CHAPTER 3

INVERSION BY PARTIAL FRACTIONS

- We now wish to develop methods for inverting the transforms to obtain the solution in the time domain
- The equations to be solved are all of the general form

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = r(t)$$

- The unknown function of time is $x(t)$, and a_n, a_{n-1}, . . . , a_1, a_0 are constants.
- The given function $r(t)$ is called the *forcing function.*
- In addition, for all problems of interest in control system analysis, the initial conditions are given.
- In other words, values of x, dx/dt, . . . , $d^{n-1}x/dt^{n-1}$ are specified at time 0.
- The problem is to determine $x(t)$ for all $t > 0$.

PARTIAL FRACTIONS

$$\frac{dx}{dt} + x = 1$$

$$x(0) = 0$$

Application of the Laplace transform yields

$$sx(s) + x(s) = \frac{1}{s}$$

$$x(s) = \frac{1}{s(s+1)}$$

The theory of partial fractions enables us to write this as

$$x(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$A = 1$$

$$B = -1$$

Now that we've found A and B, we have

$$x(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$x(t) = 1 - e^{-t}$$

$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 4 + e^{2t}$$

$$x(0) = 1 \quad x'(0) = 0 \quad x''(0) = -1$$

Taking the Laplace transform of both sides yields

$$\left[s^3x(s) - s^2 + 1\right] + 2\left[s^2x(s) - s\right] - [sx(s) - 1] - 2x(s) = \frac{4}{s} + \frac{1}{s-2}$$

Solving algebraically for *x* (*s*), we find

$$x(s) = \frac{s^4 - 6s^2 + 9s - 8}{s(s-2)(s^3 + 2s^2 - s - 2)}$$

The cubic in the denominator may be factored, and *x* (*s*) is expanded in partial fractions.

$$x(s) = \frac{s^4 - 6s^2 + 9s - 8}{s(s-2)(s+1)(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+1} + \frac{D}{s+2} + \frac{E}{s-1}$$

To find A, multiply both sides of Eq. by s and then set s= 0;
The result is

$$A = \frac{-8}{(-2)(1)(2)(-1)} = -2$$

To determine	Multiply Eq. (3.8) by	and set s to	Result
B	$s-2$	2	$B = 1/12$
C	$s+1$	-1	$C = 11/3$
D	$s+2$	-2	$D = -17/12$
E	$s-1$	1	$E = 2/3$

Accordingly, the solution to the problem is

$$x(t) = -2 + \frac{1}{12}e^{2t} + \frac{11}{3}e^{-t} - \frac{17}{12}e^{-2t} + \frac{2}{3}e^{t}$$

A comparison between this method and the classical method,

$$s^3 + 2s^2 - s - 2 = 0$$

the roots 1, 2, and 1. Thus, the complementary solution is

$$x_c(t) = C_1e^{-t} + C_2e^{-2t} + C_3e^{t}$$

Inversion of a transform that has complex roots in the denominator

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 2$$

$$x(0) = 0 \qquad x'(0) = 0$$

Application of the Laplace transform yields

$$x(s) = \frac{2}{s\left(s^2 + 2s + 2\right)}$$

The quadratic term in the denominator may be factored by use of the quadratic formula.

The roots are found to be - 1- *j* and -1+ *j.*

If we use these complex roots in the partial fraction expansion, the algebra can get quite tedious

$$x(s) = \frac{2}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

Note that the second term of the expansion has the unfactored quadratic in the denominator.

