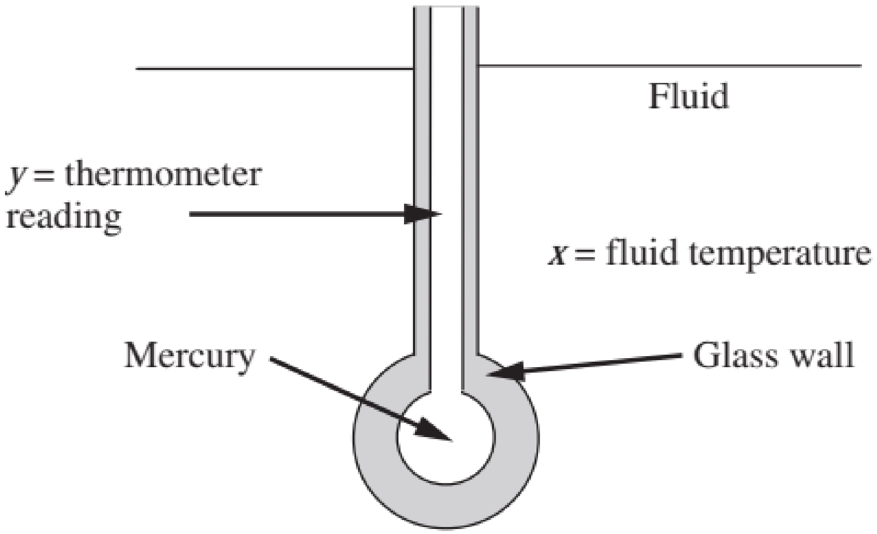
RESPONSE OF FIRST-ORDER SYSTEMS

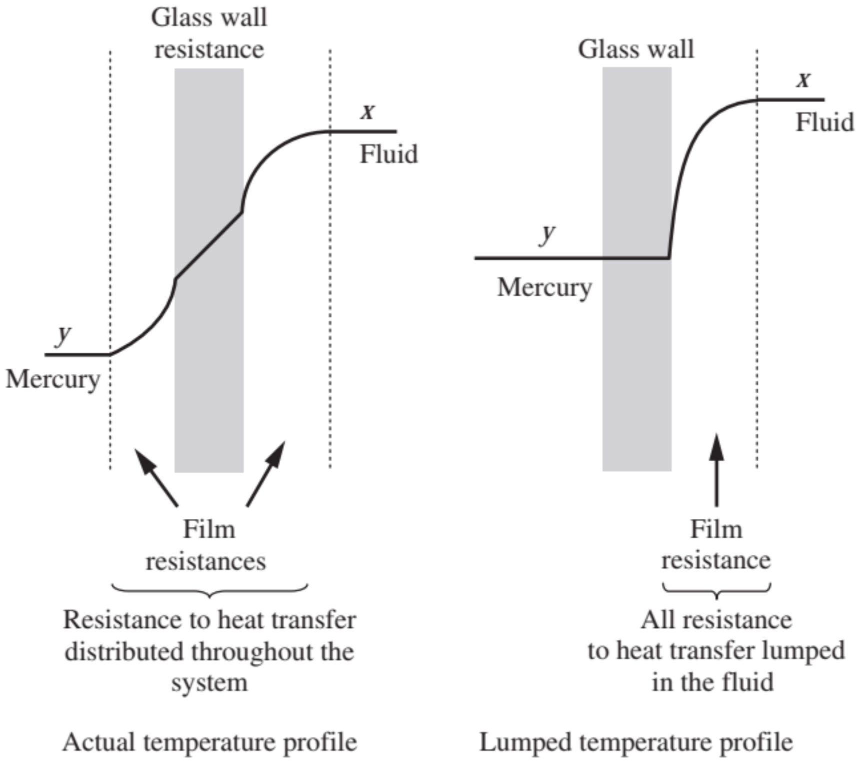
CHAPTER 4

TRANSFER FUNCTION

- **MERCURY THERMOMETER**

Transfer function for a *first-order system* by considering the unsteady-state behavior of an ordinary mercury-in-glass thermometer.





- Consider the thermometer to be located in a flowing stream of fluid for which the temperature *x* varies with time.
- Our problem is to calculate the *response* or the time variation of the thermometer reading *y* for a particular change in *x.*
- *In order that the result of the analysis of the thermometer be general and therefore applicable to other first-order systems.*
- *Symbols x and y have been selected to represent surrounding temperature and thermometer reading, respectively.*

- The following assumptions will be used in this analysis:

1. All the resistance to heat transfer resides in the film surrounding the bulb (i.e., the resistance offered by the glass and mercury is neglected).

2. All the thermal capacity is in the mercury. Furthermore, at any instant the mercury assumes a uniform temperature throughout.

3. The glass wall containing the mercury does not expand or contract during the transient response.

(In an actual thermometer, the expansion of the wall has an additional effect on the response of the thermometer reading. The glass initially expands and the cavity containing the mercury grows, resulting in a mercury reading that initially falls. Once the mercury warms and expands, the reading increases. This is an example of an inverse response. Inverse responses will be discussed in greater detail later.)

- It is assumed that the thermometer is initially at steady state. This means that, before time 0, there is no change in temperature with time.
- At time 0, the thermometer will be subjected to some change in the surrounding temperature *x* (*t*).

