HIGHER-ORDER SYSTEMS: SECOND-ORDER AND TRANSPORTATION LAG

CHAPTER 7

SECOND-ORDER SYSTEM

Transfer Function

A second-order system can arise from two first-order systems in series.

Some systems are inherently second-order, and they do not result from a series combination of two first-order systems.

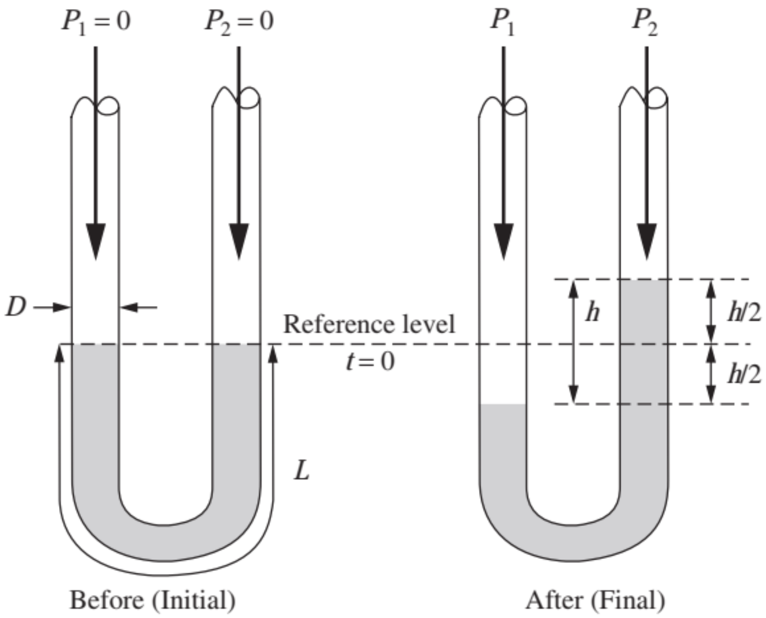
Inherently second-order systems are not extremely common in chemical engineering applications.

Most second-order systems that we encounter will result from the addition of a controller to a first-order process.

Second-order system or a *quadratic lag*

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = x(t)$$

Consider a simple manometer the pressure on both legs of the manometer is initially the same.



The length of the fluid column in the manometer is L.

At time $t = 0$, a pressure difference is imposed across the legs of the manometer.

Assuming the resulting flow in the manometer to be laminar and the steady-state friction law for drag force in laminar flow to apply at each instant, we will determine the transfer function between the applied pressure difference ΔP and the manometer reading h.

Perform a momentum balance on the fluid in the manometer:

(Sum of forces causing fluid to move) = (Rate of change of momentum of fluid)

$$\begin{pmatrix}\text{Sum of forces}\\ \text{causing fluid to move}\end{pmatrix} = \begin{pmatrix}\text{Unbalanced pressure forces}\\ \text{causing motion}\end{pmatrix} - \begin{pmatrix}\text{Frictional forces}\\ \text{opposing motion}\end{pmatrix}$$

$$\begin{pmatrix}\text{Unbalanced pressure forces}\\ \text{causing motion}\end{pmatrix} = (P_1 - P_2)\frac{\pi D^2}{4} - \rho g h \frac{\pi D^2}{4}$$

$$\begin{pmatrix}\text{Frictional forces}\\ \text{opposing motion}\end{pmatrix} = \begin{pmatrix}\text{Skin friction}\\ \text{at wall}\end{pmatrix} = \begin{pmatrix}\text{Shear stress}\\ \text{at wall}\end{pmatrix} \times \begin{pmatrix}\text{Area in contact}\\ \text{with wall}\end{pmatrix}$$

$$\begin{pmatrix}\text{Frictional forces}\\ \text{opposing motion}\end{pmatrix} = \tau_{\text{Wall}}(\pi D L) = \frac{8\mu \bar{V}}{D}(\pi D L) = \left(\frac{8\mu}{D}\right)\left(\frac{1}{2}\frac{dh}{dt}\right)(\pi D L)$$

The term for the skin friction at the wall is obtained from the Hagen-Poiseuille relationship for laminar flow .

Note that V is the average velocity of the fluid in the tube, which is also the velocity of the interface, which is equal to $\frac{1}{2}\, dh/dt$

$$(\text{Rate of change of momentum}) = \frac{d}{dt}(\text{mass} \times \text{velocity} \times \text{momentum correction factor})$$

$$= \left(\rho \frac{\pi D^2}{4} L\right)(\beta)\frac{d\bar{V}}{dt}$$

$$= \left(\rho \frac{\pi D^2}{4} L\right)(\beta)\left(\frac{1}{2}\frac{d^2 h}{dt^2}\right)$$

The momentum correction factor β accounts for the fact that the fluid has a parabolic velocity profile in the tube, and the momentum must be expressed as βmV for laminar flow.

The value of β for laminar flow is 4/3.

Substituting the appropriate terms produces the desired force balance equation for the manometer.

$$\left(\rho \frac{\pi D^2}{4} L\right)\left(\frac{4}{3}\right)\left(\frac{1}{2}\frac{d^2 h}{dt^2}\right) = (P_1 - P_2)\frac{\pi D^2}{4} - \rho g h \frac{\pi D^2}{4} - \left(\frac{8\mu}{D}\right)\left(\frac{1}{2}\frac{dh}{dt}\right)(\pi D L$$

$$\left(\rho \frac{\pi D^2}{4} L\right)\left(\frac{4}{3}\right)\left(\frac{1}{2}\frac{d^2 h}{dt^2}\right) + \left(\frac{8\mu}{D}\right)\left(\frac{1}{2}\frac{dh}{dt}\right)(\pi D L) + \rho g h \frac{\pi D^2}{4} = (P_1 - P_2)\frac{\pi D^2}{4}$$

- Dividing both sides by ρg ($\pi D^2/4$), we arrive at the standard form for a second-order system.