$$A = \frac{2}{0 + 2(0) + 2} = 1$$

$$x(s) = \frac{2}{s(s^2 + 2s + 2)} = \frac{1}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

$$2 = s^2 + 2s + 2 + Bs^2 + Cs$$

$$(B + 1)s^2 + (2 + C)s + 2 = 2$$

$$s^2: \quad B + 1 = 0 \qquad B = -1$$

$$s: \quad 2 + C = 0 \qquad C = -2$$

$$x(s) = \frac{1}{s} - \frac{s + 2}{s^2 + 2s + 2}$$

$$L\left\{e^{-at}\sin(kt)\right\} = \frac{k}{(s + a)^2 + k^2}$$

$$x(s) = \frac{1}{s} - \frac{s + 2}{(s + 1)^2 + 1}$$

$$L\left\{e^{-at}\cos(kt)\right\} = \frac{s + a}{(s + a)^2 + k^2}$$

$$x(s) = \frac{1}{s} - \frac{(s + 1) + 1}{(s + 1)^2 + 1^2} = \frac{1}{s} - \frac{s + 1}{(s + 1)^2 + 1^2} - \frac{1}{(s + 1)^2 + 1^2}$$

$$x(t) = 1 - e^{-t}(\cos t + \sin t)$$

In the next example, an exceptional case is considered; the denominator of *x* (*s*) has *repeated roots.*

Inversion of a transform with repeated roots

$$\frac{d^3x}{dt^3} + \frac{3d^2x}{dt^2} + \frac{3dx}{dt} + x = 1$$

$$x(0) = x'(0) = x''(0) = 0$$

$$x(s) = \frac{1}{s\left(s^3 + 3s^2 + 3s + 1\right)}$$

$$x(s) = \frac{1}{s(s + 1)^3} = \frac{A}{s} + \frac{B}{(s + 1)^3} + \frac{C}{(s + 1)^2} + \frac{D}{s + 1}$$

As in the previous cases, to determine *A,* multiply both sides by *s* and then set *s* to zero. This yields

$$A = 1$$

Multiplication of both sides of Eq. by $(s + 1)^3$ results in

$$\frac{1}{s} = \frac{A(s + 1)^3}{s} + B + C(s + 1) + D(s + 1)^2$$

Setting *s*= -1 in Eq. gives

$$B = -1$$

Having found *A* and *B,* we introduce these values into Eq. and place the right side of the equation over a common denominator;

$$\frac{1}{s(s+1)^3} = \frac{(s+1)^3 - s + Cs(s+1) + Ds(s+1)^2}{s(s+1)^3}$$

$$\frac{1}{s(s+1)^3} = \frac{(1+D)s^3 + (3+C+2D)s^2 + (2+C+D)s + 1}{s(s+1)^3}$$

$$1 = (1+D)s^3 + (3+C+2D)s^2 + (2+C+D)s + 1$$

$$1 + D = 0$$
$$3 + C + 2D = 0$$
$$2 + C + D = 0$$

Solving these equations gives
C= -1 and *D*= -1.

The final result is then

$$x(s) = \frac{1}{s} - \frac{1}{(s+1)^3} - \frac{1}{(s+1)^2} - \frac{1}{s+1}$$

$$x(t) = 1 - e^{-t}\left(\frac{t^2}{2} + t + 1\right)$$

The result of Example may be generalized.

$$\frac{C_1}{(s+a)^n}, \frac{C_2}{(s+a)^{n-1}}, \ldots, \frac{C_n}{s+a}$$

The other constants are determined by the method shown above Example. These terms lead to the following expression as the inverse transform:

$$\left[\frac{C_1}{(n-1)!}t^{n-1} + \frac{C_2}{(n-2)!}t^{n-2} + \cdots + C_{n-1}t + C_n\right]e^{-at}$$

QUALITATIVE NATURE OF SOLUTIONS

If we are interested *only in the form* of the solution *x* (*t*), *this information may be obtained directly from the roots of the denominator of x* (*s*).

$$\frac{d^2x}{dt^2} + \frac{2' dx}{dt} + 2x = 2 \qquad x(0) = x'(0) = 0$$

$$x(s) = \frac{2}{s\left(s^2 + 2s + 2\right)} = \frac{A}{s} + \frac{B}{s+1+j} + \frac{C}{s+1-j}$$

- Since the roots of the quadratic term are $-1 \pm j$,
- $x(t)$ must contain terms of the form $e^{-t}(C_1 \cos t + C_2 \sin t)$
- Alternatively, interested in the behavior of $x(t)$ as $t \rightarrow \infty$.
- It is clear that the terms involving sin and cos vanish because of the factor e^{-t}.
- Therefore, $x(t)$ ultimately approaches the constant, which by inspection must be unity.
- The qualitative nature of the solution $x(t)$ can be related to the location of the roots of the denominator of $x(s)$ in the complex plane.
- These roots are the roots of the characteristic equation.