By applying the unsteady-state energy balance

$$(\text{Input rate}) - (\text{Output rate}) = (\text{Rate of accumulation})$$

$$hA(x - y) - 0 = mC\frac{dy}{dt}$$

where A = surface area of bulb for heat transfer, ft^2
C = heat capacity of mercury, $Btu/(lb_m \cdot °F)$
m = mass of mercury in bulb, lb_m
t = time, h
h = film coefficient of heat transfer, $Btu/(ft^2 \cdot h \cdot °F)$

Prior to the change in *x,* the thermometer is at steady state and the derivative *dy/dt* is zero.

$$hA(x_s - y_s) = 0 \qquad t < 0$$

- Simply states that $y_s = x_s$, or the thermometer reads the true, bath temperature

$$hA[(x - x_s) - (y - y_s)] = mC\frac{d(y - y_s)}{dt}$$

Notice that $d(y - y_s)/dt = dy/dt$ because y_s is a constant.

If we define the deviation variables to be the differences between the variables and their steady-state values

$$X = x - x_s$$
$$Y = y - y_s$$

$$hA(X - Y) = mC\frac{dY}{dt}$$

let $mC/hA = \tau$,

$$X - Y = \tau\frac{dY}{dt}$$

The parameter *t* is called the *time constant* of the system and has the units of time.

From above, we have

$$\tau = \frac{mC}{hA} [=] \frac{(\text{lb}_m)\left(\frac{\text{Btu}}{\text{lb}_m \cdot {}^\circ\text{F}}\right)}{\left(\frac{\text{Btu}}{\text{ft}^2 \cdot \text{h} \cdot {}^\circ\text{F}}\right)(\text{ft}^2)} [=] \text{h}$$

- *X* is the input to the system (the bath temperature) and *Y* is the output from the system (the indicated thermometer temperature).
  Taking the Laplace transform of Eq. (4.5) gives

$$X(s) - Y(s) = \tau s Y(s) - Y(0) = \tau s Y(s)$$

Initial conditions because the initial values of *X* and *Y* are zero.
Since we start from steady state,

$$Y(0) = y(0) - y_s = y_s - y_s = 0$$

Y (0) must be zero,
and X(0) is zero for the same reason.

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1} = \frac{\text{output}}{\text{input}}$$

The expression on the right side of Eq. is called the *transfer function* of the system.
It is the ratio of the Laplace transform of the deviation in thermometer reading *(output)*
to the Laplace transform of the deviation in the surrounding temperature *(input).*

- Any physical system for which the relation between Laplace transforms of input and output deviation variables is of the form given by Eq. is called a *first-order system.*
- Synonyms for first-order systems are **first-order lag** and **single exponential** stage.
- Linear differential equation, Eq. discuss a number of other physical systems that are first-order.

  To summarize the procedure for determining the transfer function for a process:

Step 1. Write the appropriate balance equations (usually mass or energy balances or a chemical process).

Step 2. Linearize terms if necessary

Step 3. Place balance equations in deviation variable form.

Step 4. Laplace-transform the linear balance equations.

Step 5. Solve the resulting transformed equations for the transfer function, the output divided by the input

Standard Form for First-Order Transfer Functions

- The general form for a first-order system is

$$\tau \frac{dy}{dt} + y = K_p x(t)$$

- where *y* is the output variable and *x* (*t*) is the input forcing function. The initial conditions are

$$y(0) = y_s = K_p x(0) = K_p x_s$$

- Introducing deviation variables gives

$$X = x - x_s$$
$$Y = y - y_s$$

$$\tau \frac{dY}{dt} + Y = K_p X(t)$$
$$Y(0) = 0$$

$$\tau s Y(s) + Y(s) = K_p X(s)$$

We obtain the standard first-order transfer function

$$\frac{Y(s)}{X(s)} = \frac{K_p}{\tau s + 1}$$

The important characteristics of the standard form are as follows:

- The denominator must be of the form (τs +1).
- The coefficient of the s term in the denominator is the system time constant τ.
- The numerator is the steady-state gain K_p

Place the following transfer function in standard first-order form, and identify the time constant and the steady state gain.

$$\frac{Y(s)}{X(s)} = \frac{2}{s + \frac{1}{3}}$$

Rearranging to standard form,

$$\frac{Y(s)}{X(s)} = \frac{6}{3s + 1}$$

Thus, the time constant is 3, and the steady-state gain is 6.

Physical significance of the steady-state gain becomes clear if we let $X(s) = 1/s$, the unit-step function.

Then $Y(s)$ is given by

$$Y(s) = \frac{6}{s(3s+1)}$$

The ultimate value of $Y(t)$ is $t \rightarrow \infty$

$$\lim_{s \to 0}[sY(s)] = \lim_{s \to 0}\left[\frac{6}{3s+1}\right] = 6 = K_p$$

Thus the steady-state gain Kp is the steady-state value that the system attains after being disturbed by a unit-step input.

It can be obtained by setting $s = 0$ in the transfer function.

PROPERTIES OF TRANSFER FUNCTIONS.

