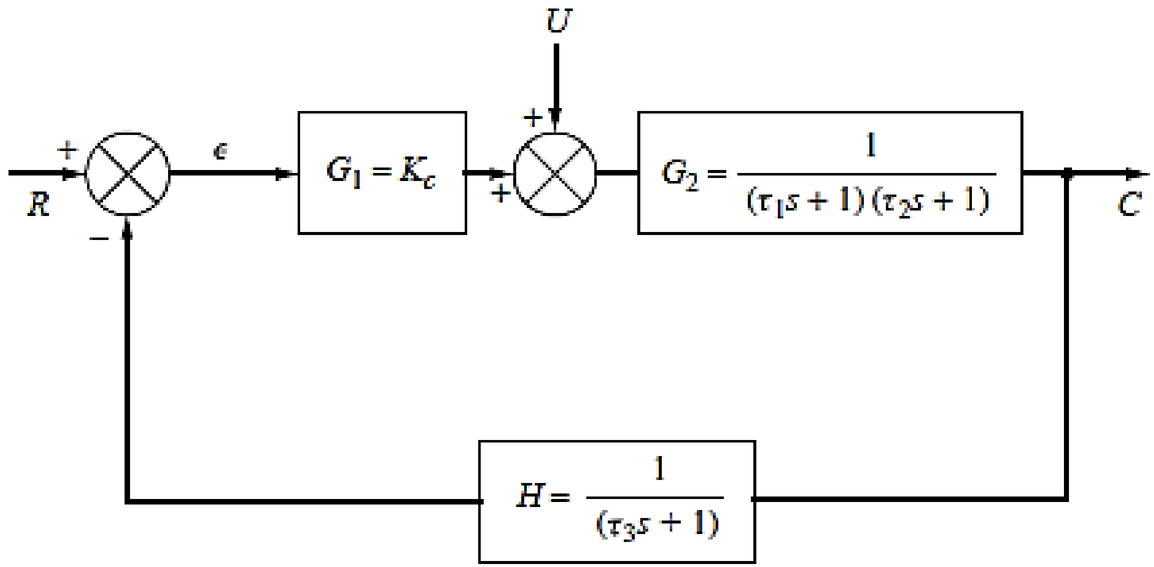
STABILITY

CHAPTER 13



$$\frac{C}{R} = \frac{G_1 G_2}{1 + G_1 G_2 H}$$

- The overall response of the control system was no higher than second-order.
- Hence, the system is inherently *stable.*
- In this chapter we consider the problem of stability in a control system only slightly more complicated than any studied previously.
- This system might represent proportional control of two stirred-tank heaters with measuring lag.
- In this discussion, only set point changes are to be considered.

$$\frac{C}{R} = \frac{K_c(\tau_3 s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1) + K_c}$$

For a unit-step change in *R,* the transform of the response is

$$C = \frac{1}{s}\frac{K_c(\tau_3 s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1) + K_c}$$

To obtain the transient response *C(t),* it is necessary to find the inverse of Eq.

This requires obtaining the roots of the denominator of Eq., which is third-order.

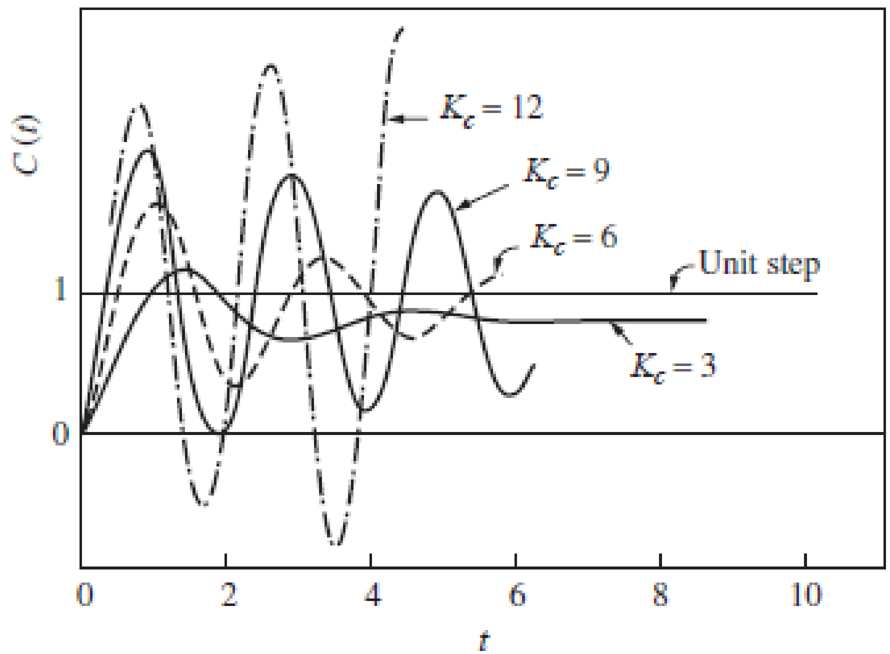
We can no longer find these roots as easily as we did for the second-order systems by use of the quadratic formula.

