Conduction

- Conduction is the transfer of heat *from one point in an object to another point* or from *one object to another* which are at *different temperatures* and *are in physical contact*, without appreciable displacement of the particles in the object

- Thermal energy may be conducted in solids by two modes –

  (i) Lattice vibrations (ii) Transport of free electrons

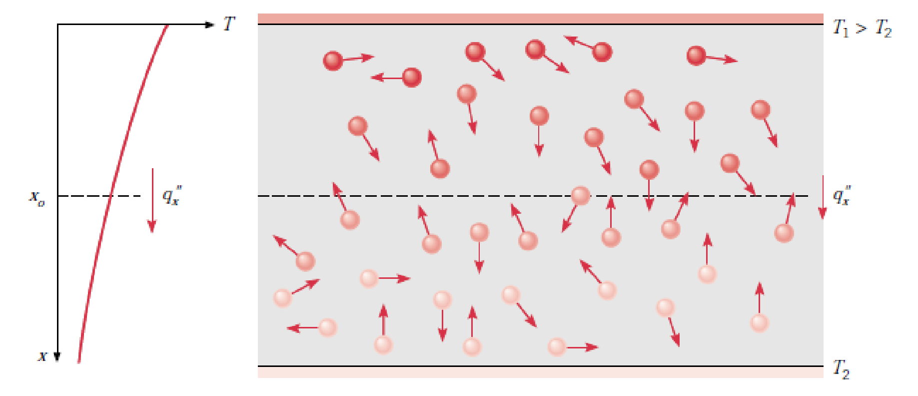
- In good electrical conductors, a large number of free electrons move about in the lattice structure of the material

  These electrons carry thermal charge from a high-temperature region to a low-temperature region just as they carry electrical charge

- Energy is also transmitted as vibrational energy in the lattice structure of the material

- In case of gases, the molecules are in constant random motion colliding with one another and exchanging energy and momentum

- Molecules in a region of high temperature have a higher kinetic energy and when they collide with molecules of a lower temperature region, they give up energy through collisions

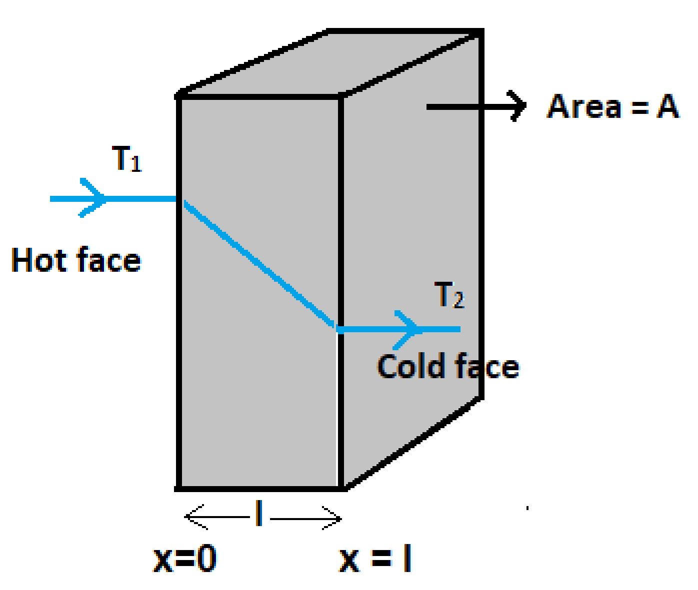


- In case of liquids, the physical mechanism of heat conduction is the same as that of gases
- However, the situation is more complex as the molecules are more closely spaced than gases and molecular force fields exert a strong influence on the energy exchange in the collision process

Basic law of heat conduction – Fourier's law

- Fourier's law was established by J.B.J. Fourier in 1822 from his experimental data on heat flux
- Consider flow of heat through a wall of thickness 'l' and area 'A'
- A source of heat exists on the left face of the wall (T_1) and a receiver of heat on the right face (T_2) – here $T_1 > T_2$
- Flow of heat (rate of heat conduction) is proportional to the change in temperature through the wall ($\frac{dT}{dx}$) and the area of the wall (A)
- The flow of heat is at right angles to the wall if the wall surfaces are isothermal and the body homogeneous and isotropic
- The rate of heat conduction Q is given by

$$Q \propto A\left(-\frac{dT}{dx}\right)$$



$$Q = kA\left(-\frac{dT}{dx}\right)$$

$$-\int_{T_1}^{T_2} dT = \frac{Q}{kA}\int_0^l dx$$

$$(T_1 - T_2) = \frac{Ql}{kA}$$

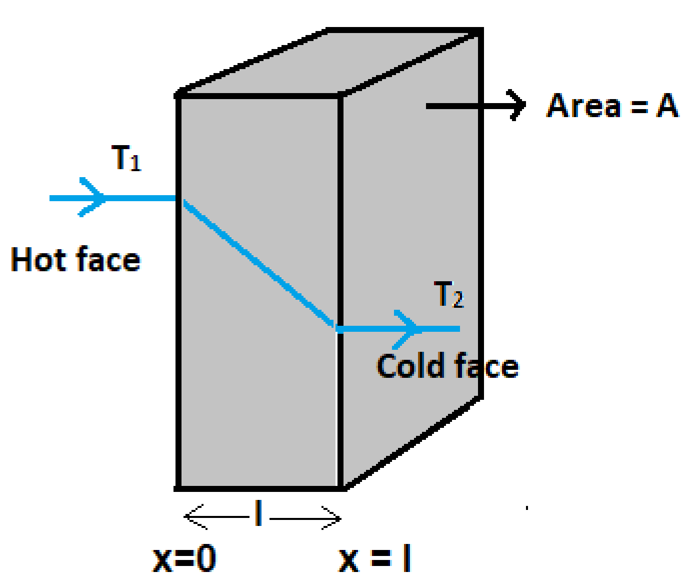
$$\boxed{Q = \frac{kA(T_1 - T_2)}{l}}$$

where k = thermal conductivity , (W/mK)

