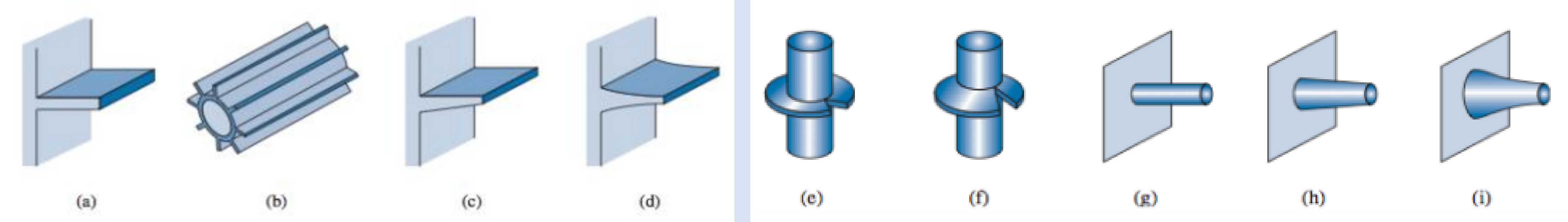
Convection

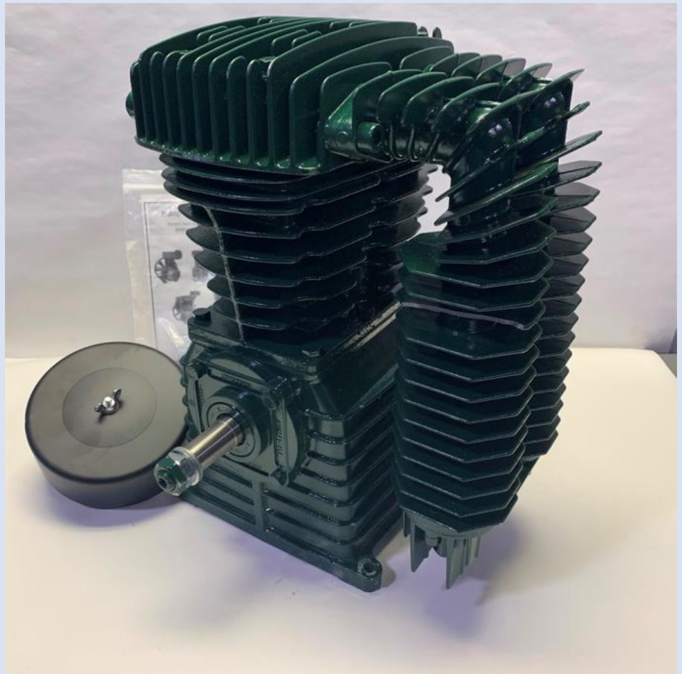
Heat transfer from extended surfaces

- In some practical operations such as the *heat exchange between a gas and a liquid*, a very large heat transfer area is required as the gas-phase heat transfer coefficient is usually very low
- A larger requirement of heat transfer area would mean a very large heat exchanger equipment
- This is often not practically feasible and alternate arrangements can be made to increase the surface area on the gas side
- When additional metal pieces are attached to ordinary heat transfer surfaces such as pipes or tubes, they *extend the surface for heat transfer*
- The rectangular metal strips or annular rings used to increase the heat transfer area are called ***fins***
- The finned surface is also known as ***extended surfaces***