Nature of terms in the solution *x* (*t*) based on roots in the denominator of *X*(*s*)

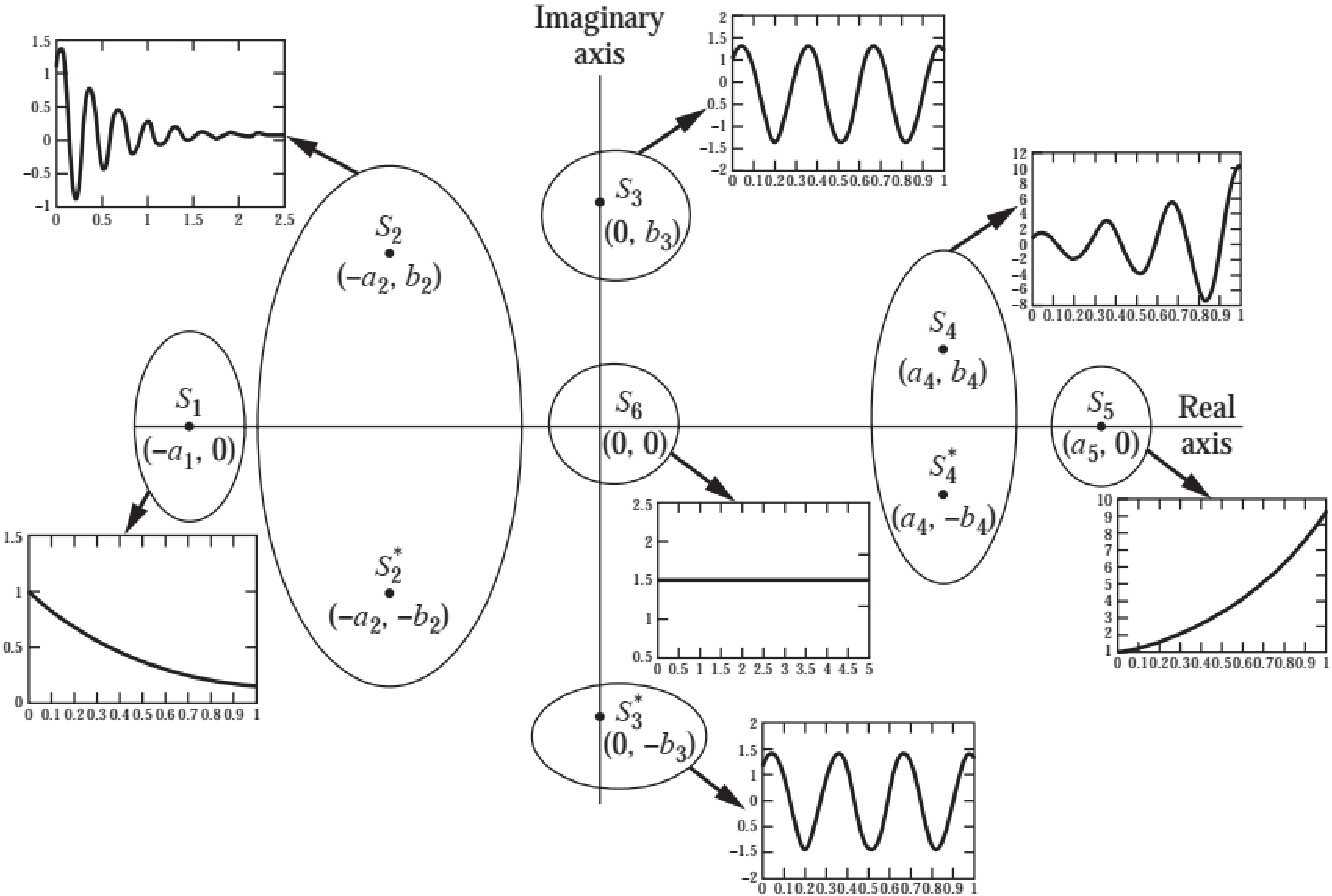
Roots in denominator of $X(s)$	Terms in $x(t)$ for $t > 0$
s_1	$C_1 e^{-a_1 t}$
s_2, s_2^*	$e^{-a_2 t}(C_1 \cos b_2 t + C_2 \sin b_2 t)$
s_3, s_3^*	$C_1 \cos b_3 t + C_2 \sin b_3 t$
s_4, s_4^*	$e^{a_4 t}(C_1 \cos b_4 t + C_2 \sin b_4 t)$
s_5	$C_1 e^{a_5 t}$
s_6	C_1

