RESPONSE OF FIRST-ORDER SYSTEMS IN SERIES

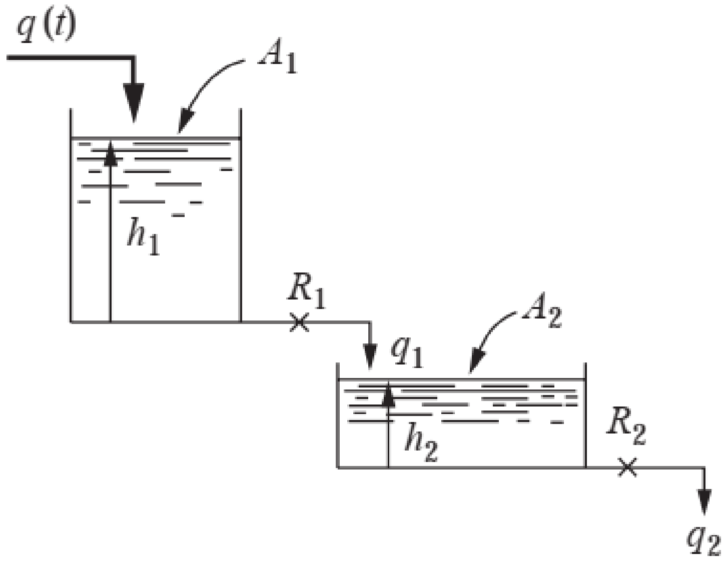
CHAPTER 6

Non-interacting system

Very often, a physical system can be represented by several first-order processes connected in series. Consider the liquid-level systems shown in Figure in which two tanks are arranged so that the outlet flow from the first tank is the inlet flow to the second tank.

The outlet flow from tank 1 discharges directly into the atmosphere before spilling into tank 2, and the flow through R_1 depends only on h_1.

The variation in h_2 in tank 2 does not affect the transient response occurring in tank 1. This type of system is referred to as a *noninteracting* system.



NONINTERACTING SYSTEM

Our problem is to find a transfer function that relates h_2 to q, that is, $H_2(s)/Q(s)$.

The approach will be to obtain a transfer function for each tank, $Q_1(s)/Q(s)$ and $H_2(s)/Q_1(s)$,

A balance on tank 1 gives

$$q - q_1 = A_1 \frac{dh_1}{dt}$$

A balance on tank 2 gives

$$q_1 - q_2 = A_2 \frac{dh_2}{dt}$$

The flow-head relationships for the two linear resistances are given by the expressions

$$\frac{Q_1(s)}{Q(s)} = \frac{1}{\tau_1 s + 1}$$

$$q_1 = \frac{h_1}{R_1} \qquad q_2 = \frac{h_2}{R_2}$$

where $Q_1 = q_1 - q_{1s}$, $Q = q - q_s$, and $\tau_1 = R_1 A_1$.

The transfer function for tank 2

$$\frac{H_2(s)}{Q_1(s)} = \frac{R_2}{\tau_2 s + 1}$$ where $H_2 = h_2 - h_{2s}$ and $\tau_2 = R_2 A_2$.

The overall transfer function $H_2(s)/Q(s)$ by multiplying

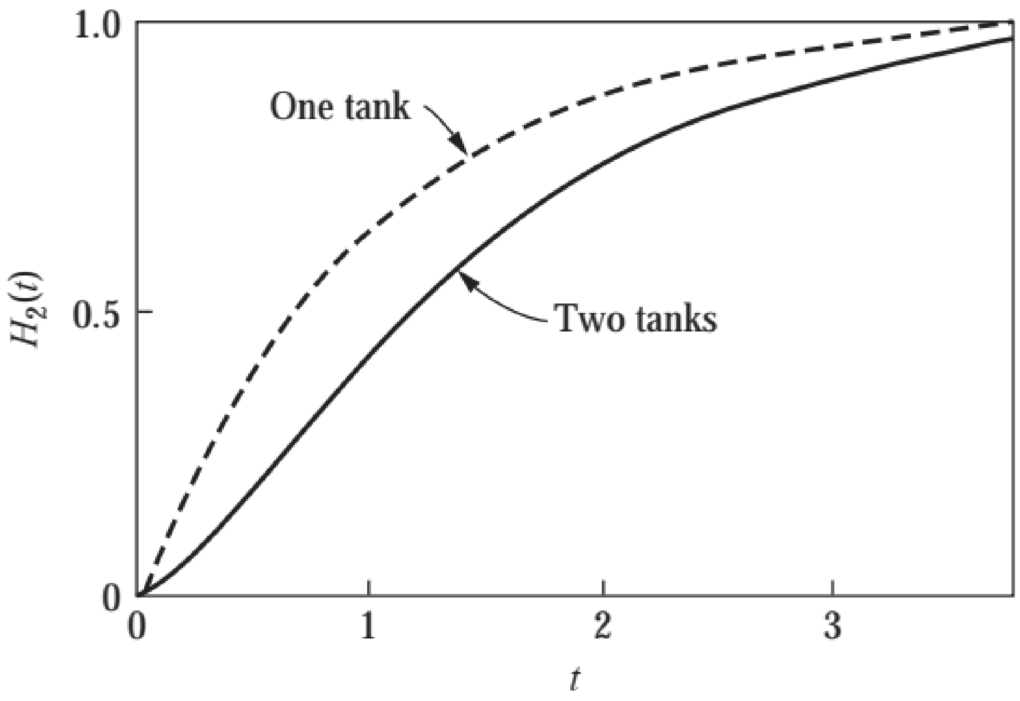
$$\frac{H_2(s)}{Q(s)} = \frac{1}{\tau_1 s + 1}\frac{R_2}{\tau_2 s + 1}$$