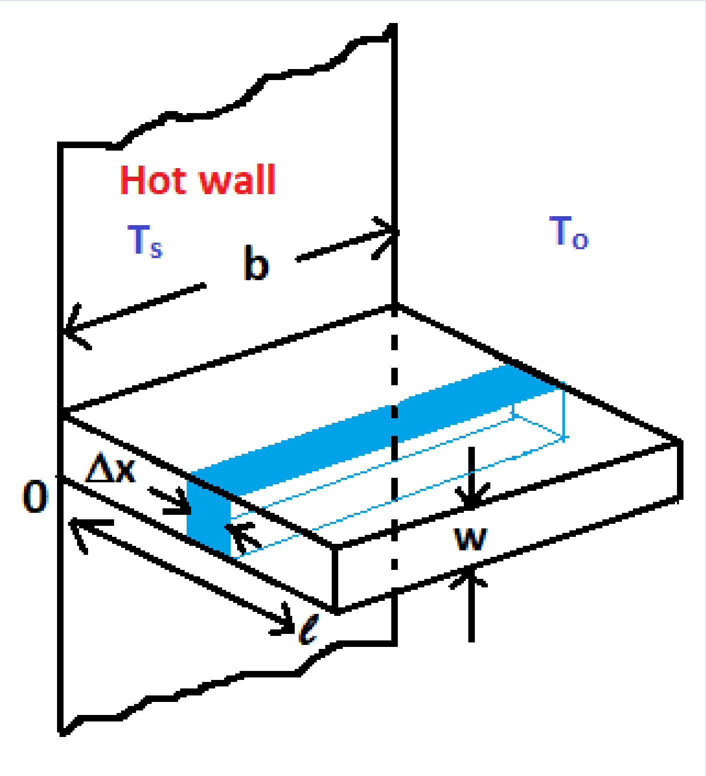
Fins are the extended surface protruding from a surface or body and they are meant for increasing the heat transfer rate between the surface and the surrounding fluid by increasing heat transfer area

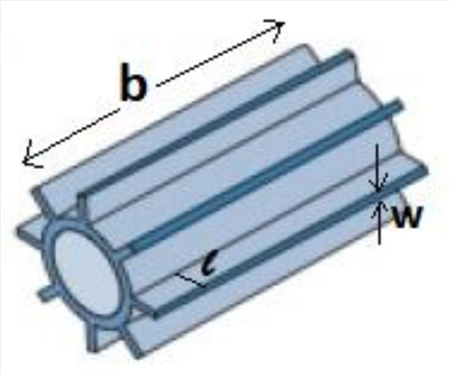


(a) Longitudinal fin – Rectangular profile
(b) Longitudinal fin – Rectangular profile
(c) Longitudinal fin - Trapezoidal profile
(d) Longitudinal fin - Concave parabolic
(e) Radial fin – Rectangular profile
(f) Radial fin – Triangular profile
(g) Pin fin – Cylindrical
(h) Pin fin – Tapered profile
(i) Pin fin – Concave parabolic



- Let us consider a fin (rectangular) that protrudes a distance 'l' from a wall as shown in the figure
- The fin has a length $'l'$, thickness, $'w'$ and breadth 'b'
- The temperature of the wall is T_S and the ambient temperature is T_o
- The temperature at any transverse section of the fin is constant
- The area for heat conduction is $= A = bw$
- The area for heat loss by convection from the exposed fin surface to the ambient $=$ $2b\Delta x + 2w\Delta x = 2(b+w)\Delta x = P\Delta x$
  where $P = perimeter = 2(b+w)$





Making a balance over the small element Δx,

Rate of heat input at x = $Aq_x|_x = -kA\left.\frac{dT}{dx}\right|_x$

Rate of heat output at $x + \Delta x$ = $Aq_x|_{x+\Delta x} = -kA\left.\frac{dT}{dx}\right|_{x+\Delta x}$

Rate of heat loss from extended surface = $hP\Delta x(T - T_o)$

Rate of heat accumulation at steady state = 0

Therefore,

$$-kA\left.\frac{dT}{dx}\right|_x - \left(-kA\left.\frac{dT}{dx}\right|_{x+\Delta x}\right) - hP\Delta x(T - T_o) = 0$$

$$-kA\left.\frac{dT}{dx}\right|_{x} - \left(-kA\left.\frac{dT}{dx}\right|_{x+\Delta x}\right) - hP\Delta x(T - T_o) = 0$$

$$Lt_{\Delta x \to 0} \quad \frac{kA\left.\frac{dT}{dx}\right|_{x+\Delta x} - kA\left.\frac{dT}{dx}\right|_{x}}{\Delta x} - hP(T - T_o) = 0$$

$$kA\frac{d^2T}{dx^2} - hP(T - T_o) = 0$$

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_o) = 0$$

Let $\theta = (T - T_o)$ and $m^2 = \frac{hP}{kA}$

$$\boxed{\frac{d^2\theta}{dx^2} - m^2\theta = 0}$$

$\Rightarrow$ **Second order linear differential equation**

General solution $\boldsymbol{\theta = C_1 e^{+mx} + C_2 e^{-mx}}$ $\Rightarrow$ **gives the temperature distribution**