$$\frac{2L}{3g}\frac{d^2 h}{dt^2} + \frac{16\mu L}{\rho D^2 g}\frac{dh}{dt} + h = \frac{P_1 - P_2}{\rho g} = \frac{\Delta P}{\rho g}$$

$$\tau^2 \frac{d^2 Y}{dt^2} + 2\zeta\tau \frac{dY}{dt} + Y = X(t)$$

$$\tau^2 = \frac{2L}{3g} \qquad \tau = \sqrt{\frac{2L}{3g}}$$

$$2\zeta\tau = \frac{16\mu L}{\rho D^2 g} \qquad \zeta = \frac{8\mu}{\rho D^2}\sqrt{\frac{3L}{2g}} \quad \text{dimensionless}$$

$$X(t) = \frac{\Delta P}{\rho g} \quad \text{and} \quad Y = h$$

If the fluid column is motionless (dY/dt =0) and located at its rest position (Y= 0) before the forcing function is applied, the Laplace transform

$$\tau^2 s^2 Y(s) + 2\zeta\tau s Y(s) + Y(s) = X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

All such systems are defined as second-order.

Note that it requires two parameters, τ and ζ, to characterize the dynamics of a second-order system in contrast to only one parameter for a first-order system.

Response of a second-order system to some of the common forcing functions, namely, step, impulse, and sinusoidal.

Step Response

If the forcing function is a unit-step function, $X(s) = \frac{1}{s}$

In terms of the manometer shown in, this is equivalent to suddenly applying a pressure difference [such that $X(t) = \Delta P / \rho g = 1$] across the legs of the manometer at time $t = 0$.
Superposition will enable us to determine easily the response to a step function of any other magnitude.

$$Y(s) = \frac{1}{s} \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

The quadratic term in this equation may be factored into two linear terms that contain the roots

$$s_a = -\frac{\zeta}{\tau} + \frac{\sqrt{\zeta^2 - 1}}{\tau} \qquad s_b = -\frac{\zeta}{\tau} \mp \frac{\sqrt{\zeta^2 - 1}}{\tau}$$

$$Y(s) = \frac{1/\tau^2}{s(s - s_a)(s - s_b)}$$

Step response of a second-order system

Case	ζ	Nature of roots	Description of response
I	< 1	Complex	Underdamped or oscillatory
II	$= 1$	Real and equal	Critically damped
III	> 1	Real	Overdamped or nonoscillatory

CASE I STEP RESPONSE FOR $\zeta < 1$.

For this case, the inversion of Eq. yields the result

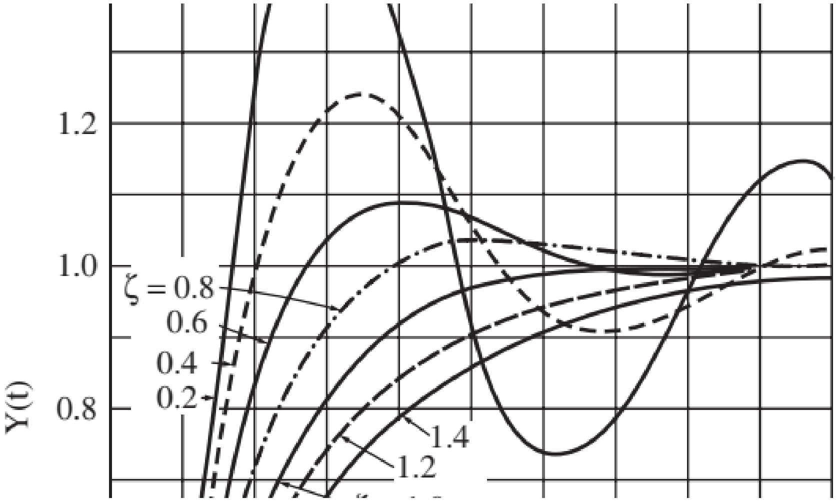
$$Y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left(\sqrt{1-\zeta^2}\,\frac{t}{\tau} + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

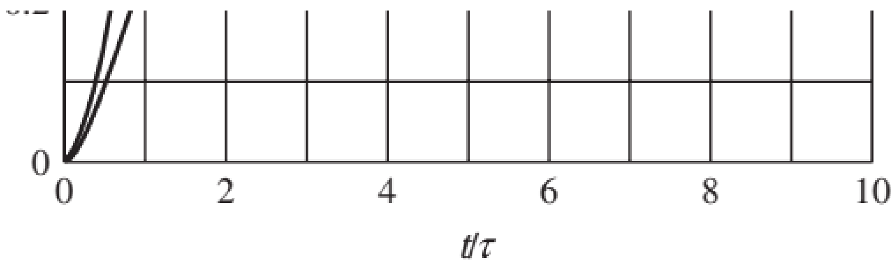
Since $\zeta < 1$, indicate a pair of complex conjugate roots in the left half-plane and a root at the origin.

$$Y(t) = C_1 + e^{-\zeta t/\tau}\left(C_2 \cos\sqrt{1-\zeta^2}\,\frac{t}{\tau} + C_3 \sin\sqrt{1-\zeta^2}\,\frac{t}{\tau} \right)$$

The constants C_1, C_2, and C_3 are found by partial fractions.

- $Y(t)$ is plotted against the dimensionless variable t/τ for several values of ζ.
- Note that for $\zeta < 1$ all the response curves are oscillatory in nature and become less oscillatory as ζ is increased.
- The slope at the origin is zero for all values of ζ.
- The response of a second-order system for $\zeta < 1$ is said to be *underdamped.*
- If we step-change the pressure difference across an underdamped manometer,
- The liquid levels in the two legs will oscillate before stabilizing.
- The oscillations are characteristic of an underdamped response.





CASE II STEP RESPONSE FOR $\zeta = 1$. For this case, the response is given by the expression

$$Y(t) = 1 - \left(1 + \frac{t}{\tau}\right)e^{-t/\tau}$$

Show that the roots s_1 and s_2 are real and equal.