For a unit-step change in *Q*, we obtain

$$H_2(s) = \frac{1}{s}\frac{R_2}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

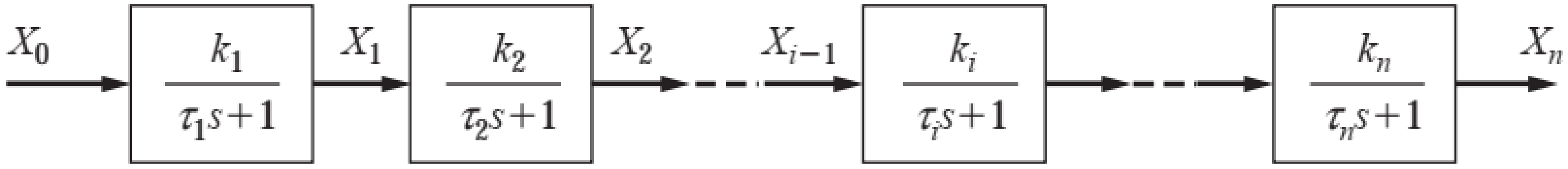
Inversion by means of partial fraction expansion gi

$$H_2(t) = R_2\left[1 - \frac{\tau_1\tau_2}{\tau_1 - \tau_2}\left(\frac{1}{\tau_2}e^{-t/\tau_1} - \frac{1}{\tau_1}e^{-t/\tau_2}\right)\right]$$



Generalization for Several Noninteracting Systems in Series

- The overall transfer function for two noninteracting first-order systems connected in series is simply the product of the individual transfer functions.
  We may now generalize this concept by considering *n* noninteracting first-order systems as represented by the block diagram



$$\frac{X_1(s)}{X_0(s)} = \frac{k_1}{\tau_1 s + 1}$$

$$\frac{X_2(s)}{X_1(s)} = \frac{k_2}{\tau_2 s + 1}$$

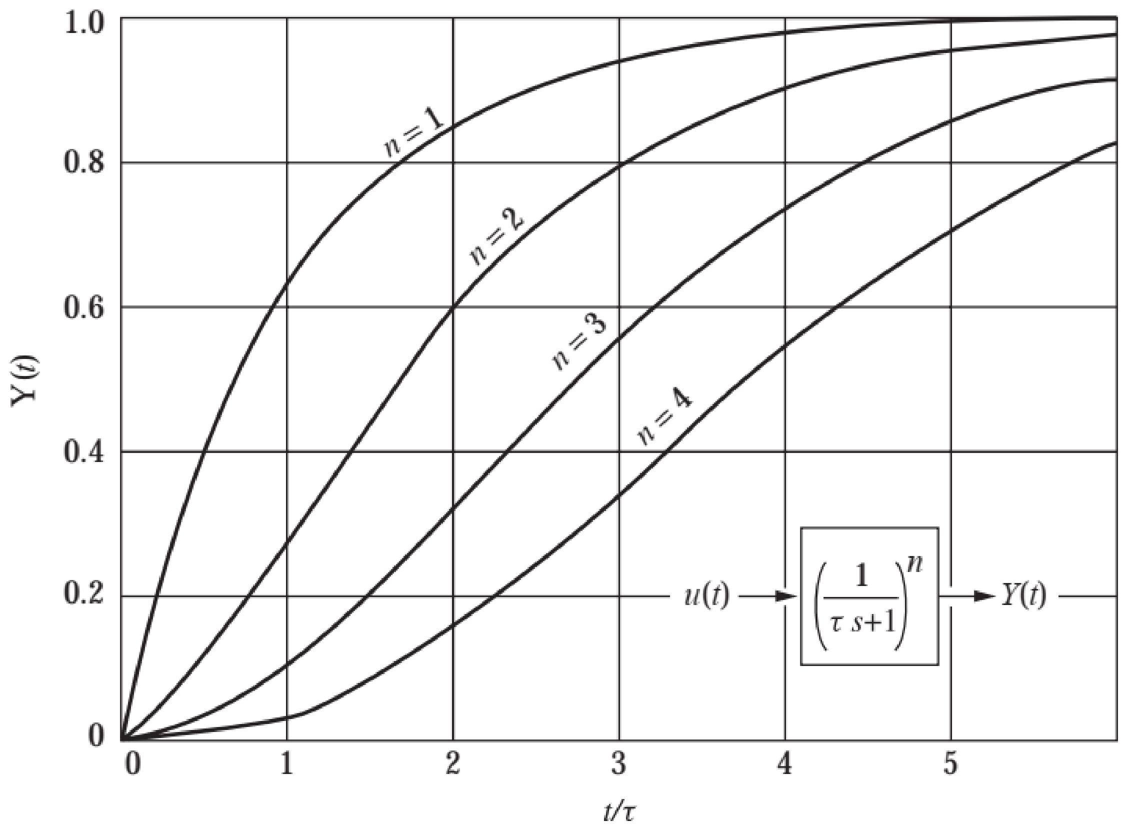
etc.