- In general, a transfer function relates two variables in a physical process;
- one of these is the cause (forcing function or input variable),
- other is the effect (response or output variable).
- In terms of the example of the mercury thermometer, the surrounding temperature is the cause or input, whereas the thermometer reading is the effect or output.
- $$\text{Transfer function} = G(s) = \frac{Y(s)}{X(s)}$$

where

$G(s)$ symbol for transfer function

$X(s)$ transform of forcing function or input, in deviation form

$Y(s)$ transform of response or output, in deviation form

- The transfer function completely describes the dynamic characteristics of the system.
- If we select a particular input variation *X* (*t*) for which the transform is *X* (*s*),
- The response of the system is simply

$$Y(s) = G(s)X(s)$$

- Taking the inverse of Y (s), we get Y (t), the response of the system.
- The transfer function results from a linear differential equation; therefore, the principle of superposition is applicable.
- This means that the transformed response of a system with transfer function G (s) to a forcing function

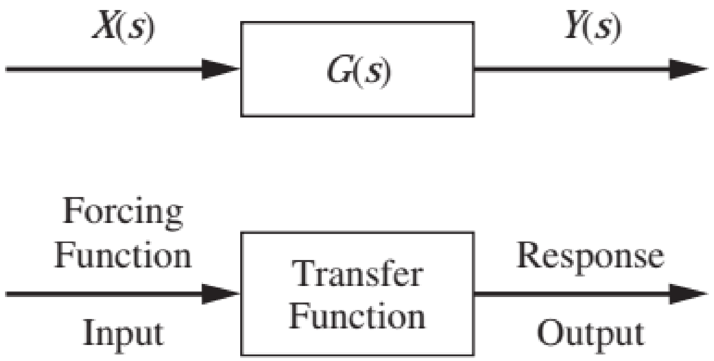
$$X(s) = a_1 X_1(s) + a_2 X_2(s)$$

Where X_1 and X_2 are particular forcing functions and a_1 and a_2 are constants, is

$$
\begin{aligned}
Y(s) &= G(s)X(s) \\
&= a_1 G(s) X_1(s) + a_2 G(s) X_2(s) \\
&= a_1 Y_1(s) + a_2 Y_2(s)
\end{aligned}
$$

- where $Y_1(s)$ and $Y_2(s)$ are the responses to X_1 and X_2 alone, respectively.

- The functional relationship contained in a transfer function is often expressed by a *block diagram* representation,



- The transfer function $G(s)$ in the box “operates” on the input function $X(s)$ to produce an output function $Y(s)$.

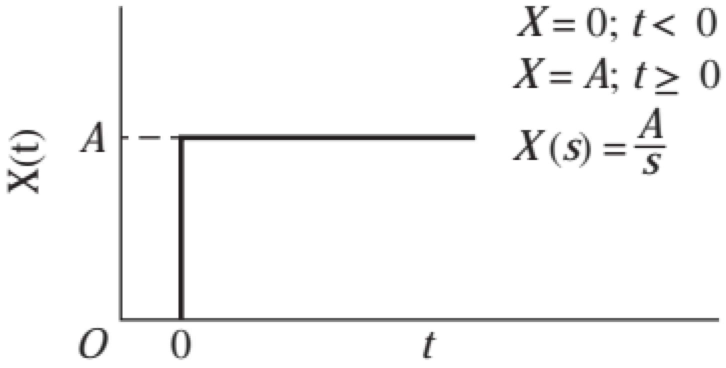
TRANSIENT RESPONSE

- Let’s explore study the transient response of the first-order system to these forcing functions.
- It is worthwhile to study its response to several common forcing functions: step, impulse, ramp, and sinusoidal.
- These forcing functions have been found to be very useful in theoretical and experimental aspects of process control.

FORCING FUNCTIONS

STEP FUNCTION.