$\frac{dT}{dx}$ = temperature gradient

$\frac{dT}{dx}$ is negative

as temp at x = 0 > temp at x = l



This is **Fourier's law of heat conduction**

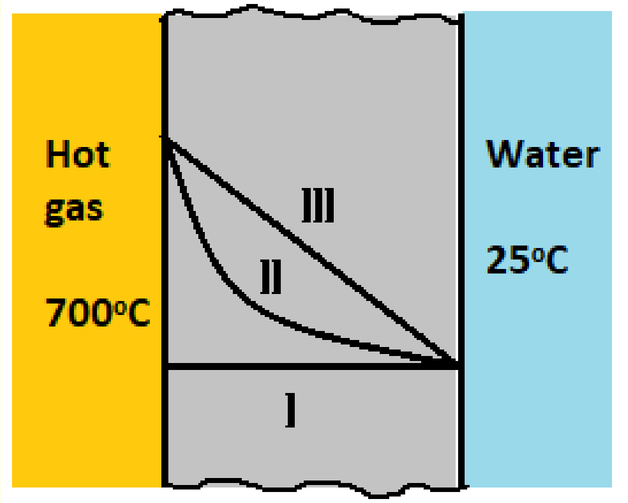
This equation can also be written as,

$$\boxed{q = \frac{Q}{A} = \frac{k(T_1 - T_2)}{l}}$$

where q is the *heat flux*

Heat flux is the heat conducted per unit area per unit time

- Let us consider the flow of heat through a water cooled furnace wall – mode of heat transfer is through conduction
- Initially the inside wall of the furnace is cold (25^{o}C) and the temperature on the water cooled side is also 25^{o}C
- The temperature distribution is a straight line (I) and is independent of both time and position
- As the furnace wall is exposed to hot combustion gases at 700^{o}C, heat flow occurs through the wall



- The temperature profile (temperature distribution) can now be represented by curve II where temperature depends on both time and position
- This process is called *unsteady state conduction* – Fourier's law can be applied at each point in the wall and at every instant of time
- If the wall is in contact with the hot gases and cold water for a very long time, there is no more change in temperature with time and the profile is shown as in line III
- This condition is called *steady state conduction*
- Under steady state condition, temperature is a function of only position and not time

$\text{I} \rightarrow T \neq f(time, position)$ $\quad$ $\text{II} \rightarrow T = f(time, position)$ $\quad$ $\text{III} \rightarrow T = f(position)\ and\ \neq f(time)$

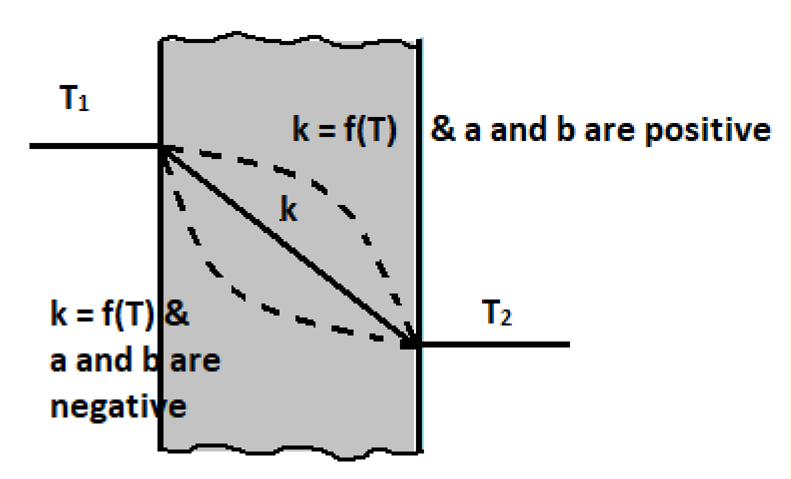
Thermal conductivity

- Thermal conductivity is the proportionality constant used in the Fourier's law
- It is a fundamental property of the material that gives a measure of the *effectivity of the material in transmitting heat through it*
- The thermal conductivity of solids are greater than those of liquids, which in turn is greater than gases ($k_{solid} > k_{liquid} > k_{gas}$)
- In solids, materials in the crystalline state have a higher thermal conductivity than those in the amorphous state – the 'orderly arrangement' of atoms in a crystalline solid allows faster transmission of thermal energy
- Solids that have higher thermal conductivity are called ***thermal conductors*** and those that are poor conductors of heat are called ***thermal insulators***
- Commercial insulators such as ceramic, diatomaceous or polymeric materials are often highly porous – high void fraction filled with an inert gas

- Thermal conductivity is a *function of temperature*
- The thermal conductivity of metals decreases with an increase in temperature and that insulating materials increases with temperature
- Thermal conductivity of liquids generally decrease with increase in temperature but are nearly independent of pressure
- Thermal conductivity of gases increase with increase temperature and decrease with molecular weight
  Pressure dependence is significant only at high pressures
- Thermal conductivity of air (gas) = 0.0262 W/mK, water (liquid)= 0.63 W/mK, silver (solid) 410 W/mK
- For heat transfer calculations, thermal conductivity at the average temperature of the material is often used
- The temperature dependence of thermal conductivity is written as

$$k = k_o(1 + aT + bT^2)$$

  where k_o = thermal conductivity at 0 K



- If the thermal conductivity of a solid is the same in all directions, the material is called ***isotropic***
- If the thermal conductivity depends on the direction, the material is called ***anisotropic***