$$\frac{X_n(s)}{X_{n-1}(s)} = \frac{k_n}{\tau_n s + 1}$$

Overall transfer function, we simply multiply the individual transfer functions;
thus

$$\frac{X_n(s)}{X_0(s)} = \prod_{i=1}^{n} \frac{k_i}{\tau_i s + 1}$$

Step response of noninteracting first-order systems in series.



INTERACTING SYSTEM

Flow through R_1 now depends on the difference between h_1 and h_2.

Tank 1 $$q - q_1 = A_1 \frac{dh_1}{dt}$$

$$q_s - q_{1s} = 0$$

At steady state,

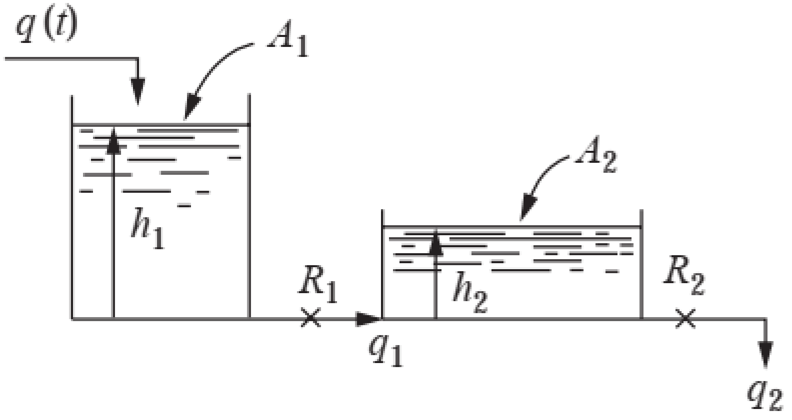
Tank 1 $$Q - Q_1 = A_1 \frac{dH_1}{dt}$$

Tank 2 $$q_1 - q_2 = A_2 \frac{dh_2}{dt}$$

At steady state,

$$q_{1s} - q_{2s} = 0$$

Tank 2 $$Q_1 - Q_2 = A_2 \frac{dH_2}{dt}$$



However, the flow-head relationship for R_1 is now

$$q_1 = \frac{1}{R_1}(h_1 - h_2)$$

The flow-head relationship for R_2 is the same as before

$$q_2 = \frac{h_2}{R_2}$$

In terms of deviation variables gives

Valve 1 $$Q_1 = \frac{H_1 - H_2}{R_1}$$

Valve 2 $$Q_2 = \frac{H_2}{R_2}$$

Transforming Eqs.

Tank 1 $$Q(s) - Q_1(s) = A_1 s H_1(s)$$

Tank 2 $$Q_1(s) - Q_2(s) = A_2 s H_2(s)$$

Valve 1 $$R_1 Q_1(s) = H_1(s) - H_2(s)$$

Valve 2 $$R_2 Q_2(s) = H_2(s)$$

The analysis has produced four algebraic equations containing five unknowns:
Q, Q_1 , Q_2 , H_1, and H_2.