The boundary conditions used to determine C_1 and C_2 is of three types

Case I : Fin is very long (infinite), temperature at the end of the fin is the same as the surrounding fluid

$$\boldsymbol{\theta = C_1 e^{+mx} + C_2 e^{-mx}}$$

At $x = 0,\quad T = T_s$ and $x = 0,\quad \theta_1 = T_s - T_o$

At $x = \infty,\quad T = T_o$ and $x = \infty,\quad \theta_2 = 0$

Using Boundary Condition 1 $\quad \theta_1 = C_1 + C_2$

Using Boundary Condition 2 $\quad 0 = C_1 e^{+\infty} + C_2 e^{-\infty}$

Since $e^{-\infty} = 0,\quad C_1 = 0$ and $C_2 = \theta_1$

$$\theta = \theta_1 e^{-mx}$$

or,

$$\frac{T - T_o}{T_s - T_o} = e^{-mx}$$

$\Rightarrow$ **temperature distribution in fin for Case I**

Heat loss from the fin, $Q = -kA\left(\frac{dT}{dx}\right)_{x=0}$

$\theta = (T - T_o)$ and $\frac{d\theta}{dx} = \frac{dT}{dx}$

$$Q = -kA\left(\frac{d\theta}{dx}\right)_{x=0}$$

$$\theta = \theta_1 e^{-mx}$$

$$\frac{d\theta}{dx} = -m\theta_1 e^{-mx}$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = -m\theta_1$$

or, $\quad Q = -kA(-m\theta_1) = kAm\theta_1$

or, $\quad Q = kA\sqrt{\frac{hP}{kA}}(T_s - T_o)$

or, $\quad Q = \sqrt{hPkA}(T_s - T_o)$

Case 2 : The end of the fin is insulated

At $x = 0,\ T = T_s$ and $x = 0,\ \theta_1 = T_s - T_o$

At $x = l,\ \frac{dT}{dx} = 0$ and $x = l,\ \frac{d\theta}{dx} = 0$

Now,

$$\theta = C_1 e^{+mx} + C_2 e^{-mx}$$

$$\frac{d\theta}{dx} = mC_1 e^{+mx} - mC_2 e^{-mx}$$

Using Boundary Condition 1 $\theta_1 = C_1 + C_2$

Using Boundary Condition 2 $0 = m\left(C_1 e^{+ml} - C_2 e^{-ml}\right)$

$$C_1 e^{+ml} = C_2 e^{-ml}$$

$$C_2 = C_1 e^{2ml}$$

$$\theta_1 = C_1 + C_2 = C_1 + C_1 e^{2ml} = C_1\left(1 + e^{2ml}\right)$$

$$C_1 = \frac{\theta_1}{(1 + e^{2ml})}$$

$$C_2 = \frac{\theta_1 e^{2ml}}{(1 + e^{2ml})} = \frac{\theta_1}{(1 + e^{-2ml})}$$

Putting the values of constants we get,

$$\theta = \frac{\theta_1}{(1 + e^{2ml})} e^{+mx} + \frac{\theta_1}{(1 + e^{-2ml})} e^{-mx}$$

$$\theta = \frac{\theta_1}{(1+e^{2ml})}e^{+mx} + \frac{\theta_1}{(1+e^{-2ml})}e^{-mx}$$