It is apparent that the roots of the denominator depend upon the particular values of the time constants and K_c .

These roots determine the nature of the transient response, according to the rules presented in Fig.

It is of interest to examine the nature of the response for the control system of Fig. as K_c is varied

- Assuming the time constants τ_1 , τ_2 , and τ_3 to be fixed.
- To be specific, consider the step response for several values of K_c $\tau_1 = 1,\ \tau_2 = \frac{1}{2},\ \text{and}\ \tau_3 = \frac{1}{3}$



- it is seen that as K_c increases, the system response becomes more oscillatory.
- In fact, beyond a certain value of K_c , the successive amplitudes of the response grow rather than decay;
- This type of response is called *unstable*

- In this chapter, the focus is on developing a clearer understanding of the concept of stability.
- In addition, we develop a quick test for detecting roots having positive real parts,

DEFINITION OF STABILITY (LINEAR SYSTEMS)

For our purposes, a *stable* system will be defined as one for which the output response is bounded for all bounded inputs. A system exhibiting an unbounded response to a bounded input is unstable.

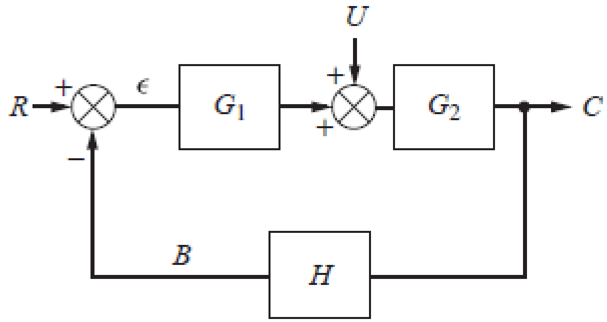
STABLE SYSTEM → a *b*ounded *i*nput produces a *b*ounded *o*utput (BIBO)

- A bounded input function is a function of time that always falls within certain bounds during the course of time.
- For example, the step function and sinusoidal function are bounded inputs.
- The function $f(t) = t$ is obviously unbounded.

- A linear mathematical model (set of linear differential equations describing the system) from which stability information is obtained is meaningful only over a certain range of variables.
- For example, a linear control valve gives a linear relation between flow and valve-top pressure only over the range of pressure (or flow) corresponding to values between which the valve is shut tight or wide open.
- When the valve is wide open, for example, further change in pressure to the diaphragm will not increase the flow.
- We often describe such a limitation by the term *saturation.*
- A physical system, when unstable, may not follow the response of its linear mathematical model beyond certain physical bounds but rather may saturate.
- However, the prediction of stability by the linear model is of utmost importance in a real control system since operation with the valve shut tight or wide open is clearly unsatisfactory control.

STABILITY CRITERION

- Translate the stability definition into a simpler criterion,



CHARACTERISTIC EQUATION

$$C = \frac{G_1G_2}{1 + G_1G_2H}R + \frac{G_2}{1 + G_1G_2H}U$$

Suppose a unit-step change in set point is applied.
Then

$$C(s) = \frac{G_1G_2}{1 + G}\frac{1}{s} = \frac{G_1G_2F(s)}{s(s - r_1)(s - r_2)(s - r_n)}$$

