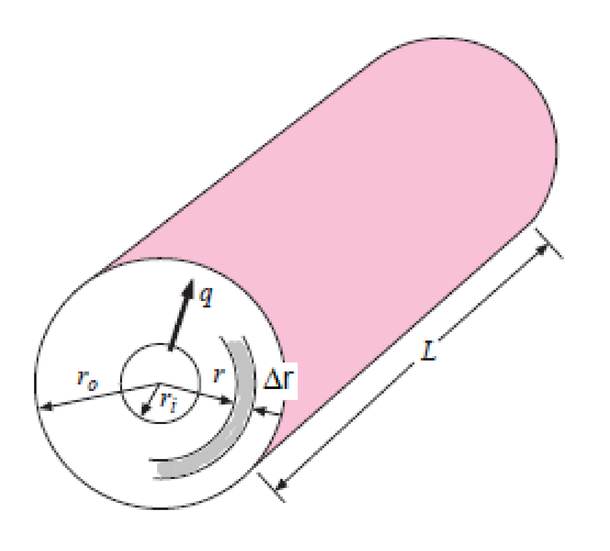
Conduction

Similar **generalized equation for heat conduction** can be developed **for cylinder and sphere** to estimate the temperature profile

Rate of heat input at radial position r = $(2\pi rL)q_r|_r$

Rate of heat output from shell at radial position =$(2\pi rL)q_r|_{r+\Delta r}$



At steady state and no heat generation,

$$(2\pi rL)q_r|_r - (2\pi rL)q_r|_{r+\Delta r} = 0$$

Dividing by (Δr) and taking limits $\Delta r \rightarrow 0$

$$\frac{(2\pi rL)q_r|_r - (2\pi rL)q_r|_{r+\Delta r}}{\Delta r} = 0$$

we get, $-\frac{d}{dr}(2\pi rLq_r) = 0$ or, $-\frac{d}{dr}(rq_r) = 0$

Integrating, $rq_r = C_1$

Now, $q_r = -k\frac{dT}{dr}$ Putting this in the previous equation, $-kr\frac{dT}{dr} = C_1$ or, $\frac{dT}{dr} = -\frac{C_1}{kr}$

or, $T = -\frac{C_1}{k}lnr + C_2$

Putting boundary conditions, at $r = r_i$ $T = T_i$ and at $r = r_o$ $T = T_o$

$$T_i = -\frac{C_1}{k} ln r_i + C_2 \quad \text{and} \quad T_o = -\frac{C_1}{k} ln r_o + C_2$$

Solving the two equations, we get,

$$C_1 = \frac{(T_i - T_o)k}{ln\left({}^{r_o}/_{r_i}\right)} \quad \text{and} \quad C_2 = T_i + \frac{(T_i - T_o) ln r_i}{ln\left({}^{r_o}/_{r_i}\right)}$$

Substituting these values of the constants, we get,

$$T = T_i - \frac{(T_i - T_o)}{ln\left({}^{r_o}/_{r_i}\right)} ln\left({}^{r}/_{r_i}\right)$$

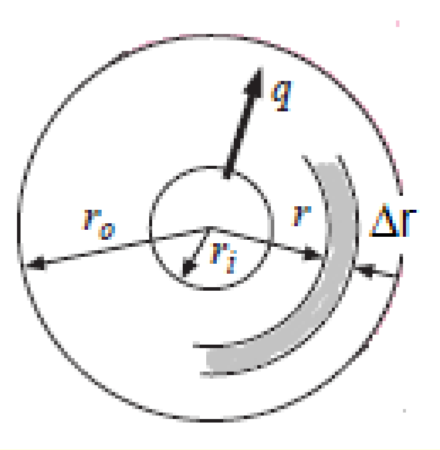
This is the **temperature profile in a hollow cylinder**

Rate of heat conduction, $Q = 2\pi r L q_r$ or, $Q = 2\pi r L \frac{C_1}{r}$

$$Q = \frac{2\pi L (T_i - T_o)k}{ln\left({}^{r_o}/_{r_i}\right)} = \frac{(T_i - T_o)}{\frac{ln\left({}^{r_o}/_{r_i}\right)}{2\pi k L}}$$

This is identical to the equation developed earlier for steady state heat conduction in hollow cylinder

Temperature profile in a sphere



Rate of heat input at radial position r = $(4\pi r^2)q_r|_r$

Rate of heat output from shell at radial position = $(4\pi r^2)\ q_r|_{r+\Delta r}$

