Introduction to unsteady state heat transfer

Introduction

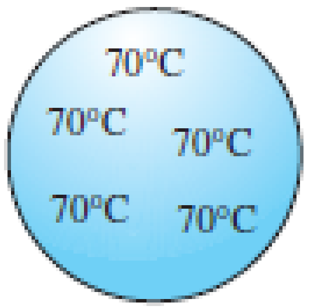
- In the earlier chapters, steady state problems were considered where the temperature at any given point and the heat flux were always constant with time
- However, processes where the temperature at any given point in the system changes with time is known as a ***unsteady state heat transfer process***

- The importance of this type of heat transfer lies in the large number of heating and cooling problems occurring in the industry
- In metallurgical industry it is necessary to predict cooling and heating rates of various geometries of metals in order to predict the time required to reach certain temperatures
- In the food industry, perishable canned foods are heated by immersing in steam baths or chilled by immersing in cold water
- In the paper industry, wood logs are immersed in steam baths before processing

The analysis of unsteady state problems (heat transfer in a body) is carried out by means of two models –

- **Lumped parameter model**
- **Distributed parameter model**

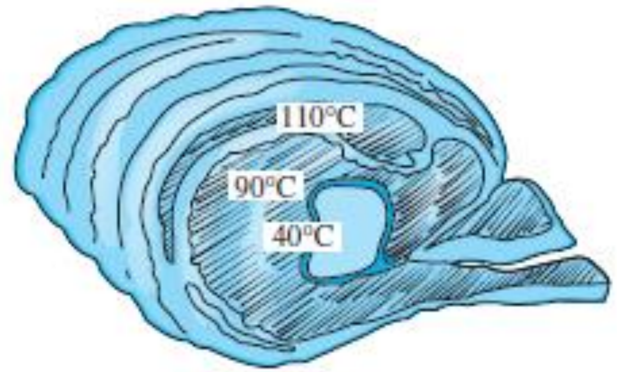
- In the **lumped parameter model**, the temperature of the body is assumed to depend only on time and not on the position



(a) Copper ball

- Eg: Body having high thermal conductivity (low resistance) or small volume – the temperatures at different positions in the body can be assumed to be equal and the only variation is with time

- In the **distributed parameter model**, the temperature in a system depends on both position and time



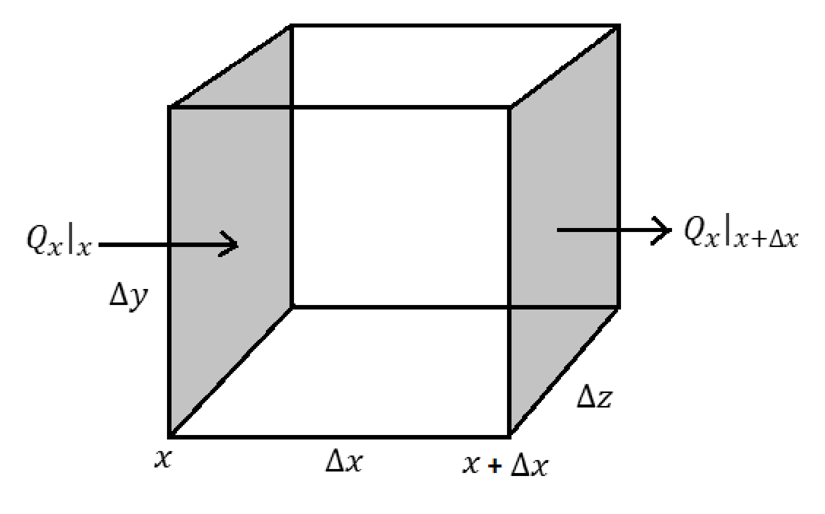
(b) Cooked food

Derivation of unsteady state conduction equation

- We consider unsteady state heat conduction in the x–direction of a solid cube, $\Delta x, \Delta y, \Delta z$ in size
- For conduction in the x–direction, we write

$$Q_x = -kA\frac{\partial T}{\partial x}$$

$\frac{\partial T}{\partial x}$ is the derivative of T with respect to x with the other variables y, z, t being constant