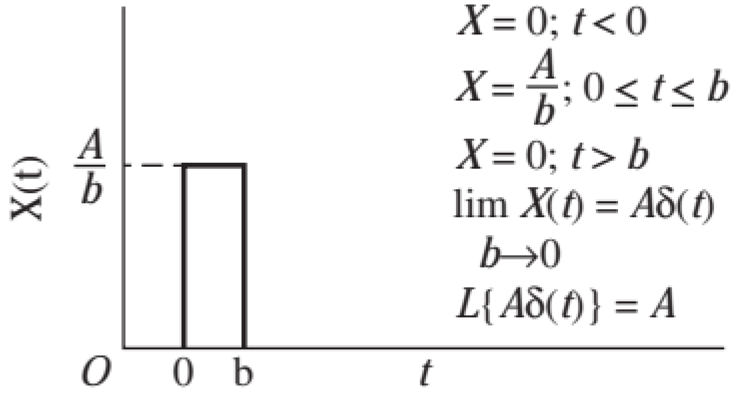
$$X(t) = Au(t)$$



$$X = 0;\ t < 0$$
$$X = A;\ t \geq 0$$
$$X(s) = \frac{A}{s}$$

IMPULSE FUNCTION

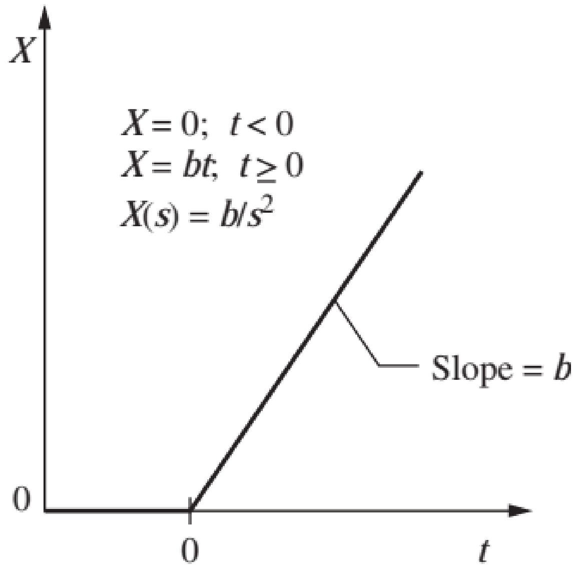
$$X(t) = A\delta(t)$$



RAMP FUNCTION

This function increases linearly with time and is described by the equations

$$X = 0 \qquad t < 0$$
$$X = bt \qquad t \geq 0$$

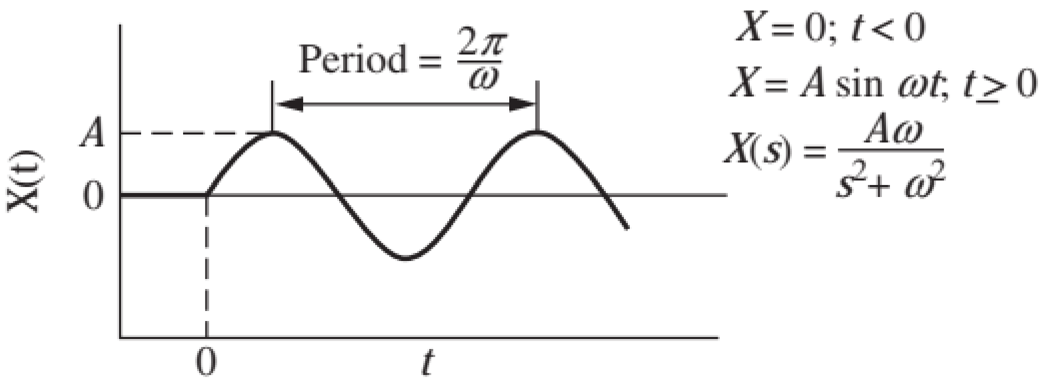


SINUSOIDAL INPUT

This function is represented mathematically by the equations

$$X = 0 \qquad t < 0$$

$$X = A\sin \omega t \qquad t \geq 0$$



where *A* is the amplitude and *w* is the radian frequency.

The radian frequency *w* is related to the frequency *f* in cycles per unit time by *w* = 2 *p f.*

This forcing function forms the basis of an important branch of control theory known as
frequency response.

Historically, a large segment of the development of control theory was based on frequency-response methods.

- If a step change of magnitude A is introduced into a first-order system, the transform of $X(t)$ is

$$X(s) = \frac{A}{s}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1}$$

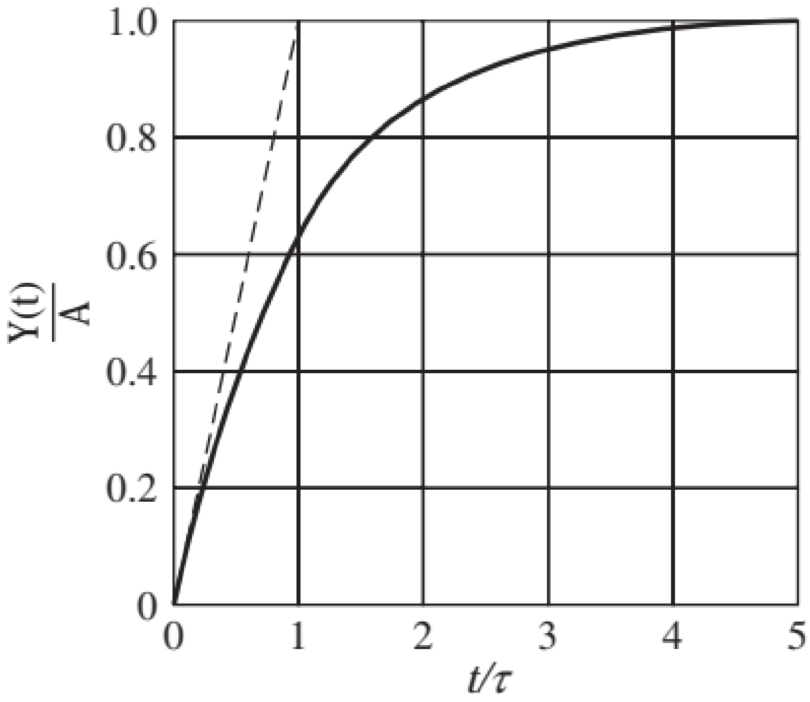
$$Y(s) = \frac{A}{s}\frac{1}{\tau s + 1}$$

$$Y(s) = \frac{A/\tau}{s(s + 1/\tau)} = \frac{C_1}{s} + \frac{C_2}{s + 1/\tau}$$

gives $C_1 = A$ and $C_2 = -A$.

$$Y(t) = 0 \qquad t < 0$$

$$Y(t) = A\left(1 - e^{-t/\tau}\right) \qquad t \geq 0$$



- The value of $Y(t)$ reaches 63.2 percent of its ultimate value when the time elapsed
  is equal to one time constant t.
- Time elapsed is $2t$, $3t$, and $4t$, the percent response is 86.5, 95, and 98, respectively.
- Response essentially completed in three to four time constants.

- A thermometer having a time constant of 0.1 min is at a steady-state temperature of 90°F (x_s) .
- At time t= 0, the thermometer is placed in a temperature bath maintained at 100°F. Determine the time needed for the thermometer to read 98°F.

$$\tau = 0.1 \text{ min} \qquad x_s = 90^\circ\text{F} \qquad A = 10^\circ\text{F}$$

- The ultimate thermometer reading will, of course, be 100°F, and the ultimate
  value of the deviation variable $Y(\infty)$ is 10°F. When the thermometer reads 98°F,

  $Y(t)$= 8°F.

$$8 = 10\left(1 - e^{-t/0.1}\right)$$

$$t = 0.161 \text{ min}$$

The appropriate values of *Y, A,* and *t* gives

where it is seen that Y/A = 0.8 at t/t = 1.6.