- According to Fourier's law,

$$Q = \frac{kA(T_1 - T_2)}{l}$$

- As heat flow is a rate process, this equation can be written as,

$$Q = \frac{(T_1 - T_2)}{\left(\frac{l}{kA}\right)}$$

$$Rate\ of\ heat\ flow = \frac{Thermal\ potential\ difference\ (driving\ force)}{Thermal\ resistance}$$

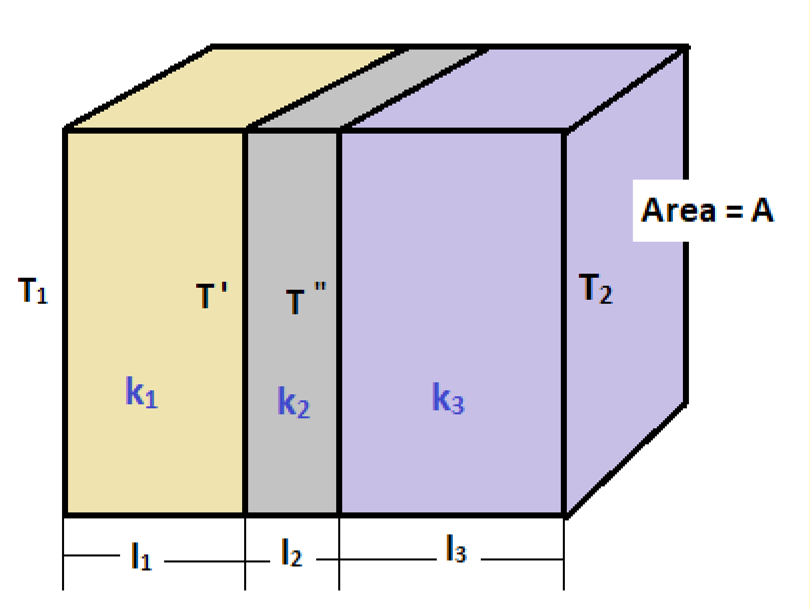
This is similar to the flow of a fluid or the flow of electric current

Steady state conduction of heat through a single solid

$$Q = \frac{(T_1 - T_2)}{\left(\frac{l}{kA}\right)}$$

Steady state conduction of heat through a composite solid

- Composite solid is a solid made up of different materials with different thermal conductivities
- For example, when three different materials (1,2 and 3) having thermal conductivities k_1, k_2, k_3 form a composite solid
- The area of heat conduction (A) is constant
- The *rates of heat flow at steady state through the individual layers are equal* ($Q_1 = Q_2 = Q_3 = Q$)
- The rates of heat flow through the walls are as follows:



$$Q_1 = Q = \frac{k_1 A (T_1 - T')}{l_1} \qquad (T_1 - T') = \frac{Q l_1}{k_1 A}$$

$$Q_2 = Q = \frac{k_2 A (T' - T'')}{l_2} \qquad (T' - T'') = \frac{Q l_2}{k_2 A}$$

$$Q_3 = Q = \frac{k_3 A (T'' - T_2)}{l_3} \qquad (T'' - T_2) = \frac{Q l_3}{k_3 A}$$

Adding the three equations,

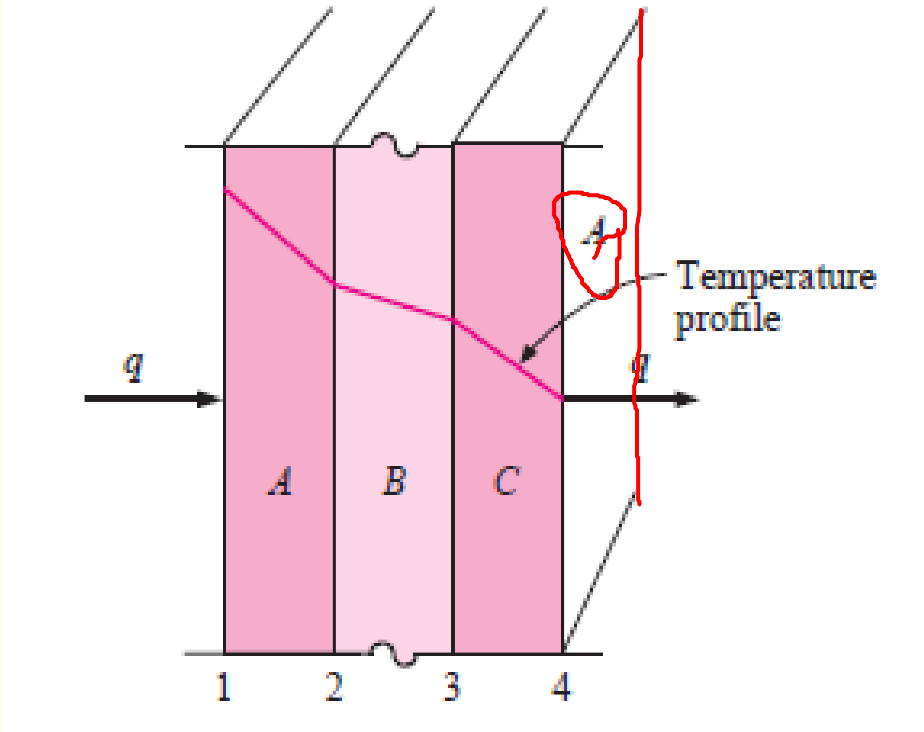
$$(T_1 - T_2) = Q\left(\frac{l_1}{k_1 A} + \frac{l_2}{k_2 A} + \frac{l_3}{k_3 A}\right)$$