where $r_1, r_2, \ldots, r_n$ are the n roots of the equation

$F(s)$ is a function that arises in the rearrangement to the right-hand form

$$C(s) = \frac{G_1 G_2}{s(1 + G)}$$

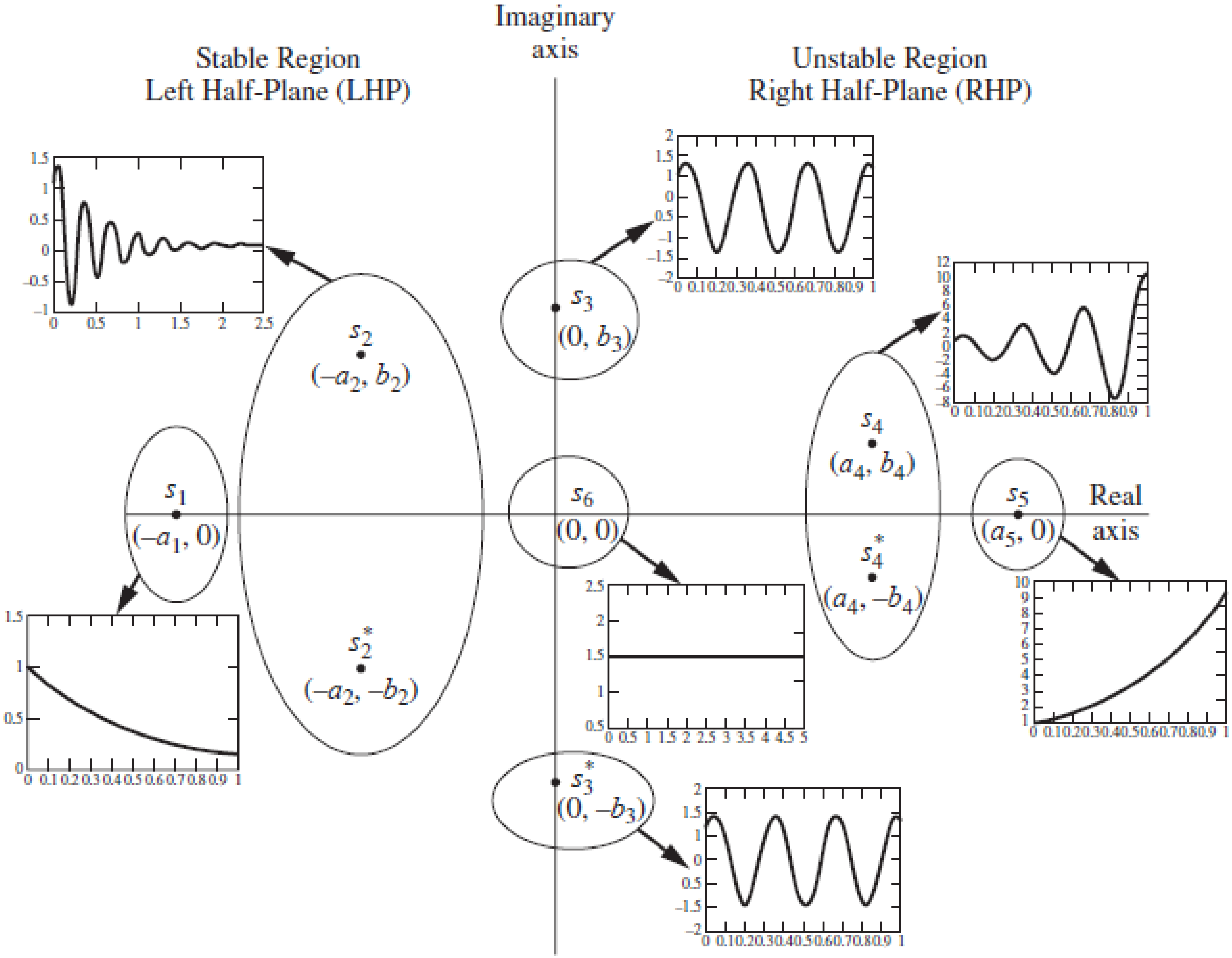
$$= \frac{K_c/[(\tau_1 s + 1)(\tau_2 s + 1)]}{s[1 + K_c/(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)]}$$

$$C(s) = \frac{K_c(\tau_3 s + 1)}{s\left[\tau_1\tau_2\tau_3 s^3 + (\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3)s^2 + (\tau_1 + \tau_2 + \tau_3)s + (1 + K_c)\right]}$$

$$C(s) = \frac{K_c(\tau_3 s + 1)/\tau_1\tau_2\tau_3}{s(s - r_1)(s - r_2)(s - r_3)}$$

$$\tau_1\tau_2\tau_3 s^3 + (\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3)s^2 + (\tau_1 + \tau_2 + \tau_3)s + (1 + K_c) = 0$$

$$F(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)}{\tau_1\tau_2\tau_3}$$



Imaginary axis
Stable Region
Left Half-Plane (LHP)
Unstable Region
Right Half-Plane (RHP)
Real axis
s1 (−a1, 0)
s2 (−a2, b2)
s2* (−a2, −b2)
s3 (0, b3)
s3* (0, −b3)
s4 (a4, b4)
s4* (a4, −b4)
s5 (a5, 0)
s6 (0, 0)