These equations may be combined to eliminate Q_1 , Q_2, and H_1 and to arrive at the desired transfer function:

$$\frac{H_2(s)}{Q(s)} = \frac{R_2}{\tau_1\tau_2 s^2 + (\tau_1 + \tau_2 + A_1R_2)s + 1}$$

The difference between the transfer function for the noninteracting system, and that for the interacting system, is the presence of the cross-product term A_1R_2 in the coefficient of *s.*

The term *interacting* is often referred to as *loading.* The second tank of is said to *load* the first tank.

The transient response of a system, consider a two-tank system for which the time constants are equal ($t_1 = t_2 = t$)

If the tanks are noninteracting, the transfer function relating inlet flow to outlet flow is

$$\frac{Q_2(s)}{Q(s)} = \left(\frac{1}{\tau s + 1}\right)^2$$

The unit-step response for this transfer function can be obtained by the usual procedure to give

$$Q_2(t) = 1 - e^{-t/\tau} - \frac{t}{\tau} e^{-t/\tau}$$

If the tanks are interacting, the overall transfer function, Aassuming further that $A_1 = A_2$)

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{\tau^2 s^2 + 3\tau s + 1}$$

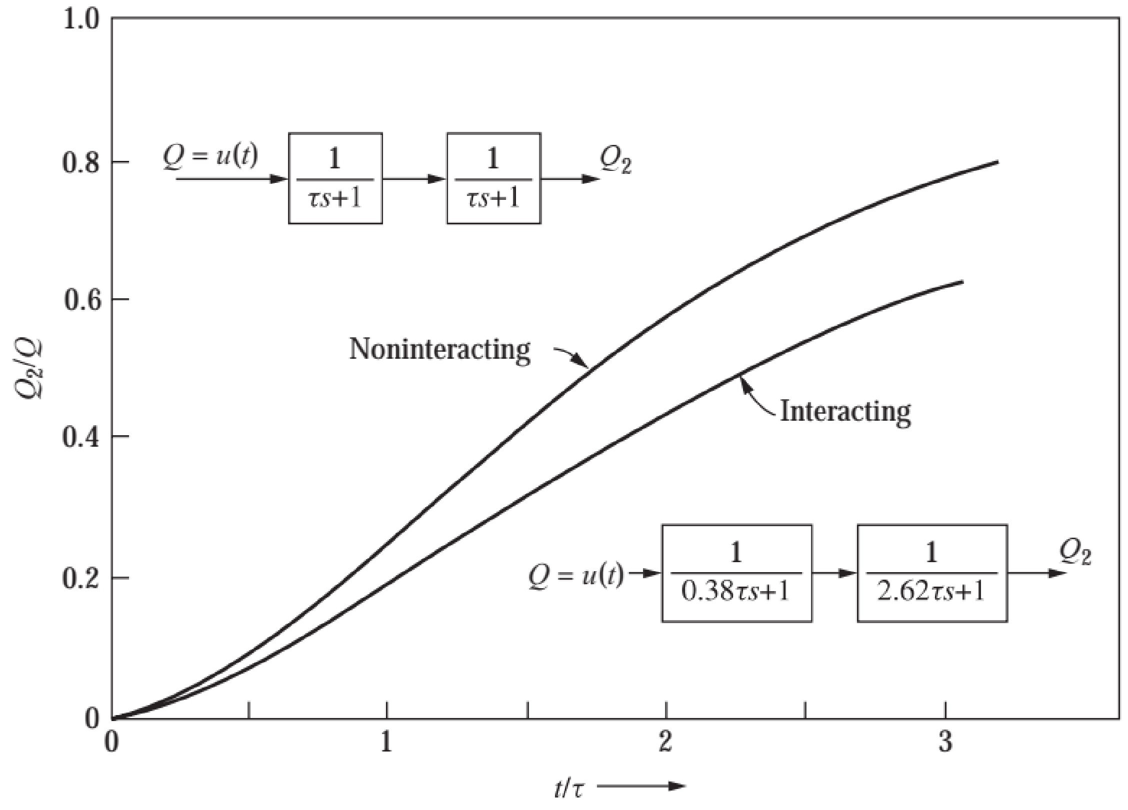
By application of the quadratic formula, the denominator of this transfer function can be written as

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{(0.38\tau s + 1)(2.62\tau s + 1)}$$

The effect of interaction has been to change the effective time constants of the interacting system.
One time constant has become considerably larger and the other smaller than the time constant *t* of either tank in the noninteracting system.

The response of $Q_2(t)$ to a unit-step change in $Q(t)$ for the interacting case

$$Q_2(t) = 1 + 0.17e^{-t/0.38\tau} - 1.17e^{-t/2.62\tau}$$



In terms of the transient response, this means that the interacting system is more sluggish than the noninteracting system.