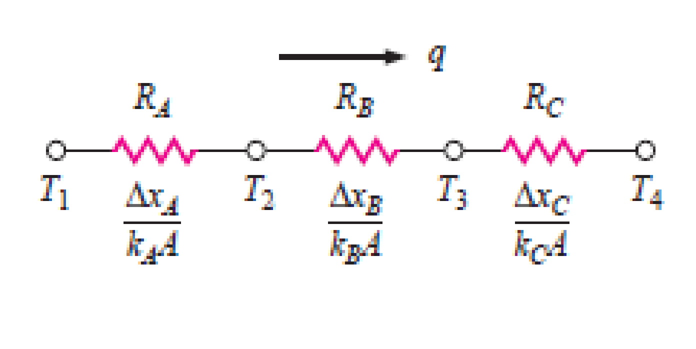
or,

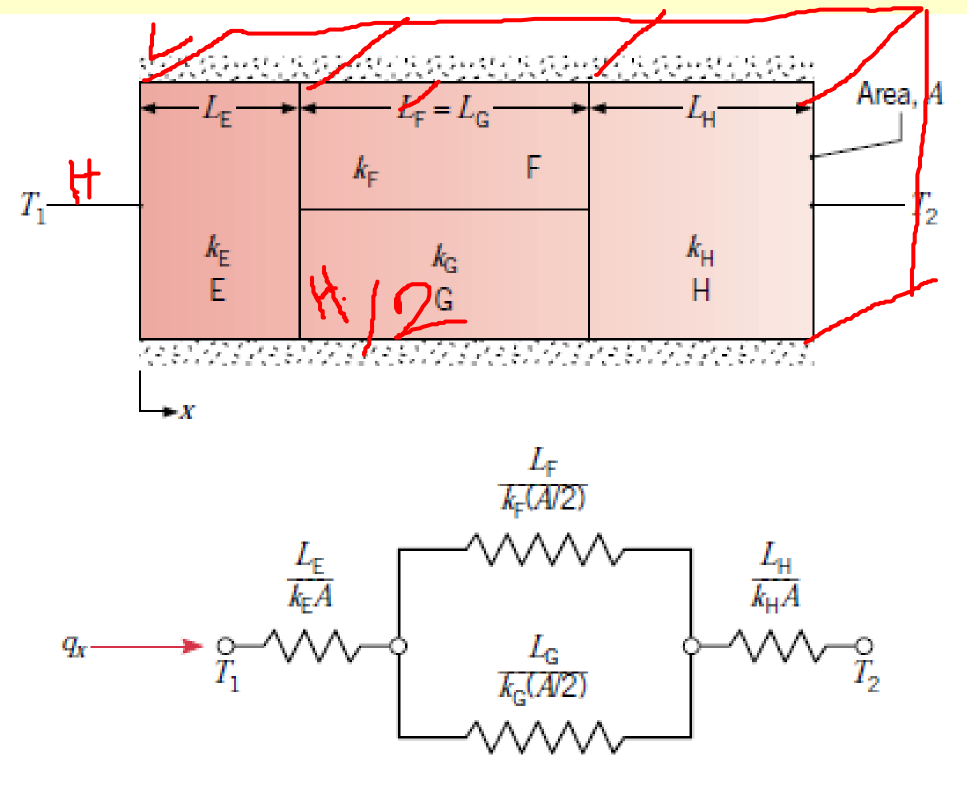
$$Q = \frac{(T_1 - T_2)}{\left(\frac{l_1}{k_1 A} + \frac{l_2}{k_2 A} + \frac{l_3}{k_3 A}\right)}$$

$$Q = \frac{\Delta T}{(R_1 + R_2 + R_3)}$$

The thermal resistances in a composite wall are in series and behave just like electrical resistances


Temperature profile
q
q
A
B
C
1
2
3
4


q
R_A
R_B
R_C
T_1
$\frac{\Delta x_A}{k_A A}$
T_2
$\frac{\Delta x_B}{k_B A}$
T_3
$\frac{\Delta x_C}{k_C A}$
T_4


L_E
$L_F = L_G$
L_H
Area, A
T_1
T_2
k_F
F
k_E
E
k_G
G
k_H
H
x
$\frac{L_F}{k_F(A/2)}$
$\frac{L_E}{k_E A}$
$\frac{L_H}{k_H A}$
q_x
T_1
$\frac{L_G}{k_G(A/2)}$
T_2

Problem

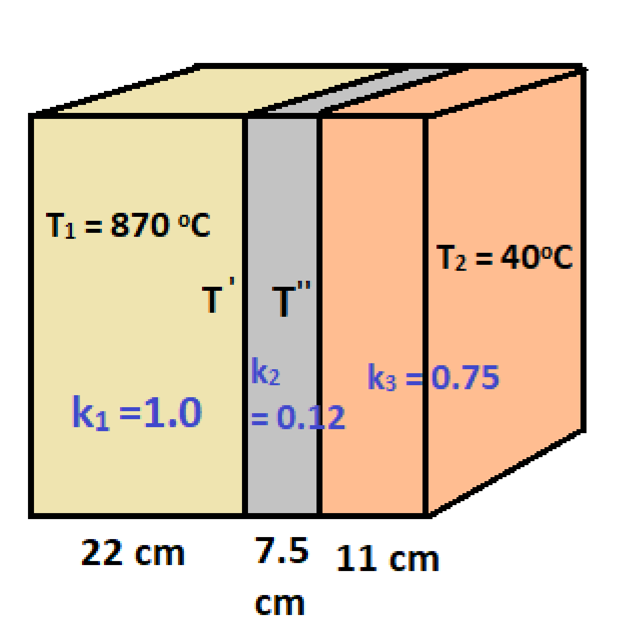
A furnace wall is made of three layers, one of fire brick, one of insulating brick and one of red brick. The inner and outer surfaces are at 870°C and 40°C, respectively. The thermal conductivities of the fire brick, insulating brick and red brick are 1.0, 0.12 and 0.75 W/m°C and the thickness are 22 cm, 7.5 cm and 11 cm.

