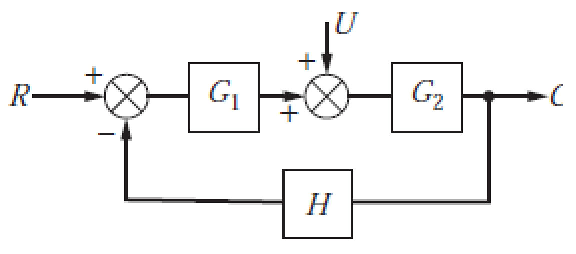
Root Locus

Chapter 14

CONCEPT OF ROOT LOCUS

- In this chapter, we develop a graphical method for finding the actual values of the roots of the characteristic equation, from which we can obtain the transient response of the system to an arbitrary forcing function.
- Illustrate the concept of a root locus diagram by considering the example presented



$$G = \frac{K_c}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)}$$

$$G(s) = \frac{K}{(s - p_1)(s - p_2)(s - p_3)}$$

$$G_1 = K_c$$

$$G_2 = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$H = \frac{1}{\tau_3 s + 1}$$

where $K = \dfrac{K_c}{\tau_1 \tau_2 \tau_3}$

$$p_1 = -\frac{1}{\tau_1} \quad p_2 = -\frac{1}{\tau_2} \quad p_3 = -\frac{1}{\tau_3}$$

- The terms p_1, p_2 , and p_3 are called the *poles* of the open-loop transfer function.
- A *pole* of $G(s)$ is any value of s for which $G(s)$ approaches infinity.
- Hence $p_1 = 1/\tau_1$ is a pole of $G(s)$.

The characteristic equation for the *closed-loop* system is

$$1 + \frac{K}{(s - p_1)(s - p_2)(s - p_3)} = 0$$

$$(s - p_1)(s - p_2)(s - p_3) + K = 0$$

Let by taking the value of $p_1 = -1$, $p_2 = -2$, $p_3 = -3$ and corresponding $K = 6K_c$

$$(s + 1)(s + 2)(s + 3) + K = 0$$

$$s^3 + 6s^2 + 11s + K + 6 = 0$$

which is third-order

- For any particular value of controller gain K_c, we can obtain the roots of the characteristic equation
- For example, if $K_c = 4.41$ ($K = 26.5$),

$$s^3 + 6s^2 + 11s + 32.5 = 0$$

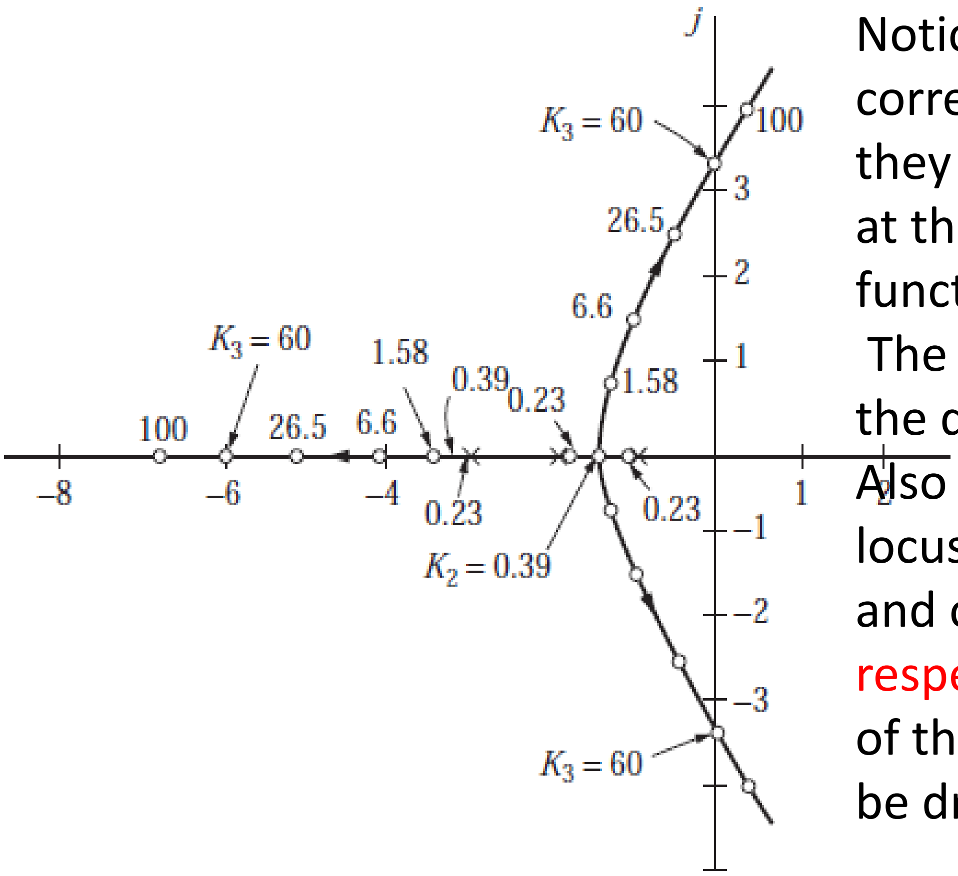
$$r_1 = -5.10$$
$$r_2 = -0.45 - j2.5$$
$$r_3 = -0.45 + j2.5$$

By selecting other values of *K*, other sets of roots are obtained

Roots of the characteristic equation $(s + 1)(s + 2)(s + 3) + K = 0$

$K = 6\ K_c$	r_1	r_2	r_3
0	−3	−2	−1
0.23	−3.10	−1.75	−1.15
0.39	−3.16	−1.42	−1.42
1.58	−3.45	−1.28 − *j*0.75	−1.28 + *j*0.75
6.6	−4.11	−0.95 − *j*1.5	−0.95 + *j*1.5
26.5	−5.10	−0.45 − *j*2.5	−0.45 + *j*2.5
60.0	−6.00	0.0 − *j*3.32	0.0 + *j*3.32
100.0	−6.72	0.35 − *j*4	0.35 + *j*4

- For convenience, we may plot the roots r_1, r_2, and r_3 on the complex plane as K changes continuously. Such a plot is called a **root locus diagram**



Notice that there are three loci or *branches* corresponding to the three roots and that they "emerge" or begin (for $K = 0$) at the poles of the open-loop transfer function (-1, -2,- 3).

The direction of increasing K is indicated on the diagram by an arrow.

Also the values of K are marked on each locus. The root locus diagram for this system and others to follow is symmetric with respect to the real axis, and only the portion of the diagram in the upper half-plane need be drawn.

- The diagram reveals two critical values of *K;* one is at *K* 2 (K=0.39) where two of the roots become equal,
- and the other is at *K* 3 (K=60) where two of the roots are pure imaginary.
- Thus, if the roots are all real, which occurs for $K < 0.39$, the response will be non-oscillatory.
- If two of the roots are complex and have negative real parts ($K_2 < K < K_3$), the response will include damped sinusoidal terms, which will produce an oscillatory response.
- If we adjust the controller gain such that $K > K3$, two of the roots are complex and have positive real parts, and the response is a growing sinusoid.