Multiplying the top and bottom of the first term by e^{-ml} and top and bottom of the second term by e^{ml}

$$\theta = \theta_1\left[\frac{e^{-m(l-x)}}{(e^{ml}+e^{-ml})} + \frac{e^{m(l-x)}}{(e^{ml}+e^{-ml})}\right] = \theta_1\left[\frac{e^{-m(l-x)}+e^{m(l-x)}}{(e^{ml}+e^{-ml})}\right]$$

$$sinh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2}$$

$$cosh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2}$$

$$tanh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{e^{\alpha}+e^{-\alpha}}$$

$$\theta = \theta_1\left[\frac{cosh[m(l-x)]}{cosh[ml]}\right]$$

$$\boxed{\frac{T-T_o}{T_s-T_o} = \frac{cosh[m(l-x)]}{cosh[ml]}}$$

$\Rightarrow$ **temperature distribution in fin for Case 2**

Heat loss from the fin, $Q = -kA\left(\frac{dT}{dx}\right)_{x=0}$

$$Q = -kA\left(\frac{d\theta}{dx}\right)_{x=0}$$

$$\frac{d\theta}{dx} = mC_1e^{+mx} - mC_2e^{-mx}$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = mC_1 - mC_2$$

$$= \frac{m\theta_1}{(1+e^{2ml})} - \frac{m\theta_1}{(1+e^{-2ml})} = -m\theta_1\left[\frac{1}{(1+e^{-2ml})} - \frac{1}{(1+e^{2ml})}\right]$$

$$= -m\theta_1\left[\frac{e^{ml}}{(e^{ml}+e^{-ml})} - \frac{e^{-ml}}{(e^{-ml}+e^{ml})}\right] = -m\theta_1\left[\frac{e^{ml}-e^{-ml}}{e^{ml}+e^{-ml}}\right] = -m\theta_1 tanh(ml)$$

$$Q = -kA\left(\frac{d\theta}{dx}\right)_{x=0} = -kA(-m\theta_1)\, tanh(ml)$$

$$Q = kA\sqrt{\frac{hP}{kA}}(T_s - T_o)tanh(ml)$$

$$\boxed{Q = \sqrt{hPkA}(T_s - T_o)tanh(ml)}$$

Case 3 : The fin is of finite length and loses heat by convection from its end

At $x = 0,\quad T = T_s$ and $x = 0,\quad \theta_1 = T_s - T_o$

At $x = l,\quad -kA\frac{dT}{dx} = hA(T - T_o)$ and $x = l,\quad -kA\frac{d\theta}{dx} = hA\theta$

Now,

$$\theta = C_1 e^{+mx} + C_2 e^{-mx}$$

$$\frac{d\theta}{dx} = mC_1 e^{+mx} - mC_2 e^{-mx}$$

Using Boundary Condition 1 $\quad \theta_1 = C_1 + C_2$

Using Boundary Condition 2 $\quad \frac{d\theta}{dx} = -\frac{h}{k}\theta$

$$m[C_1 e^{+ml} - C_2 e^{-ml}] = -\frac{h}{k}[C_1 e^{+ml} + C_2 e^{-ml}]$$

$$C_1\left[e^{+ml} + \frac{h}{mk}e^{+ml}\right] = C_2\left[e^{-ml} - \frac{h}{mk}e^{-ml}\right]$$

Now, $\quad C_1 = \theta_1 - C_2$

Replacing the value of C_1 $(= \theta_1 - C_2)$ in the previous equation, $\quad (\theta_1 - C_2)\left[e^{+ml} + \frac{h}{mk}e^{+ml}\right] = C_2\left[e^{-ml} - \frac{h}{mk}e^{-ml}\right]$

$$\theta_1 e^{+ml} + \frac{h}{mk}\theta_1 e^{+ml} - C_2 e^{+ml} - \frac{h}{mk}C_2 e^{+ml} = C_2 e^{-ml} - C_2\frac{h}{mk}e^{-ml}$$

$$C_2 = \frac{\theta_1 e^{ml}\left(1 + \frac{h}{mk}\right)}{(e^{+ml} + e^{-ml}) + \frac{h}{mk}(e^{+ml} - e^{-ml})}$$