(a) Calculate the rate of heat loss per m^2

(b) Estimate the interface temperatures

(c) What should the thickness of the insulating brick layer be in order to reduce heat loss by 50%?

(a) The three resistances can be calculated as



$$R_1 = \frac{l_1}{k_1 A} = \frac{0.22}{1 \times 1}\left(m \times \frac{m°C}{W} \times \frac{1}{m^2}\right) = 0.22\ °C/W$$

$$R_2 = \frac{l_2}{k_2 A} = \frac{0.075}{0.12 \times 1}\left(m \times \frac{m°C}{W} \times \frac{1}{m^2}\right) = 0.625\ °C/W$$

$$R_3 = \frac{l_3}{k_3 A} = \frac{0.11}{0.75 \times 1}\left(m \times \frac{m°C}{W} \times \frac{1}{m^2}\right) = 0.147\ °C/W$$

$$Total\ resistance\ , R = R_1 + R_2 + R_3 = 0.22 + 0.625 + 0.147 = 0.992\ °C/W$$

$$Q = \frac{(T_1 - T_2)}{\left(\frac{l_1}{k_1 A} + \frac{l_2}{k_2 A} + \frac{l_3}{k_3 A}\right)} = \frac{\Delta T}{(R_1 + R_2 + R_3)} = \frac{870 - 40}{0.992}\left(\frac{°C}{°C/W}\right) = 836.69\ W$$

(b) As the rate of heat conduction through each layer is equal

$$Q = \frac{(T_1 - T')}{l_1/k_1 A} = \frac{(T' - T'')}{l_2/k_2 A} = \frac{(T'' - T_2)}{l_3/k_3 A}$$

$$836.69 = \frac{(870 - T')}{0.22} \quad \Rightarrow \quad T' = 870 - (836.69 \times 0.22) = 870 - 184.07 = 685.93\ ℃$$

and $$836.69 = \frac{(T'' - 40)}{0.147} \quad \Rightarrow \quad T'' = (836.69 \times 0.147) + 40 = 122.993 + 40 = 162.99\ ℃$$

(c) 50% of $Q = 0.50 \times 836.69 = 418.345\ W$

$$Q = \frac{(T_1 - T_2)}{\left(\frac{l_1}{k_1 A} + \frac{l_2}{k_2 A} + \frac{l_3}{k_3 A}\right)}$$

$$418.345 = \frac{(870 - 40)}{\left(0.22 + \frac{l_2}{0.12 \times 1} + 0.147\right)}$$

$$153.53 + 3486.25 l_2 = 830 \quad \Rightarrow \quad l_2 = 0.194\ m = 19.4\ cm$$

Problem

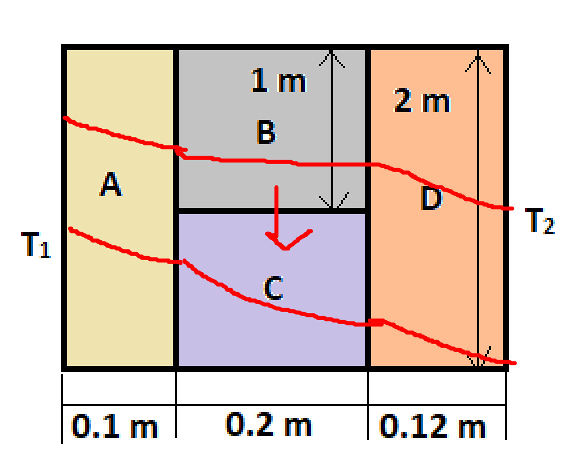
Consider a composite wall consisting of four different materials as shown in the figure.

Given: $k_A = 20$ W/m°C $k_B = 10$ W/m°C

$k_C = 7$ W/m°C $k_D = 25$ W/m°C

T_1= 120°C T_2= 50°C

Calculate the rate of heat flow per unit breadth through the assembly.



Thermal resistances of slab B (R_B) and C (R_C) are parallel and they together is in series with the resistances of A (R_A) and D (R_D)

$$Q = \frac{(T_1 - T_2)}{(R_T)} = \frac{(T_1 - T_2)}{\left(R_A + \frac{1}{\frac{1}{R_B} + \frac{1}{R_C}} + R_D\right)} = \frac{(T_1 - T_2)}{\left(\frac{l_A}{k_A A_1} + \frac{1}{\frac{1}{\frac{l_B}{k_B A_2}} + \frac{1}{\frac{l_C}{k_C A_2}}} + \frac{l_D}{k_D A_1}\right)}$$

If the breadth = P

$$R_A = \frac{l_A}{k_A A} = \frac{0.1}{20 \times 2 \times P} = \frac{0.0025}{P} \, °C/W$$

$$R_B = \frac{l_B}{k_B A} = \frac{0.2}{10 \times 1 \times P} = \frac{0.02}{P} \, °C/W$$

$$R_C = \frac{l_C}{k_C A} = \frac{0.2}{7 \times 1 \times P} = \frac{0.02857}{P} \, °C/W$$

$$R_D = \frac{l_D}{k_D A} = \frac{0.12}{25 \times 2 \times P} = \frac{0.0024}{P} \, °C/W$$

Therefore,

$$R_T = R_A + \frac{1}{\frac{1}{R_B} + \frac{1}{R_C}} + R_D$$

$$R_T = \frac{0.0025}{P} + \frac{1}{\frac{1}{\frac{0.02}{P}} + \frac{1}{\frac{0.02857}{P}}} + \frac{0.0024}{P} = \frac{0.0049}{P} + \frac{1}{\frac{P}{0.02} + \frac{P}{0.02857}}$$

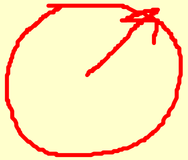
$$R_T = \frac{0.0049}{P} + \frac{0.01176}{P} = \frac{0.01666}{P}$$