The response, which is plotted in Fig. is nonoscillatory.
This condition, ζ=1, is called *critical damping* and allows the most rapid approach of the response to Y= 1 without oscillation.

CASE III STEP RESPONSE FOR $\zeta > 1$

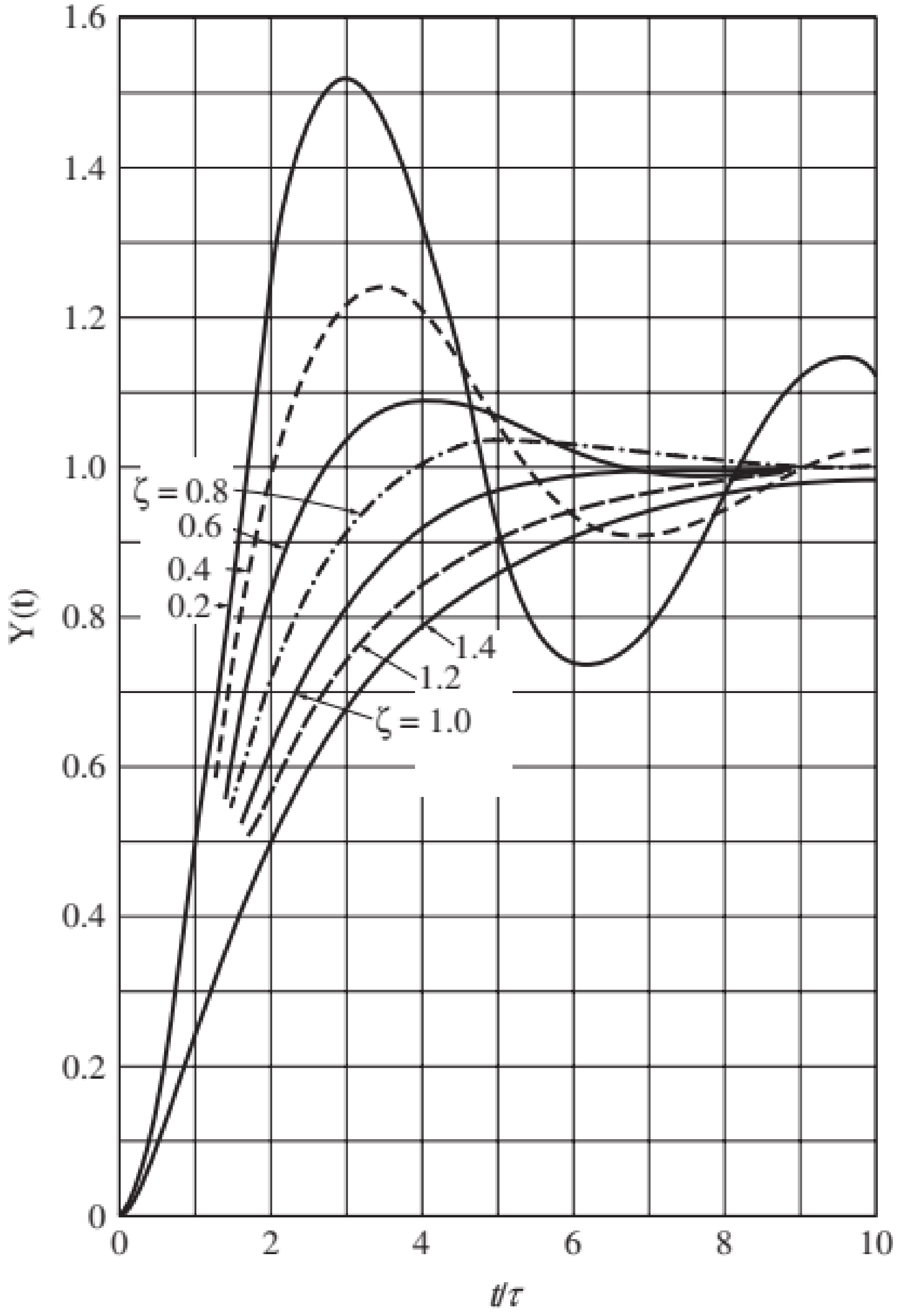
For this case, the inversion of Eq. gives the result

$$Y(t) = 1 - e^{-\zeta t/\tau}\left(\cosh\sqrt{\zeta^2 - 1}\frac{t}{\tau} + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\sinh\sqrt{\zeta^2 - 1}\frac{t}{\tau}\right)$$

where the hyperbolic functions are defined as

$$\sinh a = \frac{e^a - e^{-a}}{2} \qquad \cosh a = \frac{e^a + e^{-a}}{2}$$

- The response has been plotted for several values of ζ.
- Notice that the response is nonoscillatory and becomes more "sluggish" as ζ increases.
- This is known as an *overdamped* response.
- As in previous cases, all curves eventually approach the line $Y = 1$.