Now, $C_1 = \theta_1 - C_2$

Replacing the value of C_2 in the above equation enables us to find the value of C_1

$$C_1 = \theta_1 - C_2 = \theta_1 - \frac{\theta_1 e^{ml}\left(1+\frac{h}{mk}\right)}{\left(e^{+ml}+e^{-ml}\right)+\frac{h}{mk}\left(e^{+ml}-e^{-ml}\right)}$$

$$C_1 = \frac{\theta_1 e^{+ml}+\theta_1 e^{-ml}+\theta_1\frac{h}{mk}e^{+ml}-\theta_1\frac{h}{mk}e^{-ml}-\theta_1 e^{ml}-\theta_1\frac{h}{mk}e^{+ml}}{\left(e^{+ml}+e^{-ml}\right)+\frac{h}{mk}\left(e^{+ml}-e^{-ml}\right)}$$

$$C_1 = \frac{\theta_1 e^{-ml}\left(1-\frac{h}{mk}\right)}{\left(e^{+ml}+e^{-ml}\right)+\frac{h}{mk}\left(e^{+ml}-e^{-ml}\right)}$$

Replacing the values of C_1 and C_2 in $\theta = C_1 e^{+mx} + C_2 e^{-mx}$ we get,

$$\theta = \frac{\theta_1 e^{-ml}\left(1-\frac{h}{mk}\right)e^{+mx}}{\left(e^{+ml}+e^{-ml}\right)+\frac{h}{mk}\left(e^{+ml}-e^{-ml}\right)} + \frac{\theta_1 e^{ml}\left(1+\frac{h}{mk}\right)e^{-mx}}{\left(e^{+ml}+e^{-ml}\right)+\frac{h}{mk}\left(e^{+ml}-e^{-ml}\right)}$$

$$\frac{\theta}{\theta_1} = \frac{\left[e^{m(l-x)}+e^{-m(l-x)}\right]+\left(\frac{h}{mk}\right)\left[e^{m(l-x)}-e^{-m(l-x)}\right]}{\left(e^{+ml}+e^{-ml}\right)+\frac{h}{mk}\left(e^{+ml}-e^{-ml}\right)}$$

$$\frac{\theta}{\theta_1} = \frac{T-T_o}{T_s-T_o} = \frac{\cosh[m(l-x)]+\left(\frac{h}{mk}\right)\sinh[m(l-x)]}{\cosh[ml]+\left(\frac{h}{mk}\right)\sinh[ml]}$$

$\Rightarrow$ **temperature distribution in fin for Case 3**

Heat loss from the fin, $Q = -kA\left(\frac{dT}{dx}\right)_{x=0} = -kA\left(\frac{d\theta}{dx}\right)_{x=0}$

$$\frac{d\theta}{dx} = mC_1 e^{+mx} - mC_2 e^{-mx}$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = mC_1 - mC_2 = \frac{m\theta_1 e^{-ml}\left(1 - \frac{h}{mk}\right)}{(e^{+ml} + e^{-ml}) + \frac{h}{mk}(e^{+ml} - e^{-ml})} - \frac{m\theta_1 e^{ml}\left(1 + \frac{h}{mk}\right)}{(e^{+ml} + e^{-ml}) + \frac{h}{mk}(e^{+ml} - e^{-ml})}$$

$$= -m\theta_1\left[\frac{(e^{ml} - e^{-ml}) + \frac{h}{mk}(e^{ml} - e^{-ml})}{(e^{+ml} + e^{-ml}) + \frac{h}{mk}(e^{+ml} - e^{-ml})}\right] = -m\theta_1\left[\frac{sinh(ml) + \frac{h}{mk}cosh(ml)}{cosh(ml) + \frac{h}{mk}sinh(ml)}\right]$$

$$Q = -kA\left(\frac{d\theta}{dx}\right)_{x=0} = -kA(-m\theta_1)\left[\frac{sinh(ml)+\frac{h}{mk}cosh(ml)}{cosh(ml)+\frac{h}{mk}sinh(ml)}\right]$$

$$Q = kA\sqrt{\frac{hP}{kA}}(T_s - T_o)\left[\frac{sinh(ml) + \frac{h}{mk}cosh(ml)}{cosh(ml) + \frac{h}{mk}sinh(ml)}\right]$$

$$Q = \sqrt{hPkA}(T_s - T_o)\left[\frac{tanh(ml) + \frac{h}{mk}}{1 + \frac{h}{mk}tanh(ml)}\right]$$

