vec{F} \cdot d\vec{s} = 0} \quad -(15)$$

$\vec{F}$ conservative force can be expressed (1) as the gradient of some scalar function of position

$$\vec{F} = -\nabla V(r) \quad \text{—(16)}$$

where V is called the potential or potential energy.

$$F \cdot ds = -dV$$

$$F_S = -\frac{\partial V}{\partial S}$$

the zero level of V is arbitrary.

for the conservative system, the work done by the forces is,

$$W_{12} = V_1 - V_2 \quad \text{—(17)}$$

combine this eq. with eq. (14)

$$T_1 + V_1 = T_2 + V_2 \quad \text{—(18)}$$

Energy conservation theorem for a particle-

If the forces acting on a particle are conservative, then the total energy of the particle T+V is conserved.

Mechanics of a System of Particles -

the equation of motion for the i^{th} particle is written as.

$$\sum_j \vec{F}_{ji} + F_i^{(e)} = \dot{p}_i \quad \text{—(19)}$$

For many particle system, the external forces acting on the particles due to sources outside the system, and internal forces on some particle i due to all other

other particles in the system. (5)

where $\vec{F}_i^{(e)}$ stands for an external force and $\vec{F}_{ji}$ is the internal force on the ith particle due to the jth particle.

We are assuming here that the $\vec{F}_{ij}$ obey Newton's third law of motion in its original form; that the forces two particles on each other are equal and opposite. this is referred as the law of action and reaction.

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ji} \quad -(20)$$

$$F_{ij} + F_{ji} = 0$$

According to the law of action and reaction.

Here we define a vector $\vec{R}$ as the average of the radii vectors of the particles

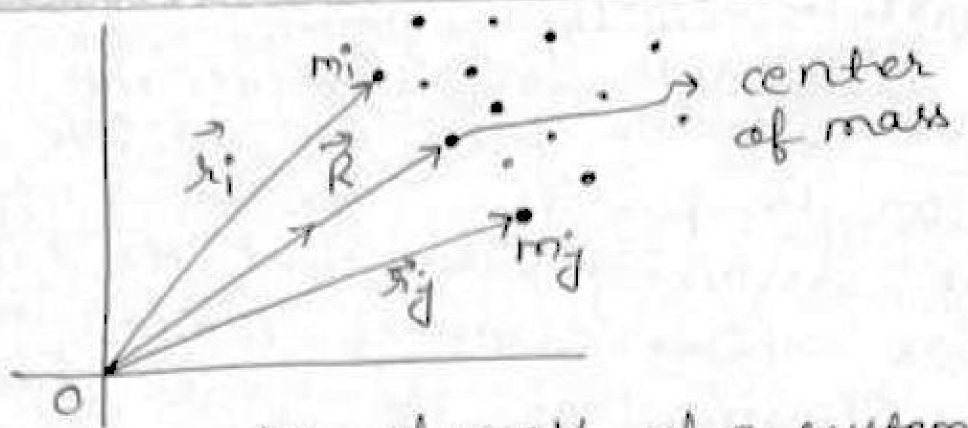
$$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\sum m_i \vec{r}_i}{M} \quad -(21)$$

the vector $\vec{R}$ defines a point known as the center of mass → eq (20) reduces,

$$M \frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} \equiv F^{(e)} \quad -(22)$$

$$P = \sum m_i \frac{dr_i}{dt} = M \frac{dR}{dt} \quad -(23)$$

$$\frac{dP}{dt} \equiv F^{(e)} \quad -(24)$$



the center of mass of a system of particles.

<u>conservation Theorem for the linear momentum of a system of particles</u> -

$$\sum_i (\vec{r}_i \times \vec{\dot{p}}_i) = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \dot{L}_i$$

$$= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \vec{r}_i \times \vec{F}_{ji} \quad \text{— (24)}$$

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \quad \text{— (25)}$$

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

$$\vec{r}_{ij} \times \vec{F}_{ji} \Rightarrow 0$$

$$\sum_i r_i \times \vec{\dot{p}}_i = \dot{L}$$

$$\sum \vec{r}_i \times \vec{F}_i^{(e)} + \vec{r}_{ij} \times \vec{F}_{ji} = \dot{L}$$

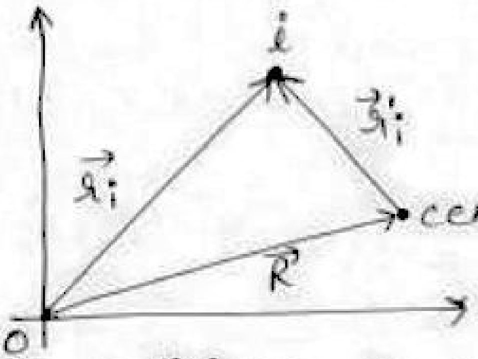
$$N^{(e)} = \dot{L}$$

$$\boxed{\frac{dL}{dt} \quad N^{(e)} = L} \quad \text{— (26)}$$

<u>Note</u> - If the internal forces between two particles, in addition to being equal and

opposite, also lie along the line joining the particles - a condition known as the strong law of action and reaction