- If there are any of the roots $r_1, r_2, \ldots, r_n$ in the right half of the complex plane, the response $C(t)$ will contain a term that grows exponentially in time and the system is unstable.
- If there are one or more roots of the characteristic equation at the origin, there is an s^m in the denominator of Eq (where $m \geq 2$) and the response is again unbounded, growing as a polynomial in time.
- This condition specifies m as greater than or equal to 2, not 1, term will invert to a term of the form $C_1 t$, which is unbounded.)
- Additionally, if there is a pair of conjugate roots of the characteristic equation on the imaginary axis, the contribution to the overall step response is a pure sinusoid, which is bounded.
- However, if the bounded input is taken as sin ωt, where ω is the imaginary part of the conjugate roots, the contribution to the overall response is a sinusoid with an amplitude that increases as a polynomial in time [the response will have a term of the form $C_1 t \sin(\omega t + \Phi)$].
- Thus, if a root lies on the imaginary axis, there is the potential for repeating the root of a bounded input (such as a step input or a sinusoid input), and the response will be unstable.
- Therefore, the right-half plane, including the imaginary axis, is the unstable region for location of roots of the characteristic equation.

- characteristic equation of a control system, which determines its stability, is the same for set point or load changes.
- The stability depends only upon $G(s)$, the open-loop transfer function.
- Furthermore, although the rules derived above were based on a step input, they are applicable to any input.

$$G_1 = 10\frac{0.5s + 1}{s} \quad \text{(PI controller)}$$

$$G_2 = \frac{1}{2s + 1} \quad \text{(stirred tank)}$$

$$H = 1 \quad \text{(measuring element without lag)}$$

$$G = G_1 G_2 H = \frac{10(0.5s + 1)}{s(2s + 1)}$$

The characteristic equation is therefore

$$1 + \frac{10(0.5s + 1)}{s(2s + 1)} = 0$$

which is equivalent to

$$s^2 + 3s + 5 = 0$$

Solving by the quadratic formula gives

$$s = \frac{-3}{2} \pm \frac{\sqrt{9 - 20}}{2}$$

$$s_1 = \frac{-3}{2} + j\frac{\sqrt{11}}{2}$$

$$s_2 = \frac{-3}{2} - j\frac{\sqrt{11}}{2}$$

Since the real part of s_1 and s_2 is negative (3/2), the system is stable.

ROUTH TEST FOR STABILITY

- The Routh test is a purely algebraic method for determining how many roots of the characteristic equation have positive real parts.
- from this it can also be determined whether the system is stable.
- The test is limited to systems that have polynomial characteristic equations.
- This means that it cannot be used to test the stability of a control system containing a transportation lag (exponential function) .
- The procedure for application of the Routh test is presented without proof.
- The proof is available elsewhere (Routh, 1905) and is mathematically beyond the scope of this text.

- The procedure for examining the roots is to write the characteristic equation in the form

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

- where a_0 is positive. (If a_0 is originally negative, both sides are multiplied by -1.)
- In this form, it is *necessary* that all the coefficients be positive if all the roots are to lie in the left half-plane.

$$a_0,\ a_1,\ a_2, \ldots, a_{n-1},\ a_n$$

- If any coefficient is negative, the system is definitely unstable, and the Routh test is not needed to answer the question of stability. (However, in this case, the Routh test will tell us the number of roots in the right half-plane.)
- If all the coefficients are positive, the system may be stable or unstable.
- It is then necessary to apply the following procedure to determine stability.

Routh Array

Arrange the coefficients of Eq. into the first two rows of the Routh array as follows:

Row				
1	a_0	a_2	a_4	a_6
2	a_1	a_3	a_5	a_7
3	b_1	b_2	b_3	
4	c_1	c_2	c_3	
5	d_1	d_2		
6	e_1	e_2		
7	f_1			
$n + 1$	g_1			

The array has been filled in for $n = 7$ to simplify the discussion.
For any other value of n, the array is prepared in the same manner.
In general, there are $n + 1$ rows.
For n even, the first row has one more element than the second row.

The elements in the remaining rows are found from the formulas

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \quad \cdots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \quad \cdots$$

- The elements for the other rows are found from formulas that correspond to those just given.
- The elements in any row are always derived from the elements of the two preceding rows.
- During the computation of the Routh array, any row can be divided by a positive constant without changing the results of the test.
  (The application of this rule often simplifies the arithmetic.)
- Having obtained the Routh array, we can apply the following theorems to determine stability.