Actually, the response for $\zeta > 1$

$$\frac{Y(s)}{X(s)} = \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

This is true for $\zeta > 1$ because the roots s_1 and s_2 are real,

may be factored into two real linear factors

$$\tau_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\tau$$

$$\tau_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\tau$$

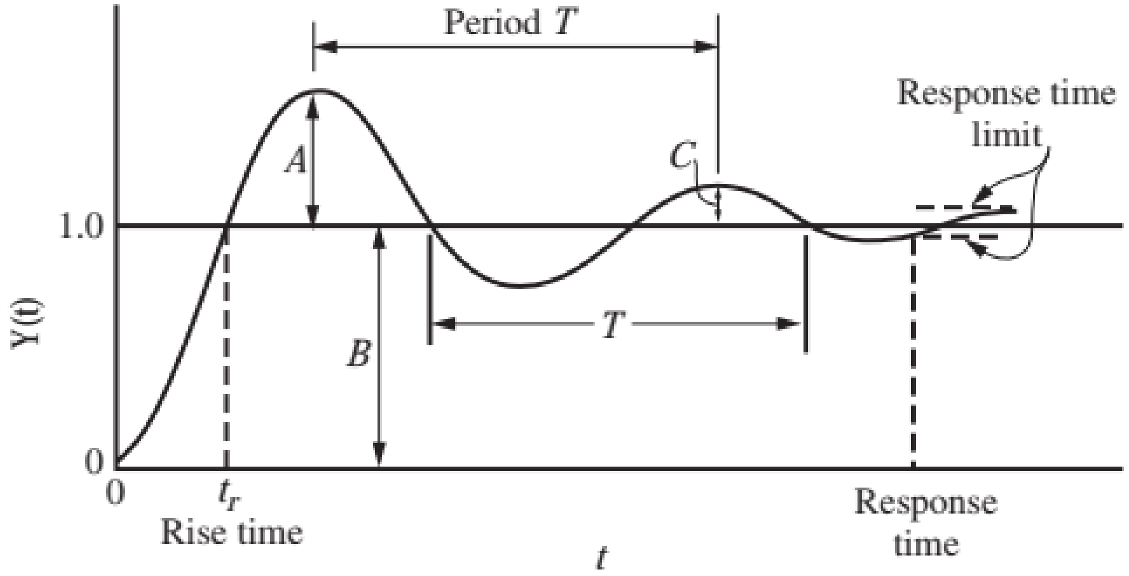
Note that if τ_1 τ_2, then $\tau = \tau_1 = \tau_2$ and $\zeta = 1$.

Terms Used to Describe an Underdamped System

Overshoot

Overshoot is a measure of how much the response exceeds the ultimate value

$$\text{Overshoot} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$



Following a step change and is expressed as the ratio *A/B* in Figure

The overshoot for a unit step is related to ζ by the expression

Why are we concerned about overshoot?

Temperature in our chemical reactor cannot be allowed to exceed a specified temperature to protect the catalyst from deactivation, or if it's a level control system,

These physical limitations, we can determine allowable values of ζ and choose our control system parameters to be sure to stay within those limits.

Decay ratio

The decay ratio is defined as the ratio of the sizes of successive peaks and is given by *C/A* in Figure.
The decay ratio is related to ζ by the expression

$$\text{Decay ratio} = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right) = (\text{overshoot})^2$$

Notice that larger ζ means greater damping, hence greater decay.

Rise time This is the time required for the response to first reach its ultimate value and is labeled t_r in Figure.

The reader can verify from Figure that t_r increases with increasing ζ.

Response time This is the time required for the response to come within 5 percent of its ultimate value and remain there.

The response time is indicated in Figure. The limits 5 percent are arbitrary, and other limits can be used for defining a response time.

Period of oscillation The radian frequency (radians/time) is the coefficient of t in the sine term; thus,

$$\text{radian frequency } \omega = \frac{\sqrt{1-\zeta^2}}{\tau}$$

Since the radian frequency ω is related to the cyclical frequency f by $\omega = 2\pi f$, it follows that

$$f = \frac{1}{T} = \frac{1}{2\pi}\frac{\sqrt{1-\zeta^2}}{\tau}$$

where T is the period of oscillation (time/cycle), T is the time elapsed between peaks. It is also the time elapsed between alternate crossings of the line $Y = 1$.

Natural period of oscillation

If the damping is eliminated [$B = 0$, $\zeta = 0$], the system oscillates continuously without attenuation in amplitude.
Under these "natural" or undamped conditions, the radian frequency is $1/\tau$, as shown by when $\zeta = 0$.
This frequency is referred to as the *natural frequency* ω_n

$$\omega_n = \frac{1}{\tau}$$

The corresponding natural cyclical frequency f_n and period T_n are related by the expression

$$f_n = \frac{1}{T_n} = \frac{1}{2\pi\tau}$$

Thus, τ has the significance of the undamped period.

Natural frequency is related to the actual frequency by the expression

$$\frac{f}{f_n} = \sqrt{1 - \zeta^2}$$

Notice that for $\zeta < 0.5$ the natural frequency is nearly the same as the actual frequency.

In summary, it is evident that ζ is a measure of the degree of damping, or the oscillatory character, and τ is a measure of the period, or speed, of the response of a second-order system

Impulse Response

If a unit impulse $\delta(t)$ is applied to the second-order system, then from Eqs. and the transform of the response is