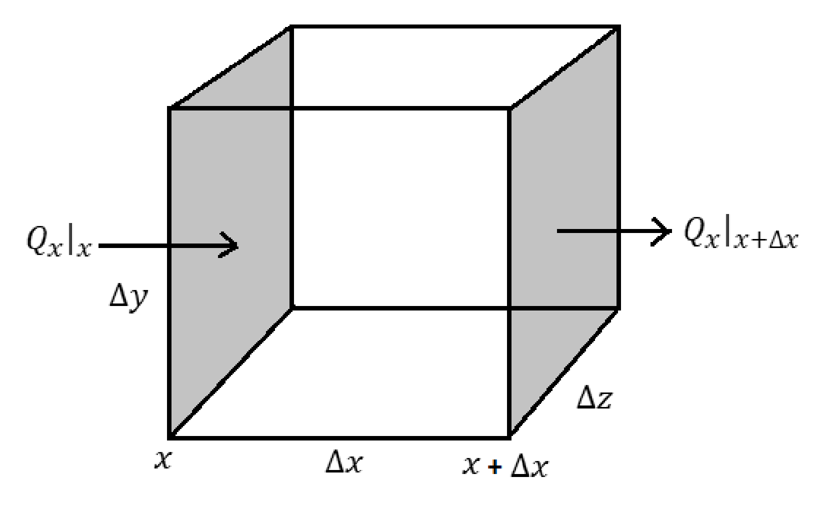
Making a heat balance on the cube,

$$Rate\ of\ heat\ input + Rate\ of\ heat\ generation = Rate\ of\ heat\ output + Rate\ of\ heat\ accumulation$$

- Rate of heat input to the cube = $Q_x|_x = -k(\Delta y \Delta z)\left.\frac{\partial T}{\partial x}\right|_x$
- Rate of heat output from the cube = $Q_x|_{x+\Delta x} = -k(\Delta y \Delta z)\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x}$
- Rate of heat generation = $(\Delta x \Delta y \Delta z)\dot{Q}$
- Rate of heat accumulation in the volume = $(\Delta x \Delta y \Delta z)\rho C_p \frac{\partial T}{\partial t}$

Making a heat balance on the cube,

$$-k(\Delta y \Delta z)\frac{\partial T}{\partial x}\bigg|_x - \left[-k(\Delta y \Delta z)\frac{\partial T}{\partial x}\bigg|_{x+\Delta x}\right] + (\Delta x \Delta y \Delta z)\dot{Q} = (\Delta x \Delta y \Delta z)\rho C_p \frac{\partial T}{\partial t}$$



Dividing by $(\Delta x \Delta y \Delta z)$ and taking $Lt_{\Delta x \to 0}$

$$k\frac{\partial^2 T}{\partial x^2} + \dot{Q} = \rho C_p \frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{Q}}{\rho C_p}$$

$$\boxed{\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{\dot{Q}}{\rho C_p}}$$

If the heat conduction is in three directions (x, y, z),

$$\boxed{\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\dot{Q}}{\rho C_p}}$$

If the rate of heat generation is zero ($\dot{Q}$ = 0), the equations become,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

or,

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

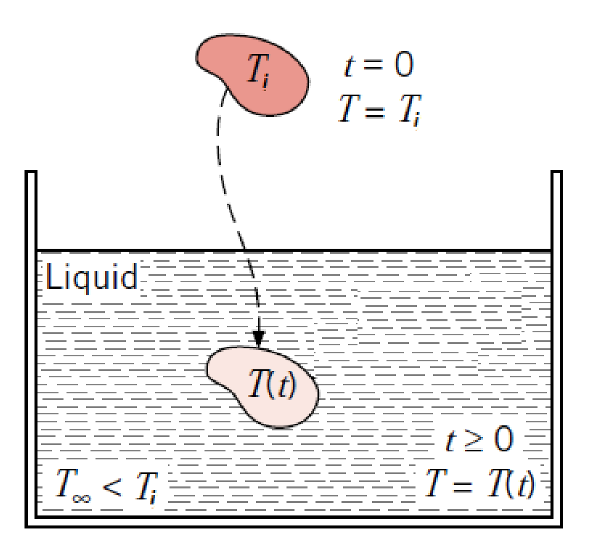
The solution to the above equations help relate the T with position x, y, z and t

Systems with negligible internal resistance (lumped parameter model)

