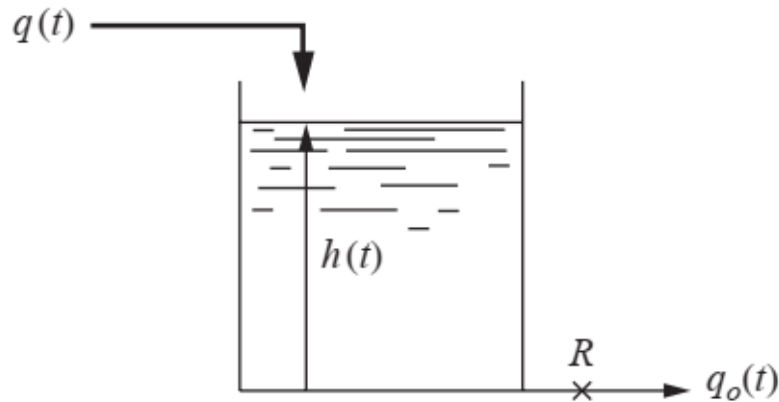


PHYSICAL EXAMPLES OF FIRST-ORDER SYSTEMS

CHAPTER 5

EXAMPLES OF FIRST-ORDER SYSTEMS

Liquid Level



Tank of uniform cross sectional area A
Flow **resistance** R such as a valve, a pipe, or a weir.

Assume that q_o , the volumetric flow rate (volume/time) related to the head h by the linear relationship

$$q_o = \frac{h}{R}$$

A specially contoured weir, called a Sutro weir, produces a linear head-flow relationship.

Turbulent flow through pipes and valves is generally proportional to h .

Flow through weirs having simple geometric shapes can be expressed as Kh^n , where K and n are positive constants.

For example, the flow through a rectangular weir is proportional to $h^{3/2}$.)

A time-varying volumetric flow q of liquid of constant density r enters the tank. Determine the transfer function that relates head to flow.

$$\left(\begin{array}{c} \text{Rate of} \\ \text{mass flow in} \end{array} \right) - \left(\begin{array}{c} \text{Rate of} \\ \text{mass flow out} \end{array} \right) = \left(\begin{array}{c} \text{Rate of accumulation} \\ \text{of mass in tank} \end{array} \right)$$

$$\rho q(t) - \rho q_o(t) = \frac{d(\rho Ah)}{dt}$$

$$q(t) - q_o(t) = A \frac{dh}{dt}$$

$$q - \frac{h}{R} = A \frac{dh}{dt}$$

Initially, the process is operating at steady state, which means that $dh/dt = 0$

$$q_s - \frac{h_s}{R} = 0$$

Subtracting Eq.

$$q - q_s = \frac{1}{R}(h - h_s) + A \frac{d(h - h_s)}{dt}$$

Define the deviation variables as

$$Q = q - q_s$$

$$H = h - h_s$$

$$Q = \frac{1}{R}H + A\frac{dH}{dt}$$

Taking the transform

$H(0)$ is zero, therefore the transform of dH/dt is simply sH

$$(s) \quad Q(s) = \frac{1}{R}H(s) + AsH(s)$$

Rearranged into the standard form of the first-order lag to give

$$\boxed{\frac{H(s)}{Q(s)} = \frac{R}{\tau s + 1}} \quad \text{where } \tau = AR.$$

Flow rate $Q(t)$ changes according to a unit-step change; thus

$$Q(t) = u(t)$$

$$Q(s) = \frac{1}{s}$$

$$H(s) = \frac{1}{s} \frac{R}{\tau s + 1}$$

Applying the final-value theorem, $H(s)$ gives

$$H(t)|_{t \rightarrow \infty} = \lim_{s \rightarrow 0} [sH(s)] = \lim_{s \rightarrow 0} \frac{R}{\tau s + 1} = R$$