IMPULSE RESPONSE

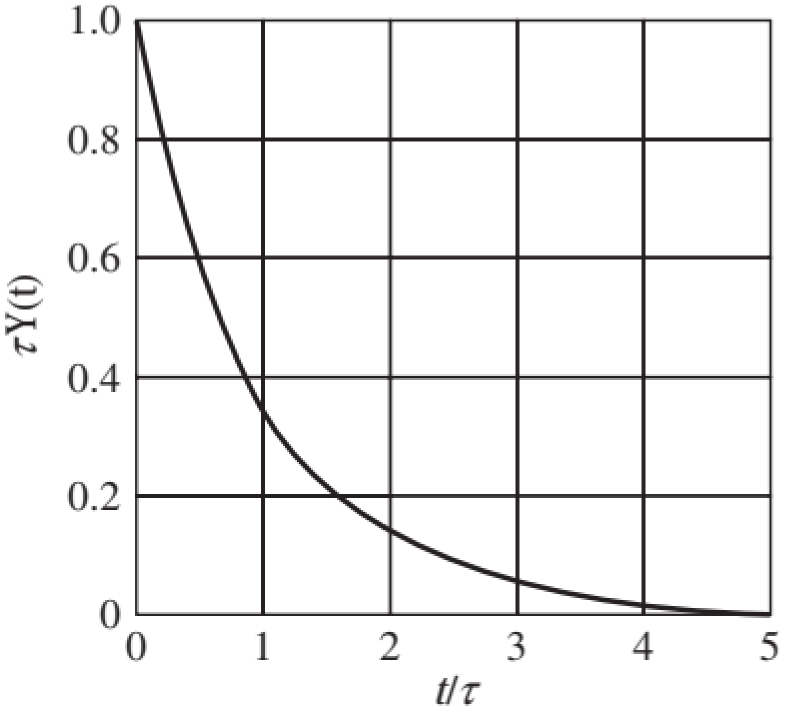
- The impulse response of a first-order system will now be developed. Anticipating the use of superposition, we consider a unit impulse for which the Laplace transform is

$$X(s) = 1$$

$$Y(s) = \frac{1}{\tau s + 1}$$

$$Y(s) = \frac{1/\tau}{s + 1/\tau}$$

$$\tau Y(t) = e^{-t/\tau}$$



RAMP RESPONSE

For a ramp input of $x(t) = bt$, where $X(s) = b / s^2$ the output is

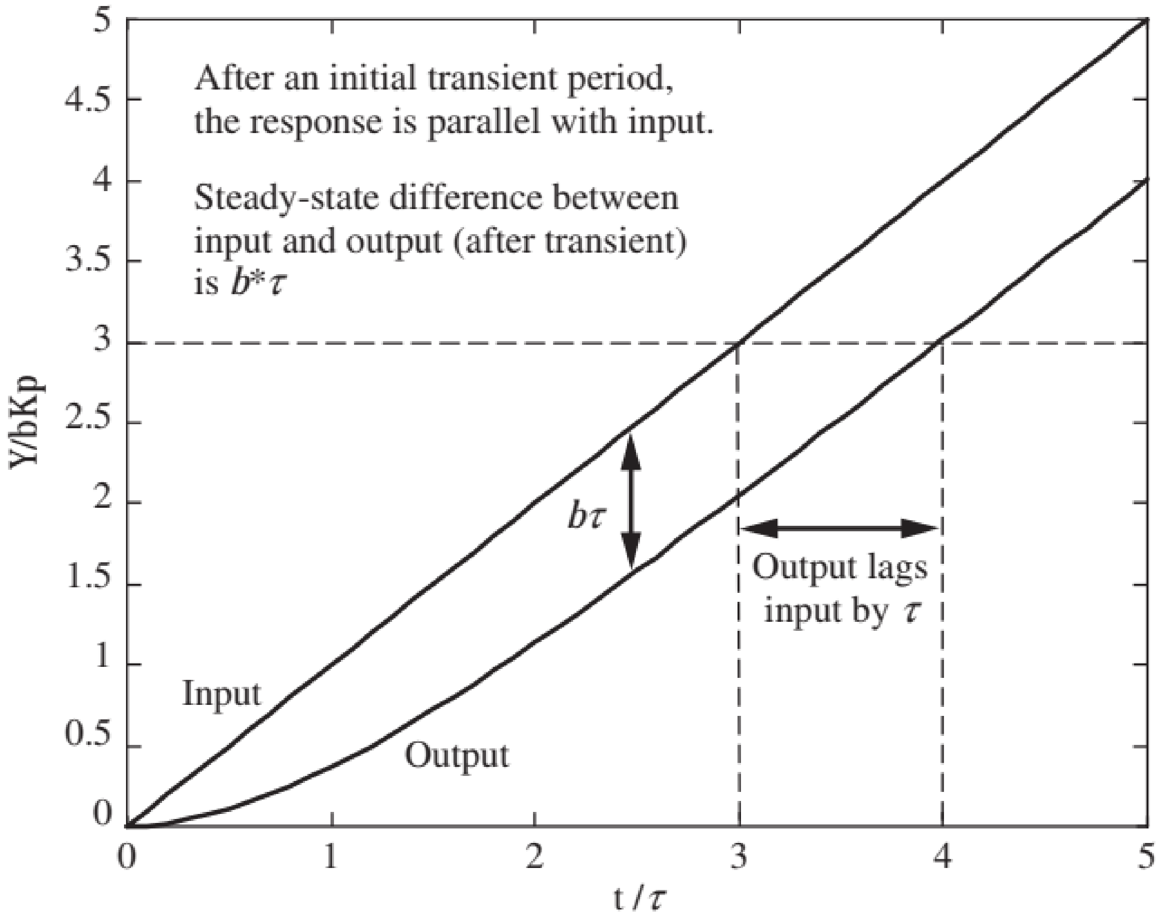
$$Y(s) = \frac{b}{s^2(\tau s + 1)}$$

Rearranging and using partial fractions yield.