Conservation Theorem for Total Angular Momentum → $\vec{L}$ is constant in time if the applied (external) torque is zero.



with the origin O as reference point, the total angular momentum of the system is,

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

let $\vec{R}$ be the radius vector from O to the center of mass, and let $\vec{r}_i'$ be the radius vector from the centre of mass to the i^{th} particle.

$$\vec{r}_i = \vec{r}_i' + \vec{R} \quad - (27)$$

$$\text{and } \vec{v}_i = \vec{v}_i' + v \qquad \vec{v} = \frac{d\vec{R}}{dt}$$

is the velocity of the center of mass relative to O, and $\vec{v}_i' = \frac{d\vec{r}_i'}{dt}$ is the velocity of the ith particle relative to the center of mass of the system.

the total angular momentum,

$$\vec{L} = \sum_i \vec{R} \times m_i\vec{v} + \sum_i \vec{r}_i' \times m_i\vec{v}_i' + \left(\sum_i m_i\vec{r}_i'\right) \times \vec{v}$$

$$+ \vec{R} \times \frac{d}{dt}\sum_i m_i r_i'$$

last two terms will vanish in this expression.

$$\boxed{\vec{L} = \vec{R} \times M\vec{v} + \sum_i \vec{r}'_i \times \vec{p}'_i} \quad (28)$$

the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass.

Now we calculate work done by all forces —

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i \quad (29)$$

can be used to reduce the equation of motion the integrals to.

$$\sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 m_i \dot{\vec{v}}_i \cdot \vec{v}_i \, dt$$

$$= \sum_i \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right)$$

$$W_{12} = T_2 - T_1$$

$$T = \frac{1}{2} \sum m_i v_i^2 \quad (30)$$

$$T = \frac{1}{2} \sum_i m_i \left(\vec{v} + \vec{v}'_i\right) \cdot \left(\vec{v} + \vec{v}'_i\right)$$

$$= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \underbrace{\vec{v} \cdot \frac{d}{dt}\left(\sum_i m_i \vec{r}'_i\right)}_{\text{this term will be vanish.}}$$

$$T = \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i v_i'^2 \quad (31)$$

this consists of two parts : the K.E. obtain if all the mass were concentrated at the center of mass + K.E. of motion about the center of mass.

$$\sum_i \int_1^2 F_i^{(e)} \cdot d\vec{s}_i = -\sum_i \int_1^2 \nabla_i V_i \cdot d\vec{s}_i$$

$$= -\sum_i V_i \Big|_1^2 ,$$

to satisfy the strong law of action and reaction, V_{ij} can be a function only of the distance b/w the particles :

$$V_{ij} = V_{ij}\,(|\vec{r}_i - \vec{r}_j|) \quad \text{— (32)}$$

$$\vec{F}_{ji} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\vec{F}_{ij} \quad \text{— (33)}$$

these two forces are equal and opposite. and lie along the line joining the two particles.

$$\nabla V_{ij}\,(|\vec{r}_i - \vec{r}_j|) = (|\vec{r}_i - \vec{r}_j)f \quad \text{— (34)}$$

f is some scalar function.

When the forces are all conservative, the second term in eq. (29) can be rewritten as a sum over pairs of particles, the terms for each pair being of the form.

$$-\int_1^2 (\nabla_i V_{ij} \cdot d\vec{s}_i + \nabla_j V_{ij} \cdot d\vec{s}_j)$$

If the difference vector $\vec{r}_i - \vec{r}_j$ is denoted by $\vec{r}_{ij}$ and if ∇_{ij} stands for gradient with respect to $\vec{r}_{ij}$, then

$$\nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij} ,$$

$$ds_i - ds_j = d\vec{r}_i - d\vec{r}_j = d\vec{r}_{ij}$$

so that the term for the ij pair has the form

$$-\int \nabla_{ij} V_{ij} \cdot d\vec{r}_{ij}$$

the total work arising from interal forces

$$-\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \nabla_{ij} V_{ij} \cdot d\vec{r}_{ij} = -\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij} \Big|_1^2 \quad \text{— (35)}$$

& total potential energy, V of the system

$$V = \sum_i V_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij} \quad \text{— (36)}$$

is such that the total energy T+V is conserved.

the second term in eq. (36) will be called the internal potential energy of the system in a rigid body, the internal forces do no work, and the internal potential must remain constant.