$$q_{o_s} = \frac{h_s}{R}$$

deviation variable $Q_o = q_o - q_{o_s}$

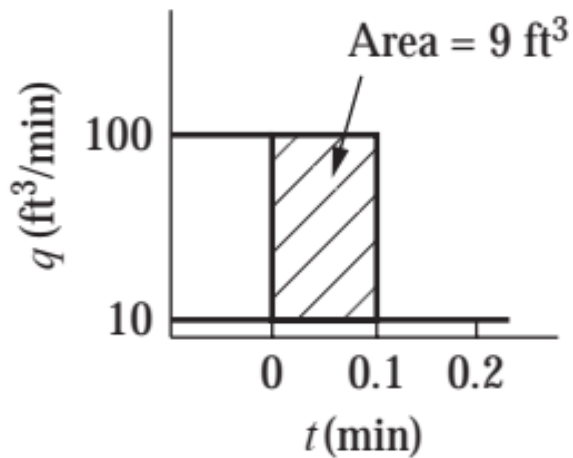
$$Q_o = \frac{H}{R}$$

$$Q_o(s) = \frac{H(s)}{R} \quad \text{and} \quad \frac{H(s)}{Q(s)} = \frac{R}{\tau s + 1}$$

combining

$$\boxed{\frac{Q_o(s)}{Q(s)} = \frac{1}{\tau s + 1}}$$

A tank having a time constant of 1 min and a resistance of $1/9$ ft/cfm is operating at steady state with an inlet flow of $10 \text{ ft}^3/\text{min}$ (or cfm). At time $t = 0$, the flow is suddenly increased to $100 \text{ ft}^3/\text{min}$ for 0.1 min by adding an additional 9 ft^3 of water to the tank uniformly over a period of 0.1 min. Plot the response in tank level and compare with the impulse response.



From the data given in this example, the transfer function of the process is

$$\frac{H(s)}{Q(s)} = \frac{1}{s + 1}$$

Input may be expressed as the difference in step functions

$$Q(t) = 90[u(t) - u(t - 0.1)]$$

$$Q(s) = \frac{90}{s}(1 - e^{-0.1s})$$

Combining this and the transfer function of the process, we obtain

$$H(s) = 10 \left[\frac{1}{s(s + 1)} - \frac{e^{-0.1s}}{s(s + 1)} \right]$$

- The first term in Eq. can be inverted as shown in Eq. to give $10(1 - e^{-t})$.
- The second term, which includes $e^{-0.1s}$, must be inverted by use of the theorem on translation of functions

the inverse of $e^{-st_0} f(s)$ is $f(t - t_0)u(t - t_0)$

$$u(t - t_0) = 0 \text{ for } t - t_0 < 0 \quad \text{or } t < t_0$$

$$L^{-1} \left\{ \frac{e^{-0.1s}}{s(s+1)} \right\} = 0 \quad \text{for } t < 0.1$$

$$= 10(1 - e^{-(t-0.1)}) \quad \text{for } t > 0.1$$

Or $10(1 - e^{-(t-0.1)})u(t - 0.1)$

The complete solution to this problem, which is the inverse of Eq.

$$H(t) = 10(1 - e^{-t})u(t) - 10(1 - e^{-(t-0.1)})u(t - 0.1)$$

which is equivalent to

$$H(t) = 10(1 - e^{-t}) \quad t < 0.1$$

$$H(t) = 10[(1 - e^{-t}) - (1 - e^{-(t-0.1)})] \quad t > 0.1$$

$$H(t) = 1.052e^{-t} \quad t > 0.1$$

$t < 0.1$

Assume Impulse input for first order system already derived expression

Inverse of $Y(s)$ can be found directly from the table of transforms and can be written in the form

$$\tau Y(t) = e^{-t/\tau}$$

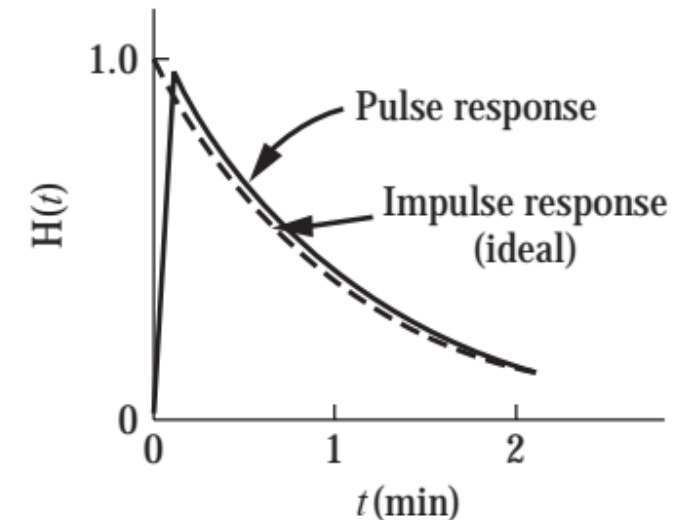
The response of the system to an impulse of magnitude 9 is given by