$$Y(s) = \frac{b}{s^2(\tau s + 1)} = \frac{b/\tau}{s^2(s + 1/\tau)}$$

$$= \frac{b}{s^2} - \frac{b\tau}{s} + \frac{b\tau}{s^2 + 1/\tau}$$

$$Y(t) = bt - b\tau\left(1 - e^{-t/\tau}\right) = b(t - \tau) + b\tau e^{-t/\tau}$$



SINUSOIDAL RESPONSE

Consider a thermometer to be in equilibrium with a temperature bath at temperature x_s.

- At some time $t = 0$, the bath temperature begins to vary according to the relationship

$$x = x_s + A \sin \omega t \qquad t > 0$$

where x = temperature of bath

x_s = temperature of bath before sinusoidal disturbance is applied

A = amplitude of variation in temperature

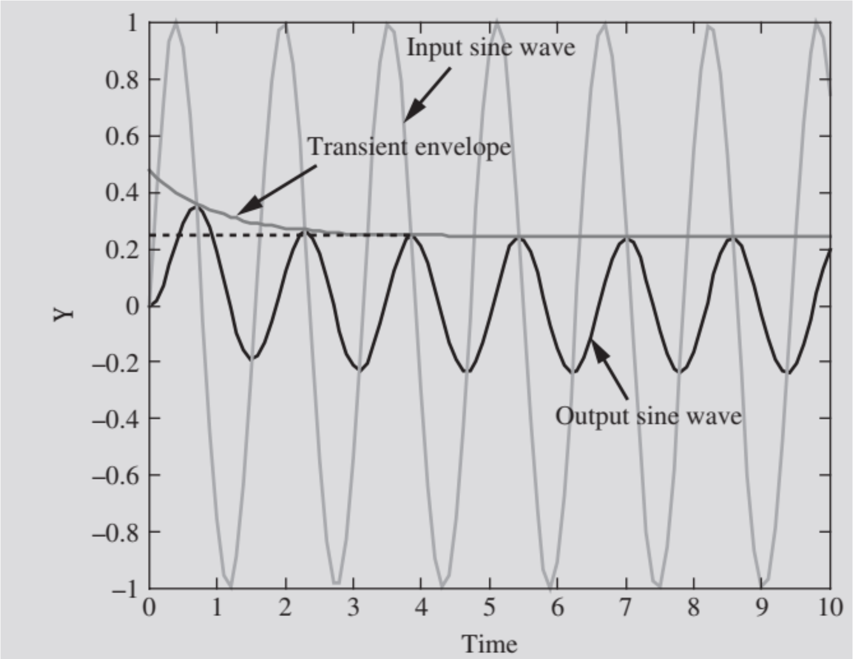
ω = radian frequency, rad/time

$$X = x - x_s$$

$$X = A \sin \omega t$$

$$X(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$Y(s) = \frac{A\omega}{s^2 + \omega^2} \frac{1/\tau}{s + 1/\tau}$$



This equation can be solved for $Y(t)$ by means of a partial fraction expansion, as described

$$Y(t) = \frac{A\omega\tau e^{-t/\tau}}{\tau^2\omega^2 + 1} - \frac{A\omega\tau}{\tau^2\omega^2 + 1}\cos\omega t + \frac{A}{\tau^2\omega^2 + 1}\sin\omega t$$

Another form by using the trigonometric identity

$$p\cos B + q\sin B = r\sin(B + \theta)$$

$$r = \sqrt{p^2 + q^2} \qquad \tan\theta = \frac{p}{q}$$

$$Y(t) = \frac{A\omega\tau}{\tau^2\omega^2 + 1}e^{-t/\tau} + \frac{A}{\sqrt{\tau^2\omega^2 + 1}}\sin(\omega t + \phi)$$

$$\phi = \tan^{-1}(-\omega\tau)$$

$$Y(t)\big|_s = \frac{A}{\sqrt{\tau^2\omega^2 + 1}}\sin(wt + \phi)$$

As $t \rightarrow \infty$, the first term on the right side of Eq. vanishes and leaves only the ultimate periodic solution, which is sometimes called the steady-state solution

- For the input forcing function with above Eq. for the ultimate periodic response, we see that

1. The output is a sine wave with a frequency ω equal to that of the input signal.
2. The ratio of output amplitude to input amplitude is $1/\sqrt{\tau^2\omega^2 + 1}$. This ratio is always smaller than 1. We often state this by saying that the signal is *attenuated.*
3. The output lags behind the input by an angle ϕ. It is clear that lag occurs, for the sign of ϕ is always negative.*

we always use the term *phase angle* (ϕ) and interpret whether there is lag or lead by the convention.

If ϕ in Eq. is negative, In terms of a recording of input and output, this means that the input peak occurs before the output peak.

If ϕ is positive in Eq., the system exhibits phase *lead,* or the output leads the input.

$\phi < 0$ phase lag

$\phi > 0$ phase lead

Phase lag can never exceed 90° and approaches this value asymptotically.