Fin efficiency

- To indicate the effectiveness of a fin in transferring a given quantity of heat, a new parameter called the **fin efficiency** is defined

$$\boldsymbol{Fin\ efficency} = \eta_f = \frac{\boldsymbol{actual\ heat\ transferred}}{\boldsymbol{heat\ that\ would\ be\ transferred\ if\ entire\ fin\ area\ were\ at\ base\ temperature\ (max\ heat\ transfer)}}$$

- The fin efficiency can be estimated for all the cases discussed previously

- The heat transfer from a fin is maximum when the fin material has infinite thermal conductivity and the temperature along the length is the base temperature

$$Q_{max} = hA_{convection}(T_s - T_o) = hPl(T_s - T_o)$$

$$P = perimeter = 2(b + w)$$

Case I:

$$Q = \sqrt{hPkA}(T_s - T_o)$$

$$Q_{max} = hPl(T_s - T_o)$$

$$\eta_f = \frac{\sqrt{hPkA}(T_s - T_o)}{hPl(T_s - T_o)} = \frac{1}{l}\sqrt{\frac{kA}{hP}}$$

$$\boxed{\eta_f = \frac{1}{ml}}$$

Case 2:

$$Q = \sqrt{hPkA}(T_s - T_o)\tanh(ml)$$

$$Q_{max} = hPl(T_s - T_o)$$

$$\eta_f = \frac{Q = \sqrt{hPkA}(T_s - T_o)\tanh(ml)}{hPl(T_s - T_o)} = \frac{1}{l}\sqrt{\frac{kA}{hP}}\tanh(ml)$$

$$\eta_f = \frac{\tanh(ml)}{ml}$$

Case 3:

$$Q = \sqrt{hPkA}(T_s - T_o)\left[\frac{\tanh(ml) + \frac{h}{mk}}{1 + \frac{h}{mk}\tanh(ml)}\right]$$

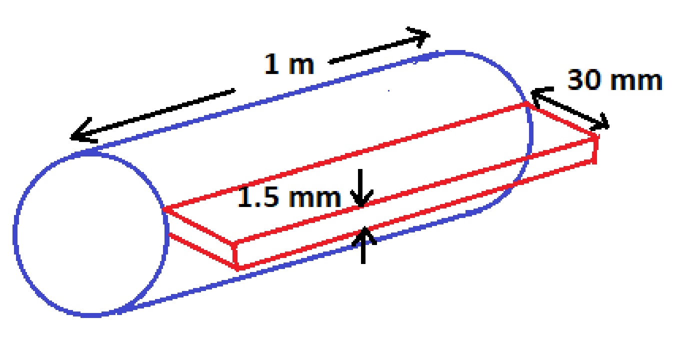
$$Q_{max} = hPl(T_s - T_o)$$

$$\eta_f = \frac{\sqrt{hPkA}(T_s - T_o)\left[\frac{\tanh(ml) + \frac{h}{mk}}{1 + \frac{h}{mk}\tanh(ml)}\right]}{hPl(T_s - T_o)} = \frac{1}{l}\sqrt{\frac{kA}{hP}}\left[\frac{\tanh(ml) + \frac{h}{mk}}{1 + \frac{h}{mk}\tanh(ml)}\right]$$

$$\eta_f = \frac{\left[\frac{\tanh(ml) + \frac{h}{mk}}{1 + \frac{h}{mk}\tanh(ml)}\right]}{ml}$$

Problem

A carbon steel pipe (actual ID = 78 mm, wall thickness = 5.5 mm) has 8 longitudinal fins (with ends insulated) of 1.5 mm thickness. Each fin extends 30 mm from the pipe wall. The thermal conductivity of the fin material is 45 W/m°C. If the wall temperature, the ambient temperature and surface heat transfer coefficients are 150°C, 28°C and 75 W/m²°C, respectively, calculate the % increase in the rate of heat transfer for the finned tube over the plain tube of 1 m each.