At steady state and no heat generation,

$$(4\pi r^2)q_r|_r - (4\pi r^2)q_r|_{r+\Delta r} = 0$$

Dividing by (Δr) and taking limits $\Delta r \rightarrow 0$

$$\frac{(4\pi r^2)q_r|_r - (4\pi r^2)q_r|_{r+\Delta r}}{\Delta r} = 0$$

we get, $$-\frac{d}{dr}(4\pi r^2 q_r) = 0$$ or, $$\frac{d}{dr}(r^2 q_r) = 0$$

Integrating, $r^2 q_r = C_1$

Now, $q_r = -k\frac{dT}{dr}$ Putting this in the previous equation, $-kr^2\frac{dT}{dr} = C_1$ or, $\frac{dT}{dr} = -\frac{C_1}{kr^2}$

or, $$T = \frac{C_1}{k}.\frac{1}{r} + C_2$$

Putting boundary conditions, at $r = r_i$ $T = T_i$ and at $r = r_o$ $T = T_o$

$$T_i = \frac{C_1}{k} \cdot \frac{1}{r_i} + C_2 \quad \text{and} \quad T_o = \frac{C_1}{k} \cdot \frac{1}{r_o} + C_2$$

Solving the two equations, we get,

$$C_1 = \frac{k(T_i - T_o)(r_i r_o)}{(r_o - r_i)} \quad \text{and} \quad C_2 = T_i - \frac{(T_i - T_o)(r_o)}{(r_o - r_i)}$$

Substituting these values of the constants, we get,

$$\boxed{T = T_i - \frac{(T_i - T_o) r_o}{(r_o - r_i)} \left(1 - \frac{r_i}{r}\right)}$$

This is the **temperature profile in a sphere**

Rate of heat conduction, $Q = 4\pi r^2 q_r$ or, $Q = 4\pi C_1$

$$Q = \frac{4\pi k (T_i - T_o)(r_i r_o)}{(r_o - r_i)} = \frac{4\pi k (T_i - T_o)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}$$

$$\boxed{Q = \frac{(T_i - T_o)}{\frac{(r_o - r_i)}{4\pi k (r_i r_o)}}}$$

Steady state heat conduction with heat generation

From the generalized equation for heat generation

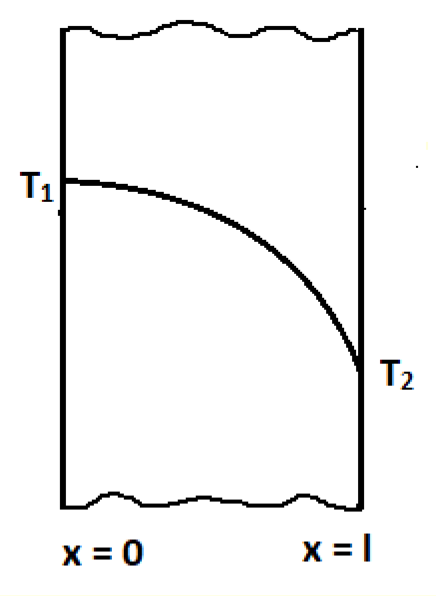
$$k\,\frac{d^2T}{dx^2} + \dot{q} = \rho C_p \frac{dT}{dt}$$

At steady state, $k\,\frac{d^2T}{dx^2} + \dot{q} = 0$

or, $\frac{d^2T}{dx^2} = -\,\frac{\dot{q}}{k}$

or, $\frac{d}{dx}\left(\frac{dT}{dx}\right) = -\,\frac{\dot{q}}{k}$

or, $\left(\frac{dT}{dx}\right) = -\,\frac{\dot{q}}{k}x + C_1$



Integrating we have, $T = -\,\frac{\dot{q}}{2k}x^2 + C_1x + C_2$

Now at, $x = 0$ $T = T_1$ and at $x = l$ $T = T_2$

Therefore, $C_2 = T_1$ and $C_1 = \frac{(T_2 - T_1)}{l} + \frac{\dot{q}l}{2k}$

Putting the values of the constant, we have

$$T = T_1 + \frac{(T_2 - T_1)x}{l} + \frac{\dot{q}x}{2k}(l - x)$$

Temperature profile in the wall will be parabolic

Rate of heat loss from the surfaces

Rate of heat loss from unit area at the surface at $x = 0$

$$-[q_x]_{x=0} = -\left[-k\frac{dT}{dx}\right]_{x=0} = -\frac{\dot{q}xk}{k} + C_1 k = \frac{(T_2 - T_1)k}{l} + \frac{\dot{q}l}{2} - \dot{q}(0)$$

$$\boxed{-[q_x]_{x=0} = \frac{(T_2 - T_1)k}{l} + \frac{\dot{q}l}{2}}$$

Rate of heat loss at $x = 0$ occurs in the opposite direction to that of increasing x, hence negative sign

Rate of heat loss from unit area at the surface at $x = l$

$$[q_x]_{x=l} = \left[-k\frac{dT}{dx}\right]_{x=l} = \frac{\dot{q}xk}{k} - C_1 k = -\frac{(T_2 - T_1)k}{l} - \frac{\dot{q}l}{2} + \dot{q}(l)$$