$$H(t)|_{\text{impulse}} = (9)\left(\frac{1}{9}e^{-t}\right) = e^{-t}$$

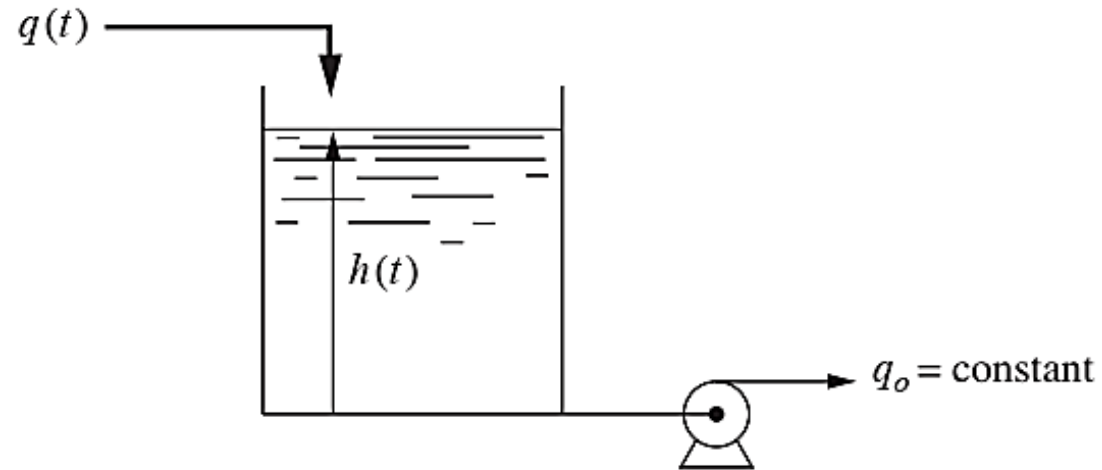
Approximation of an impulse function in a liquid-level system

The responses to step and sinusoidal forcing functions are the same for the liquid level system as for the mercury thermometer. Hence, they need not be rederived.

This is the advantage of characterizing all first-order systems by the same transfer function.



Liquid-Level Process with Constant-Flow Outlet



$q_o(t)$ is now a constant;
thus

$$q(t) - q_o = A \frac{dh}{dt}$$

At steady state $q_s - q_o = 0$

Introducing the deviation variables

$Q = q - q_s$ and $H = h - h_s$ give

$$Q = A \frac{dH}{dt}$$

$$\frac{H(s)}{Q(s)} = \frac{1}{As}$$

$$h(t) = h_s + \frac{1}{A} \int_0^t Q(t) dt$$

For instance, if a step change $Q(t) = u(t)$ were applied to the system

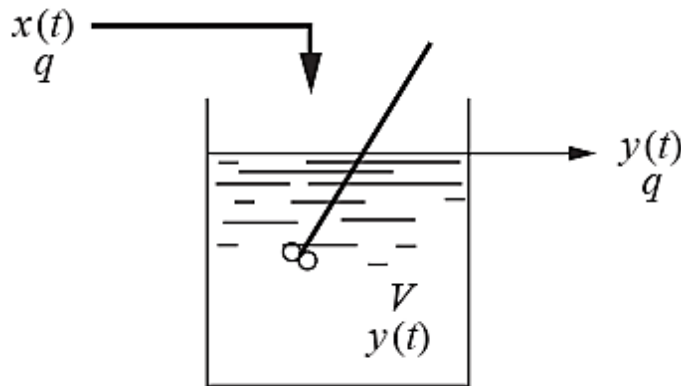
$$h(t) = h_s + t/A$$

- The step response given by Eq. is a ramp function that grows without limit.
- Such a system that grows without limit for a sustained change in input is said to have **nonregulation**.
- Systems that have a limited change in output for a sustained change in input are said to have **regulation**.
- If the inlet flow to the process is increased, the level will rise until the outlet flow becomes equal to the inlet flow, and then the level stops changing. This process is said to be self-regulating.(first example)

The transfer function for the liquid-level system with constant outlet flow given by Eq. can be considered as a special case of as $R \rightarrow \infty$.

$$\lim_{R \rightarrow \infty} \left(\frac{R}{ARs + 1} \right) = \frac{1}{As}$$

Mixing Process



Stream of solution containing dissolved salt flows at a constant volumetric flow rate q into a tank of constant holdup volume V .