- The sinusoidal response is interpreted in terms of the mercury thermometer by the following example

A mercury thermometer having a time constant of 0.1 min is placed in a temperature bath at 100°F and allowed to come to equilibrium with the bath. At time $t = 0$, the temperature of the bath begins to vary sinusoidally about its average temperature of 100°F with an amplitude of 2°F. If the frequency of oscillation is $10/\pi$ cycles/min, plot the ultimate response of the thermometer reading as a function of time.

What is the phase lag?

In terms of the symbols used in this chapter

$$\tau = 0.1 \text{ min}$$

$$x_s = 100°\text{F}$$

$$A = 2°\text{F}$$

$$f = \frac{10}{\pi} \text{ cycles/min} \qquad \omega = 2\pi f = 2\pi \frac{10}{\pi} = 20 \text{ rad/min}$$

- The amplitude of the response and the phase angle are calculated; thus

$$\frac{A}{\sqrt{\tau^2\omega^2 + 1}} = \frac{2}{\sqrt{4 + 1}} = 0.896°\text{F}$$

$$\phi = -\tan^{-1} 2 = -63.5° = -1.11 \text{ rad}$$

$$\text{Phase lag} = 63.5°$$

The response of the thermometer is therefore

$$Y(t) = 0.896 \sin(20t - 1.11)$$

$$y(t) = 100 + 0.896 \sin(20t - 1.11)$$

To obtain the lag in terms of time rather than angle, we proceed as follows:
A frequency of $10/\pi$ cycles/min means that a complete cycle (peak to peak) occurs in ($10/\pi$) 1 min.
Since one cycle is equivalent to 360° and the lag is 63.5°, the time corresponding to this lag is

$$\text{Lag time} = \frac{63.5}{360} \times (\text{time for 1 cycle})$$

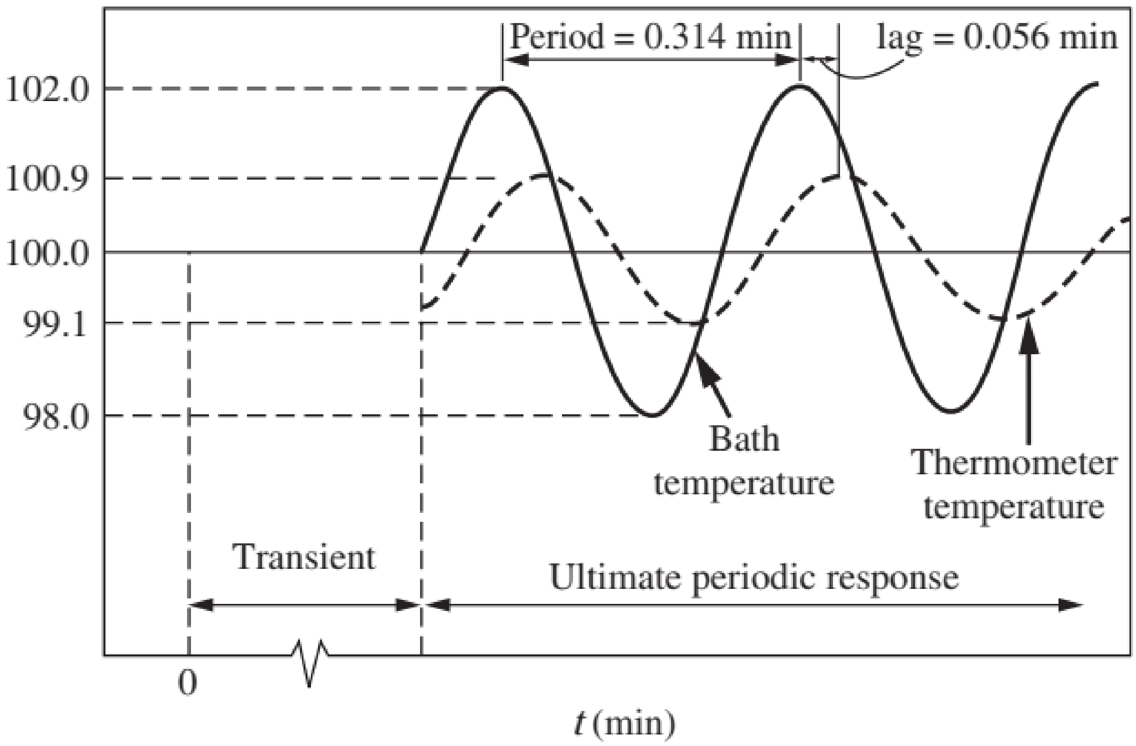
$$\text{Lag time} = \left(\frac{63.5}{360}\right)\left(\frac{\pi}{10}\right) = 0.0555 \text{ min}$$

$$y(t) = 100 + 0.896 \sin[20(t - 0.0555 \text{ min})]$$

In general, the lag in units of time is given by

$$\text{Lag time} = \frac{|\phi|}{360 f}$$

when *f* is expressed in degrees.



For all practical purposes this term becomes negligible after a time equal to about 3□.