Heat flow per unit breadth is given by,

$$Q = \frac{(T_1 - T_2)}{(R_T)} == \frac{(120 - 50)}{\left(\frac{0.01666}{P}\right)} = \frac{(120 - 50)P}{0.01666}$$

$$Q\ per\ unit\ breadth, P = \frac{Q}{P} = \frac{120 - 50}{0.01666} = 4200.55\ W = 4.2\ kW$$

Steady state heat conduction through a variable area



- In case of a plane wall, the area for heat flow is constant
- For solids of other geometries such as a cylinder (length very large) and sphere, the area depends on the radius or the radial position

Hollow cylinder

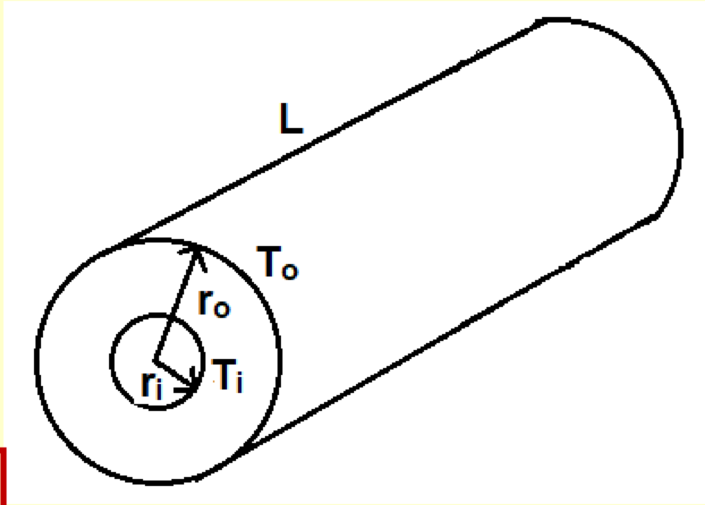
- For a cylinder with length very large compared to diameter, it may be assumed that the heat flows only in a radial direction, so that the only space coordinate needed to specify the system is 'r'
- The appropriate relation for area is introduced into the Fourier's law expression

$$Q = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$

$$\int_{r_i}^{r_o}\frac{dr}{r} = -\frac{2\pi kL}{Q}\int_{T_i}^{T_o} dT$$

$$ln\frac{r_o}{r_i} = -\frac{2\pi kL}{Q}(T_o - T_i) = \frac{2\pi kL}{Q}(T_i - T_o)$$

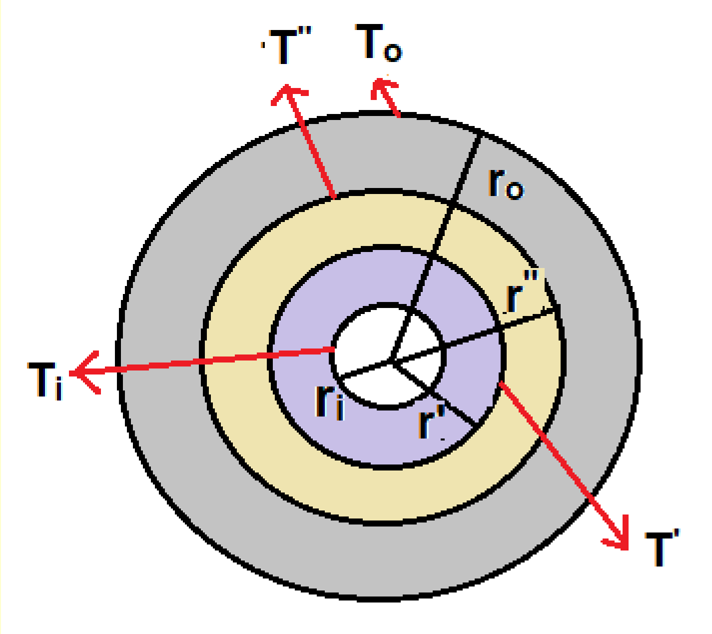
$$Q = \frac{2\pi kL(T_i - T_o)}{ln\frac{r_o}{r_i}} \quad \Rightarrow \quad Q = \frac{(T_i - T_o)}{\left(ln\frac{r_o}{r_i}\Big/2\pi kL\right)} = \frac{Driving\ force}{Resistance}$$



Steady state heat conduction through a composite cylinder made up of several layers

- The rates of heat flow at steady state through the individual layers are equal $(Q_1 = Q_2 = Q_3 = Q)$

$$Q = \frac{2\pi k_1 L(T_i - T')}{ln\left(\frac{r'}{r_i}\right)} = \frac{2\pi k_2 L(T' - T'')}{ln\left(\frac{r''}{r'}\right)} = \frac{2\pi k_3 L(T'' - T_o)}{ln\left(\frac{r_o}{r''}\right)}$$



Again, $(T_i - T') = \frac{Q\ ln\left(\frac{r'}{r_i}\right)}{2\pi k_1 L}$

$$(T' - T'') = \frac{Q ln\left(\frac{r''}{r'}\right)}{2\pi k_2 L}$$

$$(T'' - T_o) = \frac{Q\ ln\left(\frac{r_o}{r''}\right)}{2\pi k_3 L}$$

Adding the equations we get,

$$Q = \frac{(T_i - T_o)}{\frac{ln\left(\frac{r'}{r_i}\right)}{2\pi k_1 L} + \frac{ln\left(\frac{r''}{r'}\right)}{2\pi k_2 L} + \frac{ln\left(\frac{r_o}{r''}\right)}{2\pi k_3 L}}$$

Problem

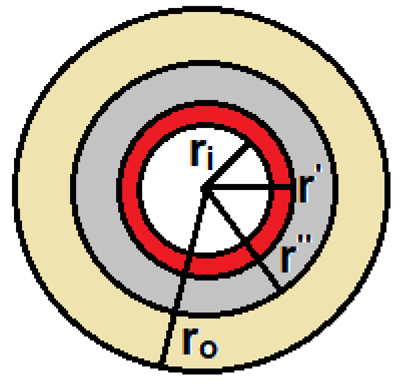
A steam pipe with inner diameter of 160 mm and outer diameter of 170 mm is covered with two layers of insulation. The thickness of the first layer of insulation is 30 mm and that of the second layer is 50 mm. The thermal conductivities of the pipe, and two insulating layers are 50, 0.15 and 0.08 W/m^{o}C The temperature of the inner surface of the steam pipe is 300^{o}C and that of the outer surface is the second insulating layer is 50^{o}C.