The concentration of the **salt in the entering stream x (mass of salt/volume) varies with time.**

It is desired to determine the transfer function relating the outlet concentration y to the inlet concentration x

If we assume the density of the solution to be constant, the flow rate in must equal the flow rate out, since the holdup volume is fixed.

We may analyze this system by writing a transient mass balance for the salt; thus

$$\left(\begin{array}{c} \text{Flow rate of} \\ \text{salt in} \end{array} \right) - \left(\begin{array}{c} \text{Flow rate of} \\ \text{salt out} \end{array} \right) = \left(\begin{array}{c} \text{Rate of accumulation} \\ \text{of salt in tank} \end{array} \right)$$

Mass balance in terms of symbols gives

$$qx - qy = \frac{d(Vy)}{dt} = V \frac{dy}{dt}$$

Introducing the deviation variables

$$X = x - x_s$$

$$Y = y - y_s$$

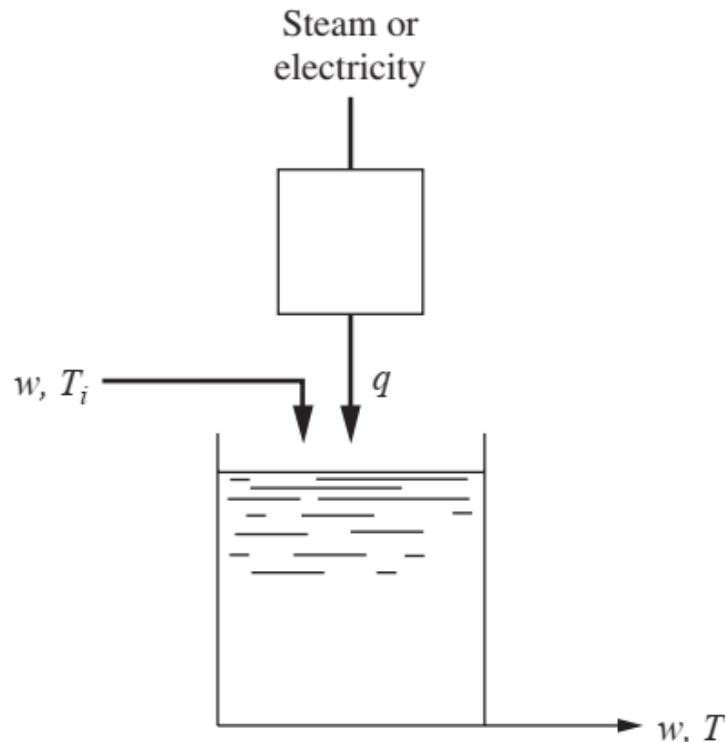
$$qX - qY = V \frac{dY}{dt}$$

Laplace transform of this expression and rearranging the result give

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1}$$

where $\tau = V/q$.

Heating Process



A stream at temperature T_i is fed to the tank.

Heat is added to the tank by means of an electric heater. The tank is well mixed, and the temperature of the exiting stream is T . The flow rate to the tank is constant at w lb/h.

A transient energy balance on the tank yields

$$\left(\begin{array}{c} \text{Rate of} \\ \text{energy flow} \\ \text{into tank} \end{array} \right) - \left(\begin{array}{c} \text{Rate of} \\ \text{energy flow} \\ \text{out of tank} \end{array} \right) + \left(\begin{array}{c} \text{Rate of} \\ \text{energy flow in} \\ \text{from heater} \end{array} \right) = \left(\begin{array}{c} \text{Rate of} \\ \text{accumulation of} \\ \text{energy in tank} \end{array} \right)$$

$$wC(T_i - T_{\text{ref}}) - wC(T - T_{\text{ref}}) + q = \rho VC \frac{d(T - T_{\text{ref}})}{dt} = \rho VC \frac{dT}{dt}$$

where T_{ref} is the reference temperature and C is the heat capacity of the fluid.