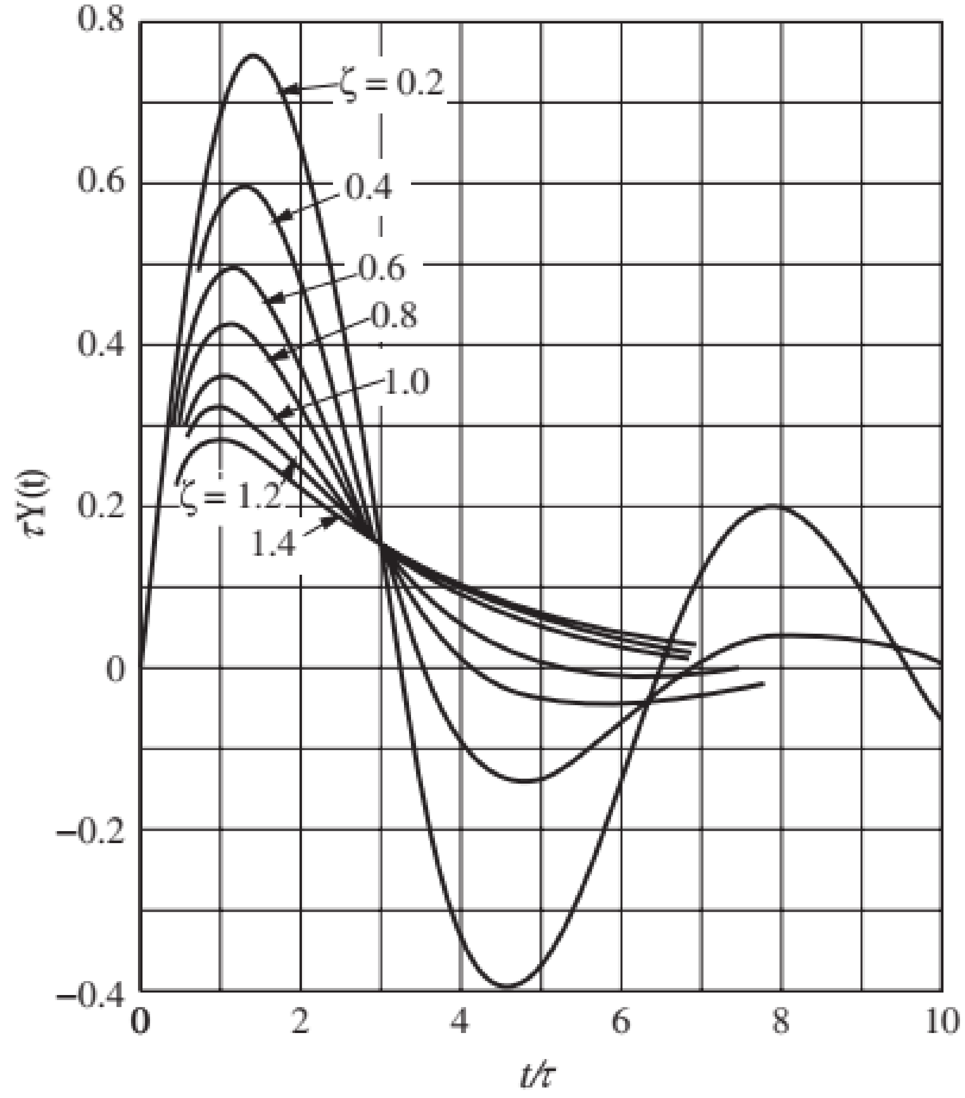
$$Y(s) = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

As in the case of the step input, the nature of the response to a unit impulse will depend on whether the roots of the denominator of Equation are real or complex.

The problem is again divided into the three cases shown in Table , and each is discussed below.

CASE I IMPULSE RESPONSE FOR $\zeta < 1$

$$Y(t) = \frac{1}{\tau}\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\sqrt{1-\zeta^2}\,\frac{t}{\tau}$$



$$Y(s)\big|_{\text{impulse}} = sY(s)\big|_{\text{step}}$$

$$Y(t)\big|_{\text{impulse}} = \frac{d}{dt}\left(Y(t)\big|_{\text{step}}\right)$$

CASE II IMPULSE RESPONSE FOR FOR $\zeta = 1$

For the critically damped case, the response is given by

$$Y(t) = \frac{1}{\tau^2} te^{-t/\tau}$$

CASE III IMPULSE RESPONSE FOR. $\zeta = 1$

$$Y(t) = \frac{1}{\tau}\frac{1}{\sqrt{\zeta^2 - 1}} e^{-\zeta t/\tau} \sinh\sqrt{\zeta^2 - 1}\,\frac{t}{\tau}$$

However, the impulse response always returns to zero.

Terms such as *decay ratio, period of oscillation,* etc., may also be used to describe the impulse response.

Sinusoidal Response

If the forcing function applied to the second-order system is sinusoidal

$$X(t) = A \sin \omega t$$

$$Y(s) = \frac{A\omega}{\left(s^2 + \omega^2\right)\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)}$$

The inversion of Equation may be accomplished by first factoring the two quadratic terms to give

$$Y(s) = \frac{A\omega/\tau^2}{(s - j\omega)(s + j\omega)(s - s_a)(s - s_b)}$$

Here s_a and s_b are the roots of the denominator of the transfer function and are given by Eqs.

For the case of an underdamped system ($\zeta < 1$), the roots of the denominator are a pair of pure imaginary roots ($+j\omega$, $-j\omega$) contributed by the forcing function and a pair of complex roots

$$\left(-\zeta/\tau + j\sqrt{1-\zeta^2}/\tau,\quad -\zeta/\tau - j\sqrt{1-\zeta^2}/\tau\right.$$

We may write the form of the response $Y(t)$

$$Y(t) = C_1 \cos \omega t + C_2 \sin \omega t + e^{-\zeta t/\tau}\left(C_3 \cos \sqrt{1-\zeta^2}\frac{t}{\tau} + C_4 \sin \sqrt{1-\zeta^2}\frac{t}{\tau}\right)$$

$$Y(t)\big|_{t\to\infty} = C_1 \cos \omega t + C_2 \sin \omega t$$

$$p \cos B + q \sin B = r \sin (B + \theta)$$

$$Y(t) = \frac{A\omega\tau}{\tau^2\omega^2 + 1} e^{-t/\tau} + \frac{A}{\sqrt{\tau^2\omega^2 + 1}} \sin(\omega t + \phi)$$

$$r = \sqrt{p^2 + q^2} \qquad \tan \theta = \frac{p}{q}$$

$$\phi = \tan^{-1}(-\omega\tau)$$

If the constants C_1 and C_2 are evaluated,

$$Y(t) = \frac{A}{\sqrt{\left[1-(\omega t)^2\right]^2 + (2\zeta\omega\tau)^2}}\sin(\omega t + \phi)$$

$$\phi = -\tan^{-1}\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}$$

$$X(t) = A\sin\omega t$$

The ratio of the output amplitude to the input amplitude is

$$\text{Amplitude ratio} = \frac{\text{output amplitude}}{\text{input amplitude}} = \frac{1}{\sqrt{\left[1-(\omega\tau)^2\right]^2 + (2\zeta\omega\tau)^2}}$$

Depending upon the values of ζ and $\omega\tau$. This is in direct contrast to the sinusoidal response of the *first-order* system, where the ratio of the output amplitude to the input amplitude is always *less than* 1.