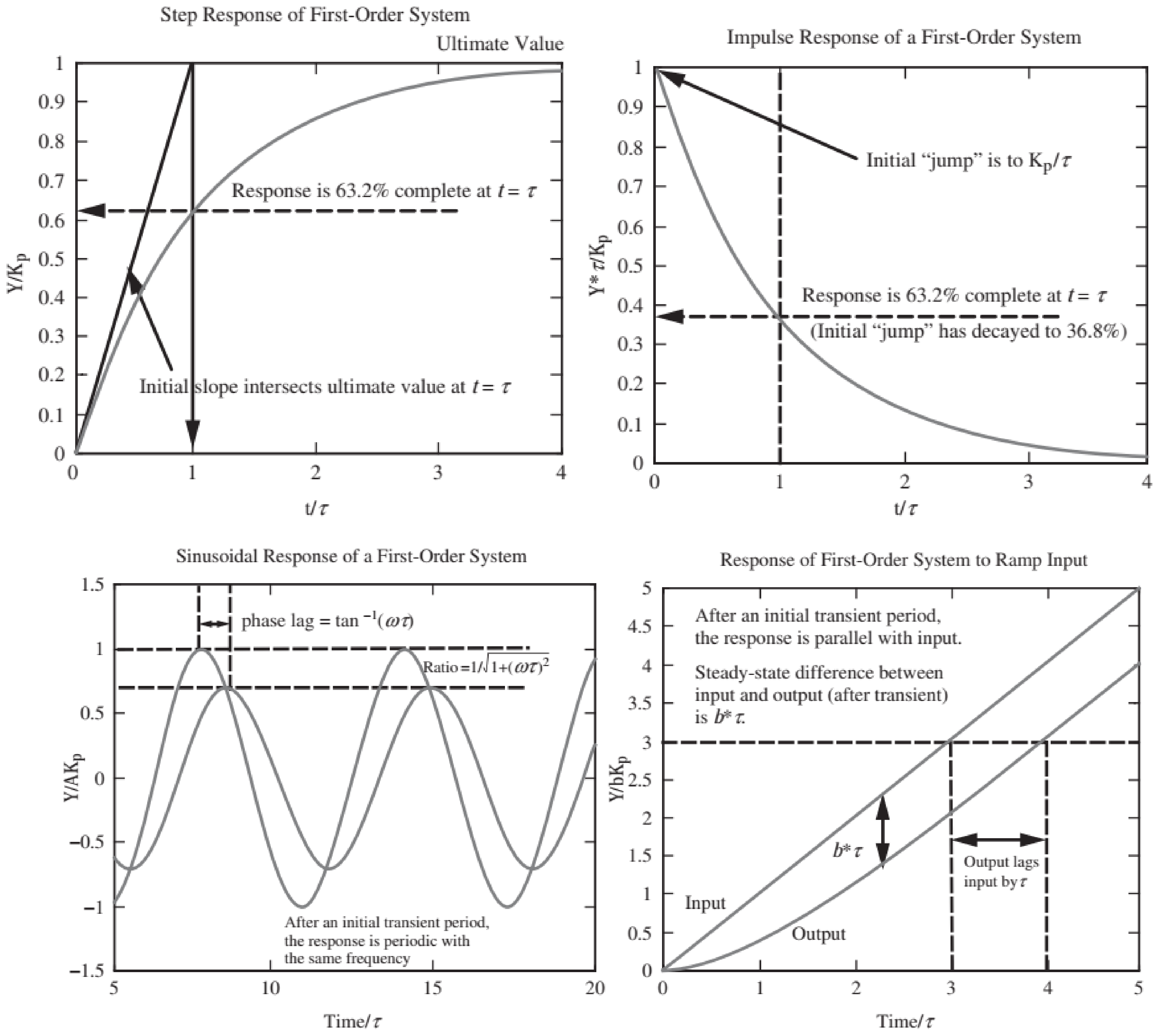
If the response were desired beginning from the time the bath temperature begins to oscillate, it would be necessary to plot the complete response as given by Eq.

Input			Output		
	$X(t)$	$X(s)$	$Y(s)$	$Y(t)$
Step	$u(t)$	$\frac{1}{s}$	$\frac{K_p}{s(\tau s + 1)}$	$Kp\,(1 - e^{-t/\tau})$
Impulse	$\delta\,(t)$	1	$\frac{K_p}{\tau s + 1}$	$\frac{K_p}{\tau} e^{-t/\tau}$
Ramp	$btu(t)$	$\frac{b}{s^2}$	$\frac{bK_p}{s^2(\tau s + 1)}$	$Kp\,[bt - b\tau\,(1 - e^{-t/\tau})]$
Sinusoid	$u(t)\,A\sin\,(\omega t)$	$\frac{A\omega}{s^2 + \omega^2}$	$\frac{A\omega K_p}{\left(s^2 + \omega^2\right)(\tau s + 1)}$	$\frac{AK_p\omega\tau}{1 + (\omega\tau)^2} e^{-t/\tau} + \frac{AK_p}{\sqrt{1 + (\omega\tau)^2}} \sin\left[\omega t + \tan^{-1}(-\omega\tau)\right]$

Deviation variables:

The difference between the process system variables and their steady-state values. When transfer functions are used, deviation values are always used. The convenience and utility of deviation variables lie in the fact that their initial values are most often zero.

$$X = x - x_s \qquad Y = y - y_s$$



Step Response of First-Order System
Ultimate Value
Response is 63.2% complete at t = τ
Initial slope intersects ultimate value at t = τ
Y/Kp
t/τ
Impulse Response of a First-Order System
Initial "jump" is to Kp/τ
Response is 63.2% complete at t = τ
(Initial "jump" has decayed to 36.8%)
Y*τ/Kp
t/τ
Sinusoidal Response of a First-Order System
phase lag = tan⁻¹(ωτ)
Ratio = 1/√1+(ωτ)²
After an initial transient period, the response is periodic with the same frequency
Y/AKp
Time/τ
Response of First-Order System to Ramp Input
After an initial transient period, the response is parallel with input.
Steady-state difference between input and output (after transient) is b*τ.
b*τ
Output lags input by τ
Input
Output
Y/bKp
Time/τ