At steady state, dT/dt is zero,

$$wC(T_{is} - T_s) + q_s = 0$$

$$wC(T_i - T_{is}) - wC(T - T_s) + q - q_s = \rho VC \frac{d(T - T_s)}{dt}$$

If we assume that T_i is constant (and so $T_i = T_{is}$) and introduce the deviation variables

$$T' = T - T_s$$

$$Q = q - q_s$$

$$-wCT' + Q = \rho VC \frac{dT'}{dt}$$

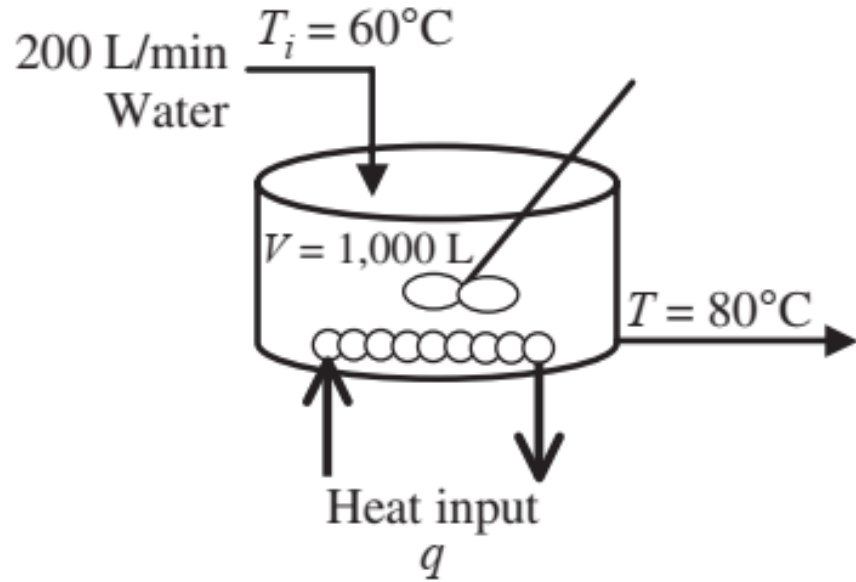
Taking Laplace transforms

$$-wCT'(s) + Q(s) = \rho VC s T'(s)$$

$$\frac{T'(s)}{Q(s)} = \frac{1/wC}{(\rho V/w)s + 1} = \frac{K}{\tau s + 1}$$

Thus, this process exhibits first-order dynamics as the tank temperature T responds to changes in the heat input to the tank.

Mixed tank heater



$$wC(T_i - T_s) - wC(T - T_s) = \rho VC \frac{d(T - T_s)}{dt}$$

In terms of deviation variables, this becomes

$$wCT_i' - wCT' = \rho VC \frac{dT'}{dt}$$

$$wCT_i'(s) - wCT'(s) = \rho VC s T'(s)$$

$$\frac{T'(s)}{T_i'(s)} = \frac{1}{(\rho V/w)s + 1} = \frac{1}{\tau s + 1}$$

Substituting in numerical values for the variables, we obtain the actual transfer function for this mixed tank heater.

$$\tau = \frac{\rho V}{w} = \frac{V}{w/\rho} = \frac{V}{v}$$

$$\frac{T'(s)}{T_i'(s)} = \frac{1}{5s + 1}$$

If the inlet temperature is stepped from 60 to 70°C, $T_i'(t) = 70 - 60 = 10$ and $T_i'(s) = 10/s$. Thus,

$$T'(s) = \frac{10}{s} \left(\frac{1}{5s + 1} \right)$$

Inverting to the time domain, we obtain the expression for $T'(t)$

$$T'(t) = 10(1 - e^{-t/5})$$

and finally, we obtain the expression for $T(t)$, the actual tank outlet temperature.