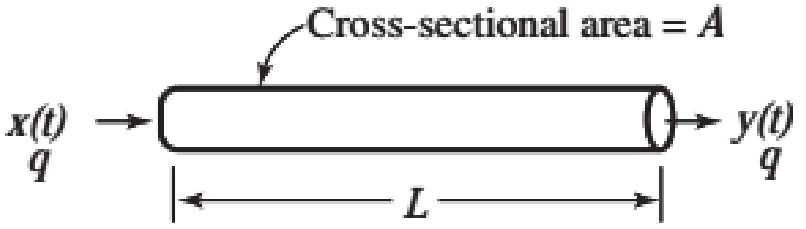
The output lags the input by phase angle $|\phi|$.

$$\phi = \text{phase angle} = -\tan^{-1}\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}$$

TRANSPORTATION LAG

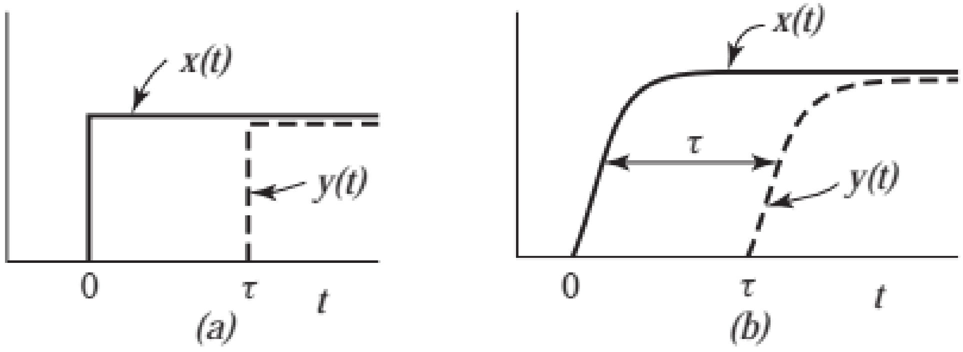
A phenomenon that is often present in flow systems is the *transportation lag.* Synonyms for this term are *dead time* and *distance velocity lag.*

System with transportation lag.



If a step change were made in $x(t)$ at $t = 0$, the change would not be detected at the end of the tube until τ later, where τ is the time required for the entering fluid to pass through the tube.

response $y(t)$ at the end of the pipe would be identical with $x(t)$ but again delayed by τ



The transportation lag parameter τ is simply the time needed for a particle of fluid to flow from the entrance of the tube to the exit, and it can be calculated from the expression

$$\tau = \frac{\text{volume of tube}}{\text{volumetric flow rate}} \qquad \tau = \frac{AL}{q}$$

Relationship between y (t) and x (t) is $\quad y(t) = x(t - \tau)$

Deviation variables $X = x - x_s$, and $Y = y - y_s$ give

$$Y(t) = X(t - \tau)$$

If the Laplace transform of $X(t)$ is $X(s)$, then the Laplace transform of $X(t - \tau)$ is $e^{-s\tau} X(s)$.

$$Y(s) = e^{-s\tau} X(s)$$

$$\frac{Y(s)}{X(s)} = e^{-s\tau}$$

We shall see in a later chapter that the presence of a transportation lag in a control system can make it much more difficult to control.

In general, such lags should be avoided if possible by placing equipment close together. They can seldom be entirely eliminated.

APPROXIMATION OF TRANSPORT LAG.

The transport lag is quite different from the other transfer functions (first-order, second-order, etc.)

It is not a rational function (i.e., a ratio of polynomials.)

- The transport lag can also be difficult to simulate by computer.
- For these reasons, several approximations of transport lag that are useful in control calculations are presented here.
- One approach to approximating the transport lag is to write $e^{-\tau s}$ as

$1/e^{\tau s}$ and to express the denominator as a Taylor series;

the result is

$$e^{-\tau s} = \frac{1}{e^{\tau s}} = \frac{1}{1 + \tau s + \tau^2 s^2/2 + \tau^3 s^3/3! + \cdots}$$

Keeping only the first two terms in the denominator gives

$$e^{-\tau s} \cong \frac{1}{1 + \tau s}$$

This approximation, which is simply a first-order lag, is a crude approximation of a transport lag.

An improvement can be made by expressing the transport lag as

$$e^{-\tau s} = \frac{e^{-\tau s/2}}{e^{\tau s/2}}$$

Expanding numerator and denominator in a Taylor series and keeping only terms of first-order give

$$e^{-\tau s} \cong \frac{1 - \tau s/2}{1 + \tau s/2} \qquad \text{first-order Padé}$$

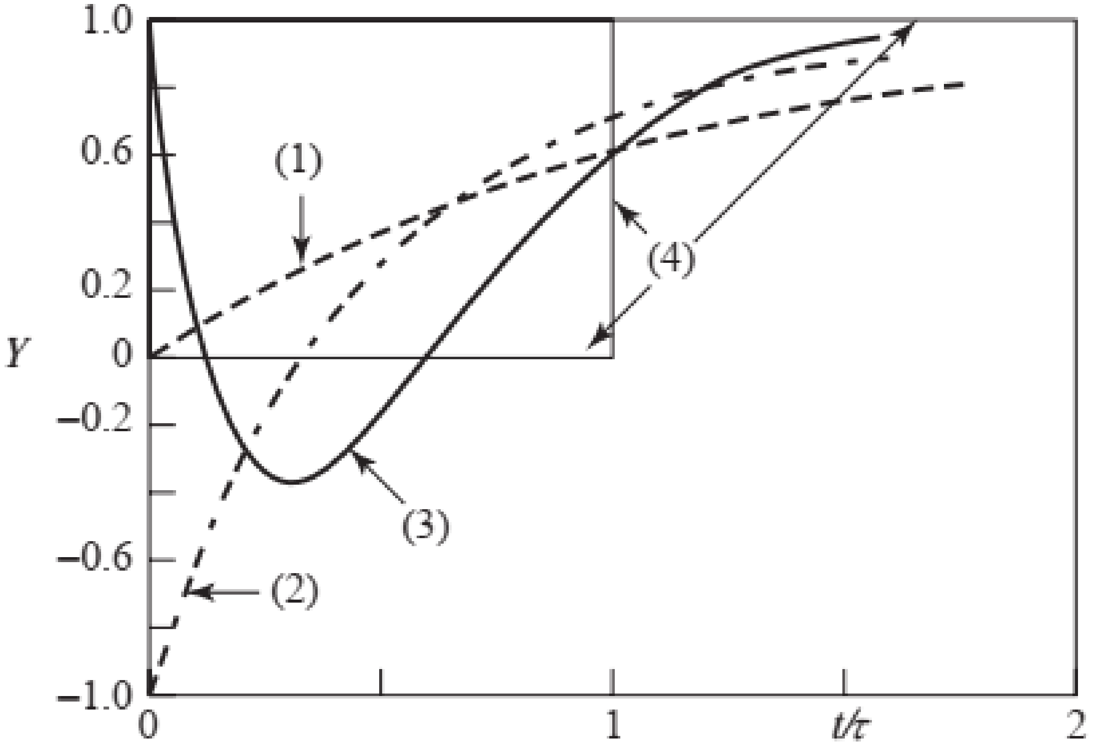
This expression is also known as a *first-order Padé* approximation.