THEOREMS OF THE ROUTH TEST

- **Theorem 1.** The necessary and sufficient condition for all the roots of the characteristic equation to have negative real parts (stable system) is that all elements of the first column of the Routh array (a_0, a_1, b_1, c_1, etc.) be positive and nonzero.

- **Theorem 2.** If some of the elements in the first column are negative, the number of roots with a positive real part (in the right half-plane) is equal to the number of sign changes in the first column.

- **Theorem 13.3.** If *one* pair of roots is on the imaginary axis, equidistant from the origin, and all other roots are in the left half-plane, then all the elements of the n^{th} row will vanish and none of the elements of the preceding row will vanish. The location of the pair of imaginary roots can be found by solving the equation

$$Cs^2 + D = 0$$

where the coefficients *C* and *D* are the elements of the array in the $(n-1)^{st}$ row as read from left to right, respectively.

- The algebraic method for determining stability is limited in its usefulness in that all we can learn from it is whether a system is stable.
- It does not give us any idea of the degree of stability or the roots of the characteristic equation.

Example 13.2. Given the characteristic equation

$$s^4 + 3s^2 + 5s^2 + 4s + 2 = 0$$

determine the stability by the Routh criterion.

Since all the coefficients are positive, the system may be stable. To test this, form the following Routh array:

Row			
1	1	5	2
2	3	4	
3	11/3	6/3	
4	26/11	0	
5	2		

Since there is no change in sign in the first column, there are no roots having positive real parts, and the system is stable.

Using $\tau_1 = 1$, $\tau_2 = \frac{1}{2}$, and $\tau_3 = \frac{1}{3}$,

Determine the values of K_c for which the control system is stable.
For the value of K_c for which the system is on the threshold of instability, determine the roots of the characteristic equation.

The characteristic equation 1+ $G(s)$ =0 becomes

$$1 + \frac{K_c}{(s+1)(s/2+1)(s/3+1)} = 0$$

Rearrangement of this equation for use in the Routh test gives

$$s^3 + 6s^2 + 11s + 6(1 + K_c) = 0$$

The Routh array is

Row		
1	1	11
2	6	$6(1 + K_c)$
3	$10 - K_c$	
4	$6(1 + K_c)$	

Since the proportional sensitivity of the controller K_c is a positive quantity, we see that the fourth entry in the first column, $6(1 + K_c)$, is positive.

$$10 - K_c > 0$$

$$K_c < 10$$

It is concluded that the system will be stable only if $Kc < 10$

- At *Kc* =10, the system is on the verge of instability, and the element in the *n* th (third) row of the array is zero.
- According to Theorem 3, the location of the imaginary roots is obtained by solving

$$Cs^2 + D = 0$$

$$6s^2 + 66 = 0$$

$$s = \pm j\sqrt{11}$$

Therefore, two of the roots on the imaginary axis are located at $\sqrt{11}$ and $-\sqrt{11}$

The third root can be found by expressing Eq. (13.11) in factored form

$$(s - s_1)(s - s_2)(s - s_3) = 0$$

where s_1 , s_2 , and s_3 are the roots. Introducing the two imaginary roots ($s_1 = j$ 11 and $s_2 = -j$ 11) into Eq. and multiplying out the terms give

$$s^3 - s_3 s^2 + 11s - 11s_3 = 0$$

- Comparing this equation with Eq. we see that s_3 = - 6.

Let Using the new parameters given above in this equation leads to

$$s^4 + 6s^3 + 11s^2 + 36s + 120 = 0$$

Notice that the order of the characteristic equation has increased from three to four as a result of adding integral action to the controller.
The Routh array becomes

Row			
1	1	11	120
2	6	36	
3	5	120	
4	−108		
5	120		

Because there are two sign changes in the first column, we know from Theorem 2 of the Routh test that two roots have positive real parts.

- In next chapter we will discuss for obtaining more information about the actual location of the roots of the characteristic equation.
- This will enable us to predict the form of the curves for various values of K_c .
- The advantage of these tools is that they are graphical and are easy to apply compared with standard algebraic solutions of the characteristic equation.
- There are two distinct approaches to this problem: *root locus methods and frequency-response methods.*
- Root locus methods allow rapid determination of the location of the roots of the characteristic equation as functions of parameters such as K_c.
- However, they are difficult to apply to systems containing transportation lags.
- Also, they require a reasonably accurate knowledge of the theoretical process transfer function.