$$T(t) = T_s + T'(t) = 80 + 10(1 - e^{-t/5})$$

In summarizing the previous examples of first-order systems, the time constant for each has been expressed in terms of system parameters; thus

$$\tau = \frac{mC}{hA} \quad \text{for thermometer}$$

$$\tau = AR \quad \text{for liquid-level process}$$

$$\tau = \frac{V}{q} \quad \text{for mixing process}$$

$$\tau = \frac{\rho V}{w} \quad \text{for heating process}$$

LINEARIZATION

- Most physical systems of practical importance are nonlinear.
- Characterization of a dynamic system by a transfer function can be done only for linear systems (those described by linear differential equations).
- A very important technique for such approximation is illustrated by the following discussion of the liquid-level system.
- Flow out of the tank follows a square root relationship

$$q_o = Ch^{1/2}$$

where C is a constant

- For a liquid of constant density and a tank of uniform cross-sectional area A , a material balance around the tank gives

$$q(t) - q_o(t) = A \frac{dh}{dt}$$

nonlinear differential equation

$$q - Ch^{1/2} = A \frac{dh}{dt}$$

Method of Linearization

Taylor series expansion truncated to include only the linear terms.

Thus for a single variable function

$$\underline{f(x)} = f(x_s) + \left. \frac{df}{dx} \right|_{x_s} (x - x_s) + (\text{higher-order terms})$$

For functions of two variables, we have

$$f(x, y) = f(x_s, y_s) + \left. \frac{\partial f}{\partial x} \right|_{(x_s, y_s)} (x - x_s) + \left. \frac{\partial f}{\partial y} \right|_{(x_s, y_s)} (y - y_s) \\ + (\text{higher-order terms})$$

Consider the differential equation describing the dynamics of a system

$$\frac{dy}{dt} + \underbrace{f(y)}_{\substack{\text{nonlinear} \\ \text{term}}} = x(t)$$

Linearizing the nonlinear term gives

$$\frac{dy}{dt} + \underbrace{f(y_s) + \left. \frac{\partial f}{\partial y} \right|_{y_s} (y - y_s)}_{\text{linearized approximation}} = x(t)$$

Writing this equation again for the steady-state case gives

$$\frac{dy_s}{dt} + f(y_s) + \left. \frac{\partial f}{\partial y} \right|_{y_s} (y_s - y_s) = x_s$$

we can convert the original differential equation to deviation variables:

where $X = x - x_s$ and $Y = y - y_s$.

$$\frac{d(y - y_s)}{dt} + \left. \frac{\partial f}{\partial y} \right|_{y_s} (y - y_s) = x - x_s$$

$$\frac{dY}{dt} + \left. \frac{\partial f}{\partial y} \right|_{y_s} Y = X$$

- At this point, we cannot proceed as before and take the Laplace transform.
- This is due to the presence of the nonlinear term $h^{1/2}$, for which there is no simple transform.
- This difficulty can be circumvented by linearizing the nonlinear term.
- By means of a Taylor series expansion, the function $q_o (h)$ may be expanded around the steady-state value h_s ;

thus

$$q_o = q_o (h_s) + q'_o (h_s)(h - h_s) + \frac{q''_o (h_s)(h - h_s)^2}{2} + \dots$$

where $q'_o(h_s)$ is the first derivative of q_o evaluated at h_s , $q''_o(h_s)$ is the second derivative, etc. If we keep only the linear term, the result is

$$q_o \cong q_o (h_s) + q'_o (h_s)(h - h_s)$$

Taking the derivative of q_o with respect to h in Eq. and evaluating the derivative at $h = h_s$ give

$$q'_o(h_s) = \frac{1}{2} Ch_s^{-1/2}$$

$$q_o = q_{o_s} + \frac{1}{R_1}(h - h_s)$$

where $q_{o_s} = q_o(h_s)$ and $1/R_1 = \frac{1}{2}Ch_s^{-1/2}$

$$q - q_{o_s} - \frac{h - h_s}{R_1} = A \frac{dh}{dt}$$

$$q_s = q_{o_s}$$

$$A \frac{dh}{dt} + \frac{h - h_s}{R_1} = q - q_s$$

Introducing deviation variables

$$Q = q - q_s \text{ and } H = h - h_s$$

$$\frac{H(s)}{Q(s)} = \frac{R_1}{\tau s + 1}$$

$$R_1 = \frac{2h_s^{1/2}}{C} \quad \tau = R_1 A$$