(a) Determine the quantity of heat lost per metre length of steam pipe.

(b) Estimate the interface temperatures between the pipe and layers

(a) For a composite cylinder

$$Q = \frac{(T_i - T_o)}{\frac{ln\left(\frac{r'}{r_i}\right)}{2\pi k_1 L} + \frac{ln\left(\frac{r''}{r'}\right)}{2\pi k_2 L} + \frac{ln\left(\frac{r_o}{r''}\right)}{2\pi k_3 L}}$$



Here, $r_i = 0.08$ m, $r' = 0.085$ m, $r'' = 0.085 + 0.03 = 0.115\ m$, $r_o = 0.115 + 0.05 = 0.165$ m

$$Q = \frac{(300 - 50)}{\frac{ln\left(\frac{0.085}{0.08}\right)}{2\pi \times 50 \times L} + \frac{ln\left(\frac{0.115}{0.085}\right)}{2\pi \times 0.15 \times L} + \frac{ln\left(\frac{0.165}{0.115}\right)}{2\pi \times 0.08 \times L}} = \frac{(250) \times 1}{1.93 \times 10^{-4} + 0.3207 + 0.7182} = 240.6\ W$$

(b) It was seen earlier.

$$Q = \frac{2\pi k_1 L\left(T_i - T'\right)}{ln\left(\frac{r'}{r_i}\right)} = \frac{2\pi k_2 L\left(T' - T''\right)}{ln\left(\frac{r''}{r'}\right)} = \frac{2\pi k_3 L\left(T'' - T_o\right)}{ln\left(\frac{r_o}{r''}\right)}$$

$$240.6 = \frac{2\pi \times 50 \times 1 \times \left(300 - T'\right)}{ln\left(\frac{0.085}{0.08}\right)} = \frac{2\pi \times 0.15 \times 1 \times \left(T' - T''\right)}{ln\left(\frac{0.115}{0.085}\right)} = \frac{2\pi \times 0.08 \times 1 \times \left(T'' - 50\right)}{ln\left(\frac{0.165}{0.115}\right)}$$

Thus, $240.6 = \frac{2\pi \times 50 \times 1 \times \left(300 - T'\right)}{ln\left(\frac{0.085}{0.08}\right)}$

$$T' = 300 - 0.046 = 299.954℃$$

and $240.6 = \frac{2\pi \times 0.08 \times 1 \times \left(T'' - 50\right)}{ln\left(\frac{0.165}{0.115}\right)}$

$$T'' = 172.80 + 50 = 222.8\ ℃$$

Sphere

- The rate of heat transfer in a hollow sphere can be estimated similar to that of a cylinder
- Here the area of a sphere is introduced into the Fourier's law expression

$$Q = -kA\frac{dT}{dr} = -k4\pi r^2\frac{dT}{dr}$$

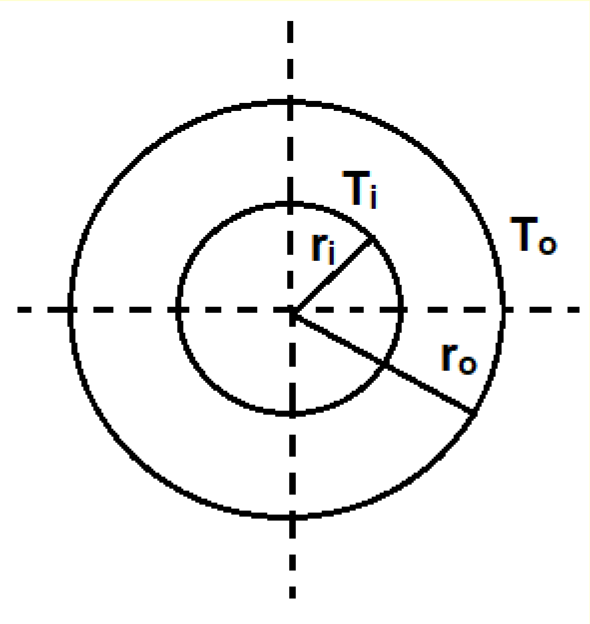
$$\int_{r_i}^{r_o}\frac{dr}{r^2} = -\frac{4\pi k}{Q}\int_{T_i}^{T_o} dT$$

$$-\left[\frac{1}{r_o} - \frac{1}{r_i}\right] = -\frac{4\pi k}{Q}(T_o - T_i)$$

$$\left[\frac{1}{r_i} - \frac{1}{r_o}\right] = \frac{4\pi k}{Q}(T_i - T_o)$$

$$Q = \frac{4\pi k r_i r_o (T_i - T_o)}{(r_o - r_i)}$$

$$Q = \frac{(T_i - T_o)}{\frac{(r_o - r_i)}{4\pi k r_i r_o}} = \frac{Driving\ force}{Resistance}$$