The rate of heat transfer from a plain pipe, $Q_o = hA(\Delta T) = h(2\pi r_o L)(T_s - T_o)$

$$Q_o = h(2\pi r_o L)(T_s - T_o) = 75 \times 2 \times \pi \times 0.0445 \times 1 \times (150 - 28) = 2558.36\ W$$

The rate of heat transfer from a finned pipe, $Q_t = Q_o + Q_f$ (Q_f = heat transfer from fins)

$$r_o = r_i + t$$
= 39+5.5 = 44.5 mm

For a finned pipe, $m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{75 \times 2.003}{45 \times 0.0015}} = 47.18$

$$P = 2(b + w)$$
$= 2(1 + 0.0015)$
= 2.003 m

For a fin with insulated ends, $\eta_f = \frac{tanh(47.18 \times 0.03)}{47.18 \times 0.03} = 0.6278$

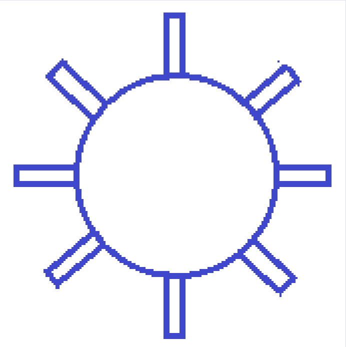
$$A = bw$$
$= 1 \times 0.0015$ = 0.0015 m

$$Q_f = \sqrt{hPkA}(T_s - T_o)tanh(ml)$$

$$Q_f = \sqrt{75 \times 2.003} \times 45 \times 0.0015 \times (150 - 28) tanh(47.18 \times 0.03) = 345.288\ W$$

For eight fins, the total heat transfer due to fins is = $8 \times 345.288 = 2761.83\ W$

Now, the area of contact of a fin with the pipe wall = $1\ \mathrm{m} \times 0.0015\ m = 0.0015\ \mathrm{m}^2$

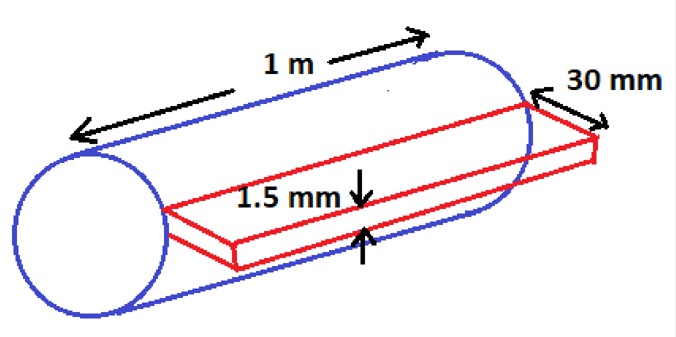


Area of contact of eight fins = $8 \times 0.0015 = 0.012\ \mathrm{m}^2$

Free outside area of finned pipe $= (2 \times \pi \times 0.0445 \times 1) - 0.012 = 0.2796 - 0.012 = 0.2676\ \ \mathrm{m}^2$

Therefore, the rate of heat transfer from the free outside area of pipe,

$Q_o = h(A_o)(T_s - T_o) = 75 \times 0.2676 \times (150 - 28) = 2448\ W$



The total rate of heat transfer from a finned pipe, $Q_t = Q_o + Q_f$

$Q_t = 2448 + 2761.83 = 5209.83\ W$

Percentage (%) in heat transfer using finned pipe $= \frac{5209.83 - 2558.36}{2558.36} \times 100 = 103.64\%$