If any of these roots are repeated, the term given in Table is multiplied by a power series in t

$$K_1 + K_2 t + K_3 t^2 + \cdots + K_r t^{r-1}$$

where r is the number of repetitions of the root and the constants $K_1, K_2, \ldots, K_r$ can be evaluated by partial fraction expansion.

Location of typical roots of characteristic equation



Any proper fraction may be resolved into a number of partial fractions subject to the following rules.

$$\frac{F(s)}{\underbrace{(as+b)^n}_{\text{root repeated } n \text{ times}}} = \underbrace{\frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \cdots + \frac{A_n}{(as+b)^n}}_{\text{there are } n \text{ partial fractions in expansion}}$$

$$\frac{F(s)}{\cdots(as^2+bs+c)\cdots} = \frac{As+B}{\underbrace{as^2+bs+c}_{\substack{\text{numerator is polynomial of one}\\ \text{less degree than denominator}}}} + \cdots$$

$$\frac{F(s)}{\cdots\underbrace{(as^2+bs+c)^n}_{\text{yields}\rightarrow}\cdots} =$$

$$\underbrace{\frac{A_1 s + B_1}{as^2+bs+c} + \frac{A_2 s + B_2}{(as^2+bs+c)^2} + \cdots + \frac{A_n s + B_n}{(as^2+bs+c)^n} + \cdots}_{n \text{ terms in expansion}}$$

FINAL-VALUE THEOREM

- If $f(s)$ is the Laplace transform of $f(t)$, then

$$\lim_{t \to \infty} [f(t)] = \lim_{s \to 0} [sf(s)]$$

provided that $sf(s)$ does not become infinite for any value of s satisfying **Re(s) ≥ 0**.

$$\int_0^\infty \frac{df}{dt} e^{-st}\, dt = sf(s) - f(0)$$

$$\lim_{s \to 0} \int_0^\infty \frac{df}{dt} e^{-st}\, dt = \lim_{s \to 0} [sf(s)] - f(0)$$

$$\int_0^\infty \frac{df}{dt} dt = \lim_{s \to 0} [sf(s)] - f(0)$$

$$\lim_{t \to \infty} [f(t)] - f(0) = \lim_{s \to 0} [sf(s)] - f(0)$$

- Find the final value of the function *x* (*t*) for which the Laplace transform is

$$x(s) = \frac{1}{s\left(s^3 + 3s^2 + 3s + 1\right)}$$

- Direct application of the final-value theorem yields

$$\lim_{t \to \infty}[x(t)] = \lim_{s \to 0} \frac{1}{s^3 + 3s^2 + 3s + 1} = 1$$

$$x(t) = 1 - e^{-t}\left(\frac{t^2}{2} + t + 1\right)$$

Find the final value of the function *x* (*t*) for which the Laplace transform is

$$x(s) = \frac{s^4 - 6s^2 + 9s - 8}{s(s - 2)\left(s^3 + 2s^2 - s - 2\right)}$$

$$sx(s) = \frac{s^4 - 6s^2 + 9s - 8}{(s+1)(s+2)(s-1)(s-2)}$$

Since this becomes infinite for *s*= 1 and *s* = 2, the conditions of the theorem are
not satisfied.
Note that we [illegible] und that

$$x(t) = -2 + \frac{1}{12}e^{2t} + \frac{11}{3}e^{-t} - \frac{17}{12}e^{-2t} + \tfrac{2}{3}e^{t}$$