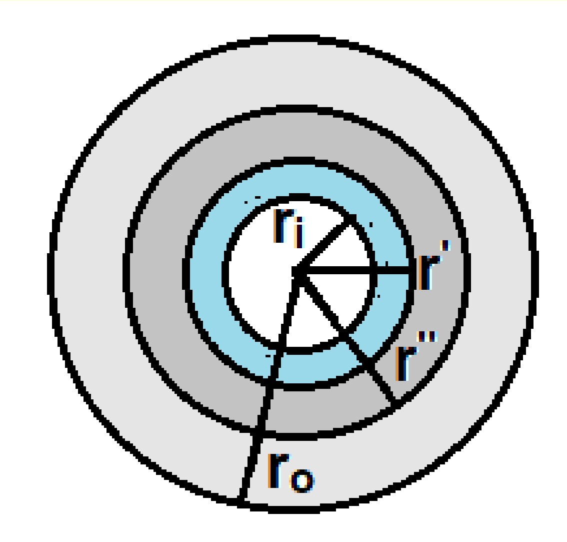


- For a composite sphere,

$$Q = \frac{(T_i - T_o)}{\frac{(r' - r_i)}{4\pi k_1 r_i r'} + \frac{(r'' - r')}{4\pi k_2 r'' r'} + \frac{(r_o - r'')}{4\pi k_3 r_o r''}}$$

- The rates of heat transfer and interface temperatures can be determined from this composite sphere heat transfer expression

$$Q = \frac{(T_i - T')}{\frac{(r' - r_i)}{4\pi k_1 r_i r'}} = \frac{(T' - T'')}{\frac{(r'' - r')}{4\pi k_2 r'' r'}} = \frac{(T'' - T_o)}{\frac{(r_o - r'')}{4\pi k_3 r_o r''}}$$

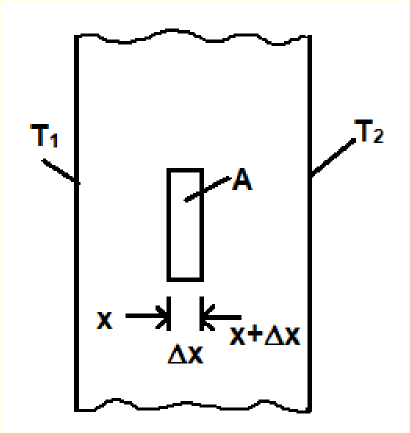


Development of the one dimensional heat conduction equation (generalized)

The generalized equation can be developed from the basic energy balance equation

Let us consider an elementary volume or control volume at a distance x and write a balance equation (shell balance)

$$Rate\ of\ heat\ input\ to\ the\ element\ at\ x - Rate\ of\ heat\ output\ from\ the\ element\ at\ x + \Delta x + Rate\ of\ heat\ generation\ in\ the\ control\ vol = Rate\ of\ heat\ accumulation\ in\ the\ control\ vol$$



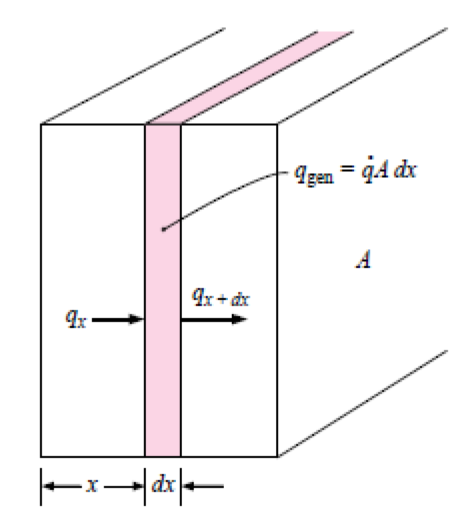
If q_x is the heat flux (heat flux = rate of heat flow per unit area) in the x direction

Rate of heat input = $Aq_x|_x$

Rate of heat output = $Aq_x|_{x+\Delta x} = Aq_x + \left(\frac{dq_x}{dx}\right)\Delta x A$

Rate of heat generation = $\dot{q}A\Delta x$ ($\dot{q}$ =rate of heat generation per unit volume)

Rate of heat accumulation = $mC_p\frac{dT}{dt} = (A\Delta x)\rho C_p\frac{dT}{dt}$



Heat balance, $[Aq_x] - \left[Aq_x + \left(\frac{dq_x}{dx}\right)\Delta x A\right] + \dot{q}A\Delta x = (A\Delta x)\rho C_p\frac{dT}{dt}$

Dividing by $(A\Delta x)$, we have $-\left(\frac{dq_x}{dx}\right) + \dot{q} = \rho C_p\frac{dT}{dt}$

Now, $q_x = -k\frac{dT}{dx}$, $\frac{d}{dx}\left[k\frac{dT}{dx}\right] + \dot{q} = \rho C_p\frac{dT}{dt}$

$$k\frac{d^2T}{dx^2} + \dot{q} = \rho C_p\frac{dT}{dt}$$

One dimensional heat conduction equation

$$k\frac{d^2T}{dx^2} + \dot{q} = \rho C_p \frac{dT}{dt}$$

At steady state, $\frac{dT}{dt} = 0$

No heat generation, $\dot{q} = 0$

$\therefore \quad k\frac{d^2T}{dx^2} = 0$ and

if k is independent of temperature, $\frac{d^2T}{dx^2} = 0$

To solve this, two boundary conditions are needed, at $x = 0$ $\quad T = T_1$

and at $x = l$ $\quad T = T_2$

Integrating, $\quad \int \frac{d^2T}{dx^2} = 0$

$$\frac{dT}{dx} = C_1$$

$$T = C_1 x + C_2$$

Putting boundary conditions, $\quad C_2 = T_1$ and $C_1 = \frac{T_2 - T_1}{l}$

or, $$T = \left(\frac{T_2 - T_1}{l}\right)x + T_1$$

The **temperature distribution inside the wall**, i.e., temperature of the wall at any position can be determined from this equation