Another well-known approximation for a transport lag is the second-order Padé approximation:

$$e^{-\tau s} \cong \frac{1 - \tau s/2 + \tau^2 s^2/12}{1 + \tau s/2 + \tau^2 s^2/12} \qquad \text{second-order Padé}$$

Equation (7.48) is not merely the ratio of two Taylor series; it has been optimized to give a better approximation.

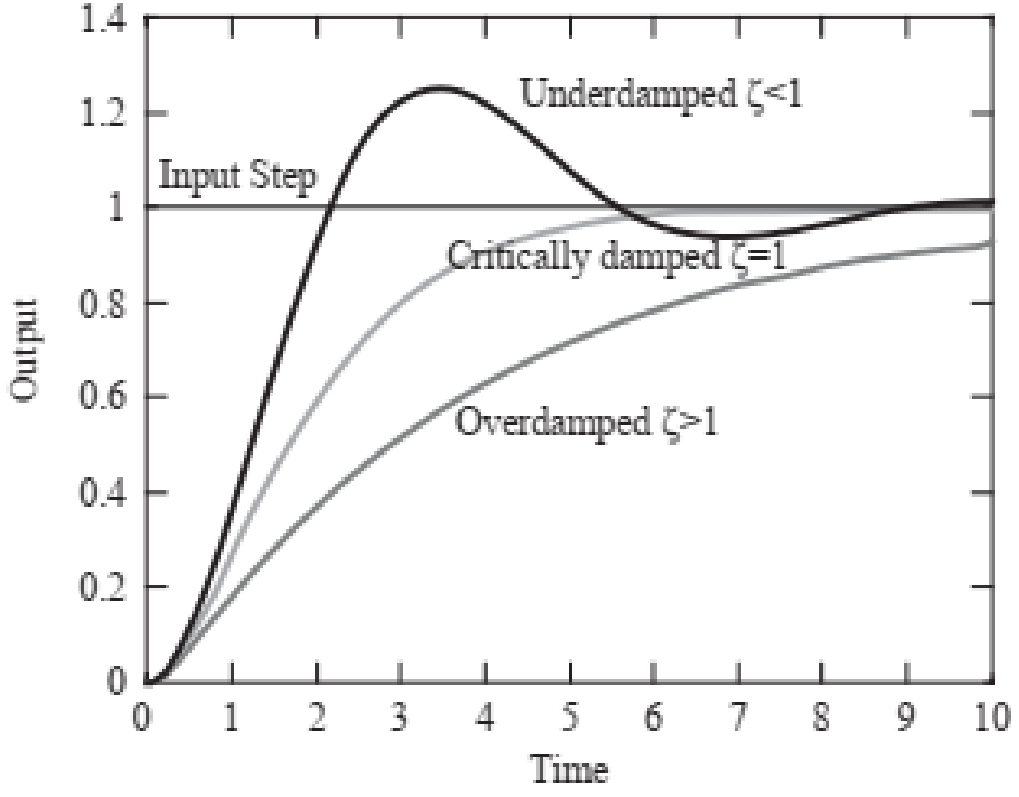
- The step responses of the three approximations of transport lag presented in Figure. The step response of $e^{\tau s}$ is also shown for comparison.
- Notice that the response for the first-order Padé approximation drops to 1 before rising exponentially toward 1.
- The response for the second order Padé approximation jumps to 1 and then descends to below 0 before returning gradually back to 1.