INITIAL-VALUE THEOREM

$$\lim_{x \to 0}[f(t)] = \lim_{s \to \infty}[sf(s)]$$

Find the initial value *x*(0) of the function that has the transform

$$x(s) = \frac{s^4 - 6s^2 + 9s - 8}{s(s-2)\left(s^3 + 2s^2 - s - 2\right)}$$

$$sx(s) = \frac{s^4 - 6s^2 + 9s - 8}{s^4 - 5s^2 + 4}$$

$$sx(s) = \frac{1 - 6/s^2 + 9/s^3 - 8/s^4}{1 - 5/s^2 + 4/s^4}$$

$$X(0) = 1$$

TRANSLATION OF TRANSFORM

If $L\{f(t)\} = f(s)$, then

$$L\{e^{-at} f(t)\} = f(s + a)$$

Proof

$$L\{e^{-at} f(t)\} = \int_0^\infty f(t) e^{-at} e^{-st}\, dt = \int_0^\infty f(t) e^{-(s+a)t}\, dt = f(s + a)$$

Find $L\{e^{-at} \cos kt\}$. Since

$$L\{\cos kt\} = \frac{s}{s^2 + k^2}$$

- A primary use for this theorem is in the inversion of transforms. For example, by using this theorem the transform

$$x(s) = \frac{1}{(s + a)^2}$$

as $L\{t\} = \frac{1}{s^2}$

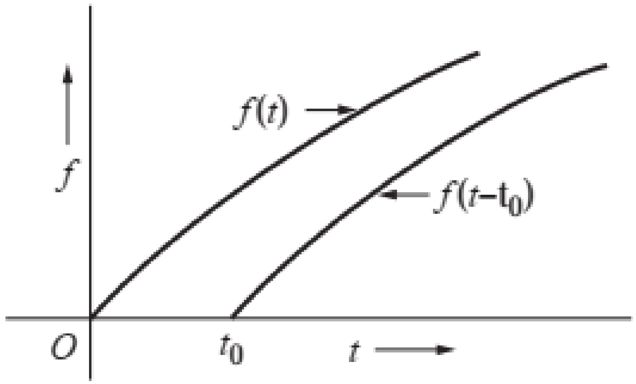
$$x(t) = te^{-at}$$

TRANSLATION OF FUNCTION

If $L\{f(t)\} = f(s)$, then

$$L\{f(t - t_0)\} = e^{-st_0} f(s)$$

$$f(t) = 0 \quad \text{for } t < 0$$



Proof.

$$L\{f(t - t_0)\} = \int_0^\infty f(t - t_0)e^{-st}\, dt$$

$$= e^{-st_0} \int_{-t_0}^\infty f(t - t_0)e^{-s(t-t_0)}\, d(t - t_0)$$

TRANSFORM OF AN INTEGRAL

- If *L* { *f*(*t*)} = *f*(*s*), then

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{f(s)}{s}$$

- This important theorem is closely related to the theorem on differentiation.
- Since the operations of differentiation and integration are inverses of each other when applied to the time functions,

$$\frac{d}{dt}\int_0^t f(t)\,dt = \int_0^t \frac{df}{dt}\,dt = f(t)$$

- It is to be expected that these operations when applied to the transforms will also be inverses.

$$s\,\frac{f(s)}{s} = \frac{1}{s}\,sf(s) = f(s)$$

Solve the following equation for *x* (*t*).