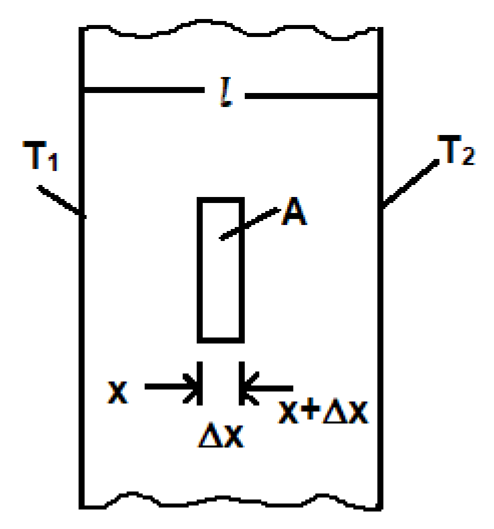
$$\boxed{[q_x]_{x=l} = \frac{\dot{q}l}{2} - \frac{(T_2 - T_1)k}{l}}$$

Average temperature of a solid

- For a solid undergoing heat transfer, there is a temperature distribution in the solid (plane wall, cylinder, sphere)
- It is sometimes necessary to determine the average temperature of a solid

Plane wall

For a plane wall of thickness, l, area, A, specific heat, C_p, and density, ρ and having a temperature distribution, $T = T(x)$



Amount of heat energy contained in an element of thickness, dx is = $(Adx)\rho C_p T$

Total energy in the wall can be obtained by integrating this quantity

Again, if T_{av} is the average temperature of the wall, the total amount of heat energy in it is = $(Al)\rho C_p T_{av}$

$$\therefore\ (Al)\rho C_p T_{av} = \int_0^l (Adx)\rho C_p T(x) = A\rho C_p \int_0^l T(x)dx$$

$$\therefore\ \boxed{T_{av} = \frac{1}{l}\int_0^l T(x)dx}$$

Cylinder

For a cylinder of length, L, radius, R, specific heat, C_p, and density, ρ and having a temperature distribution, $T = T(r)$

Amount of heat energy contained in the volume of an elementary shell of thickness, dr is = $(2\pi r dr L)\rho C_p T(r)$

Total energy in the wall can be obtained by integrating this quantity

Again, if T_{av} is the average temperature of the wall, the total amount of heat energy in it is = $(\pi R^2 L)\rho C_p T_{av}$

$$\therefore (\pi R^2 L)\rho C_p T_{av} = \int_0^R (2\pi r dr L)\rho C_p T(r) = 2\pi L \rho C_p \int_0^R rT(r)dr$$

$$\therefore \quad \boxed{T_{av} = \frac{2}{R^2}\int_0^R rT(r)dr}$$

Sphere

For a sphere, the average temperature can be estimated by,

$$\boxed{T_{av} = \frac{3}{R^3}\int_0^R r^2 T(r)dr}$$

Problem

The steady state temperature distribution in a 0.3 m thick plane wall is $T = 600 + 2500x - 12000x^2$

where T is in °C and x is measured from one surface of the wall. One dimensional steady state heat conduction occurs in the wall along the x-direction. The thermal conductivity of the material of the wall is 23.5 W/m °C

(a) What are the surface temperatures and the average temperature of the wall?

(b) Calculate the maximum temperature in the wall and it's location.

(c) Calculate the heat fluxes at the surface

(d) Do you think there is heat generation in the wall? If so, what is the average volumetric rate of heat generation?

(a) The temperature at $x = 0$ (left surface) is, $T(x = 0) = 600$°C and

The temperature at $x = 0.3$ (right surface) is, $T(x = 0.3) = 600 + 2500(0.3) - 12000(0.3)^2 = 270$°C

Average temperature of the wall, $T_{av} = \frac{1}{l}\int_0^l T(x)dx = \frac{1}{0.3}\int_0^{0.3}(600 + 2500x - 12000x^2)dx$

$$T_{av} = \frac{1}{0.3}\left[600 \times 0.3 + 250 \times \frac{(0.3)^2}{2} - 12000 \times \frac{(0.3)^3}{3}\right] = \frac{1}{0.3}[180 + 112.5 - 10.8] = \frac{184.5}{0.3} = 615°\text{C}$$

(b) Maximum temperature in the wall occurs when $\frac{dT}{dx} = 0$

$\frac{dT}{dx} = 2500 - 24000x = 0$ or $x = 0.1042\ m$

$T(maximum) = 600 + 2500(0.1042) - 12000(0.1042)^2 = 730.2$°C

(c) Temperature gradient in the wall $= \frac{dT}{dx} = 2500 - 24000x$

Heat flux at $x = 0$, $[q_x]_{x=0} = -\left[k\frac{dT}{dx}\right]_{x=0} = -23.5 \times 2500 = -58750\ W/m^2$

Heat flux at $x = 0.3$, $[q_x]_{x=0.3} = -\left[k\frac{dT}{dx}\right]_{x=0.3} = -23.5[2500 - 24000 \times 0.3\ ] = 110450\ W/m^2$

(d) Heat flow at left wall is opposite to that in the right wall

As heat is conducted from both walls, there must be a source of heat in the wall

Total heat loss = $110450 + 58750 = 169200\ W/m^2$

If surface area is 1 m^2, volume per unit surface area is = 0.3 m^3/m^2

Therefore, average volumetric rate of heat generation = $\frac{169200}{0.3}\frac{W/m^2}{m^3/m^2} = 564000\ W/m^3$