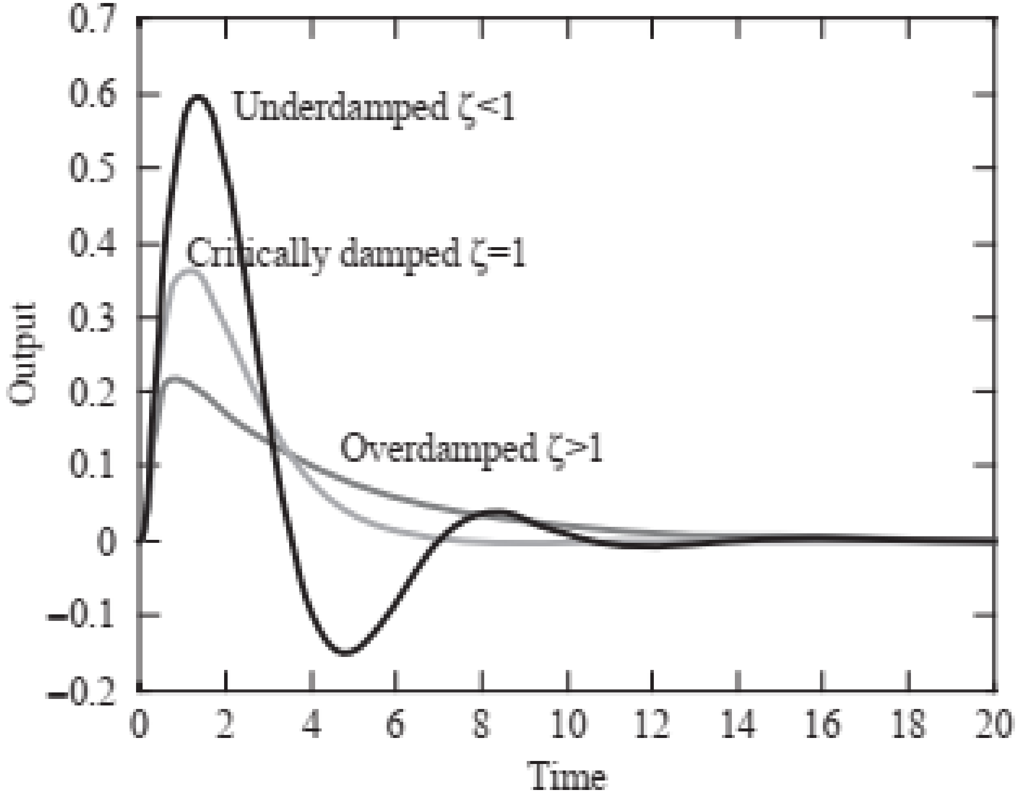
Step response to approximation of the transport lag $e^{-\tau s}$:
(1) $\frac{1}{\tau s + 1}$; (2) first-order Padé; (3) second-order Padé ; (4) $e^{-\tau s}$.

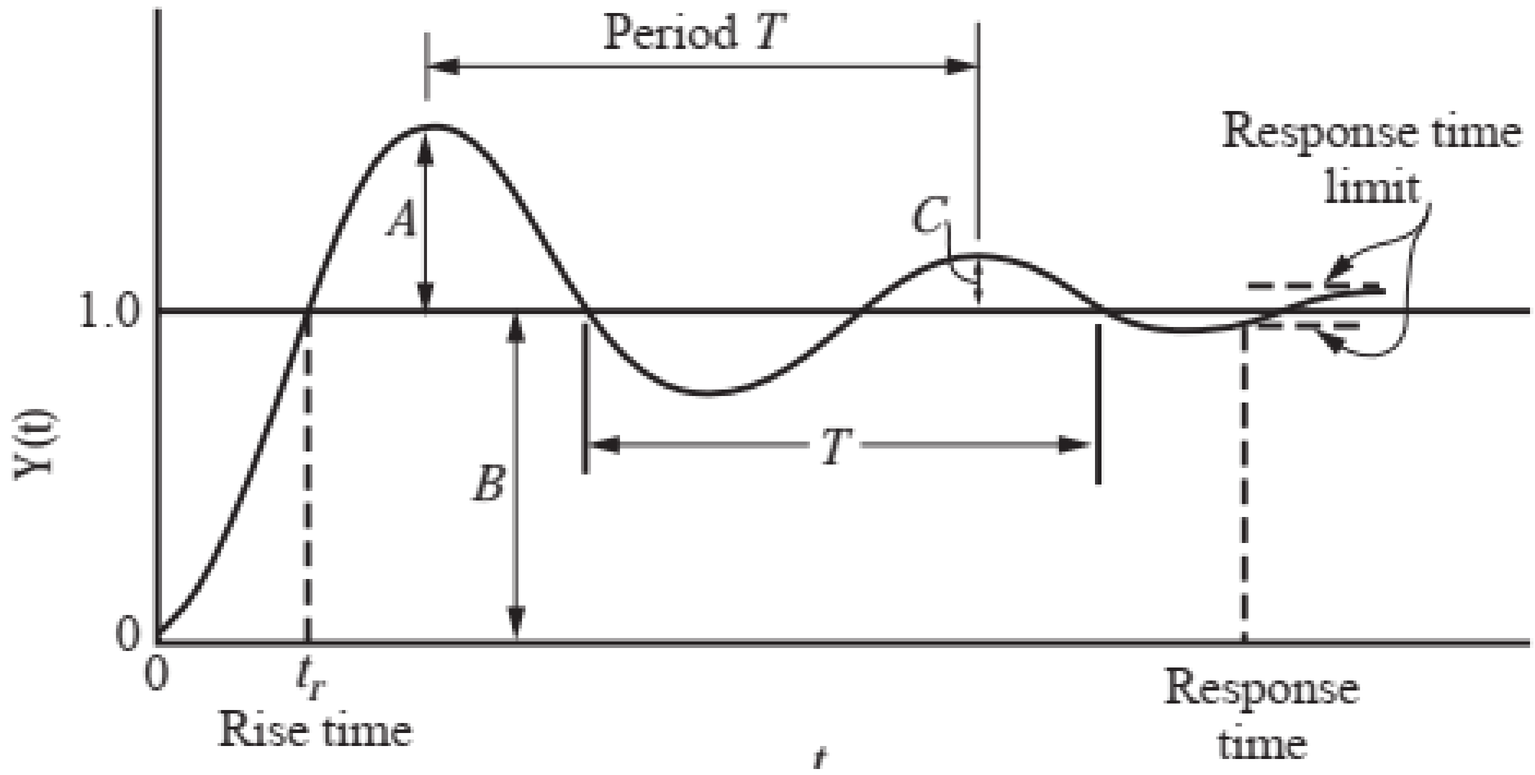
- Although none of the approximations for $e^{-\tau s}$ is very accurate, the approximation for $e^{-\tau s}$ is more useful when it is multiplied by several first-order or second-order transfer functions.
- In this case, the other transfer functions filter out the high-frequency content of the signals passing through the transport lag, with the result that the transport lag approximation,
- when combined with other transfer functions, provides a satisfactory result in many cases. The accuracy of a transport lag can be evaluated most clearly in terms of frequency response, a topic covered later in this book.

Sample Second-Order System Response
to a Unit-Step Input



Sample Second-Order System Response
to a Unit Impulse Input

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Response time limit
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C
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