$$\frac{dx}{dt} = \int_0^t x(t)\,dt - t$$

$$x(0) = 3$$

$$sx(s) - 3 = \frac{x(s)}{s} - \frac{1}{s^2}$$

$$x(s) = \frac{3s^2 - 1}{s\left(s^2 - 1\right)} = \frac{3s^2 - 1}{s(s + 1)(s - 1)}$$

$$x(s) = \frac{1}{s} + \frac{1}{s + 1} + \frac{1}{s - 1}$$

$$x(t) = 1 + e^{-t} + e^{t}$$

Relationship between unit step and unit impulse: $\int_0^\infty \delta(t)\,dt = u(t)$

and $\frac{du(t)}{dt} = \delta(t)$

Use the theorem for the transform of an integral to determine the transform of the unit-step function if we know that *L* {*d* (*t*)} = 1

$$u(t) = \int^\infty \delta(t)\,dt, \text{ then}$$

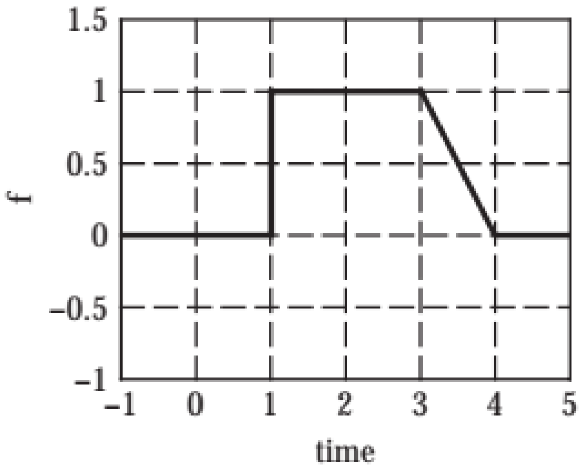
$$L\{u(t)\} = L\left\{\int_0^\infty \delta(t)\,dt\right\} = \frac{1}{s} L\{\delta(t)\} = \frac{1}{s}$$

cross check: since we know that *du* (*t*)/ *dt* = *d* (*t*), then

$$L\{\delta(t)\} = L\left\{\frac{du(t)}{dt}\right\} = sL\{u(t)\} = s \cdot \frac{1}{s} = 1$$

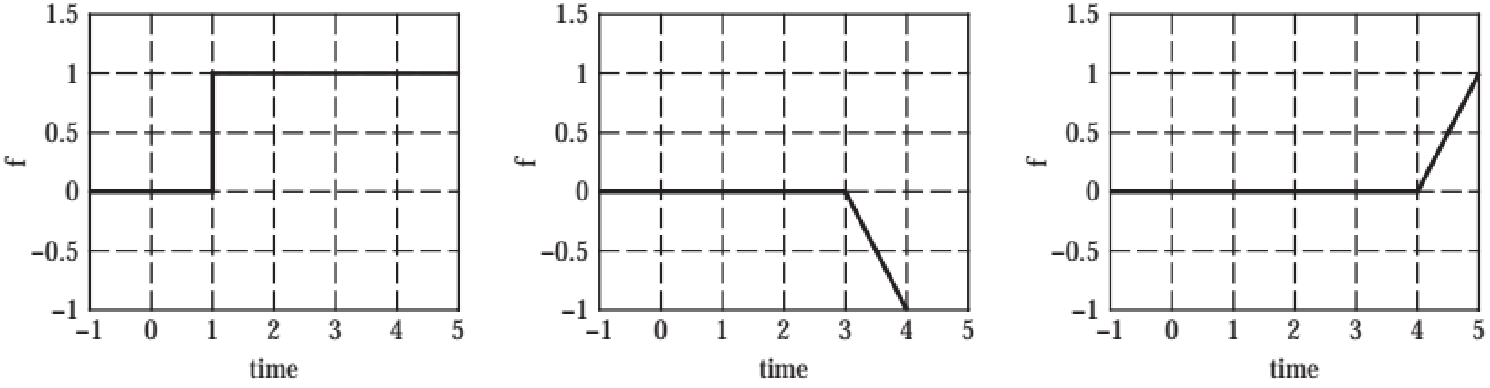
CUSTOM INPUTS

- We can produce “custom” input signals by appropriately constructing them using standard input signals.
- These custom inputs are frequently useful when we analyze a process disturbance.



We can consider this input signal to be constructed from several individual “pieces,”

$$f(t) = u(t-1) - (t-3)u(t-3) + (t-4)u(t-4)$$



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