- Here, unsteady or transient heat conduction is considered for systems that may be considered uniform in temperature - this type of analysis is called the *lumped-heat-capacity model* or *lumped parameter model*
- A solid is considered which has a very high thermal conductivity (or very low internal conductive resistance compared to the external surface resistance)
- Heat is transferred from the external fluid to the surface of the solid
- The smaller the physical size of the body, the more realistic the assumption of a uniform temperature throughout

Systems with negligible internal resistance (lumped parameter model)

- A hot solid body of steel has a temperature of T_i at $t = 0$ (initial temperature)
- This body is suddenly immersed into a large bath of cold water which has a temperature of T_∞ (held constant with time)
- Heat transfer coefficient, h also remains constant with time
- Making a heat balance on the small object for a small interval of time dt, the heat transfer from the bath to the object must be equal to the change in internal energy of the object
- If the solid body has a volume = V, area = A, density = ρ, heat capacity = C_p, heat balance is written as,



$$hA(T_\infty - T) = \rho V C_p \frac{dT}{dt}$$

$$\int_{T_i}^{T} \frac{dT}{(T_\infty - T)} = \frac{hA}{\rho V C_p} \int_{0}^{t} dt$$

$$ln\left[\frac{T_\infty - T_i}{T_\infty - T}\right] = \frac{hA}{\rho V C_p} t$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\left(\frac{hA}{\rho V C_p}\right)t}$$

This is called the ***lumped parameter model***

Lumped parameter model

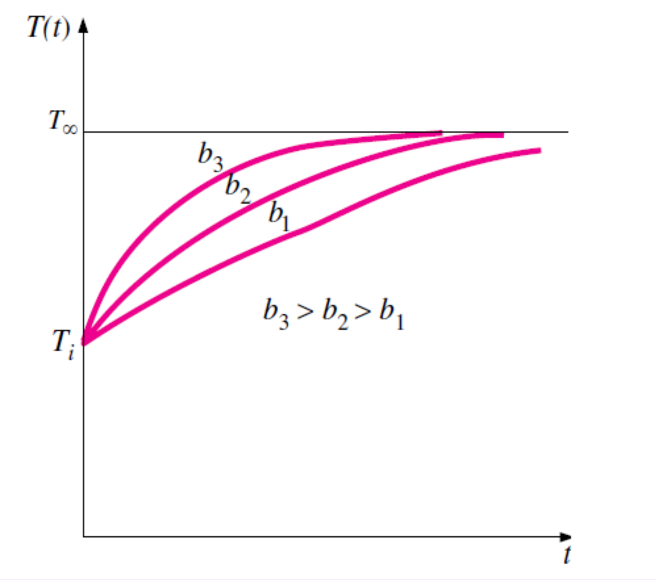
$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\left(\frac{hA}{\rho V C_p}\right)t}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-bt}$$

The reciprocal of ***b*** has time unit, and is called the **time constant**

where $b = \left(\frac{hA}{\rho V C_p}\right)$

- The temperature of a body approaches the ambient temperature T_∞ exponentially
- The temperature of the body changes rapidly at the beginning, but slowly later on
- A large value of *b* indicates that the body approaches T_∞ in a short time
- The larger the value of the exponent *b*, the higher the rate of decay in temperature
- *b* is proportional to the surface area, but inversely proportional to the mass and the specific heat of the body
- This is not surprising since it takes longer to heat or cool a larger mass, especially when it has a large specific heat



Problem

A 2 kg copper ball at 200°C cools down in ambient air at 29°C. If it requires one hour to cool the ball down to 35°C, calculate the average value of the surface heat transfer coefficient.

As copper is an excellent conductor of heat, it may be assumed that the temperature in the ball remains uniform at any instant.

For copper, $\rho = 8950$ kg/m^3, C_p = 0.383 kJ/kg°C, $k = 0.377$ W/m°C

The following derived expression for the lumped parameter model is used to solve the problem,

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\left(\frac{hA}{\rho V C_p}\right)t}$$

Area $(A) = 4\pi r^2$ Volume $(V) = \frac{4}{3}\pi r^3$ $\frac{A}{V} = \frac{3}{r}$

Now, $V = \frac{4}{3}\pi r^3 = \frac{M}{\rho}$ $r^3 = \frac{2}{8950} \times \frac{3}{4\pi}$ $r = 0.0376\, m$

$$\frac{35 - 29}{200 - 29} = exp\left[-\left(\frac{h \times 3}{8950 \times 0.0376 \times 0.383 \times 1000}\right)3600\right]$$

$$ln\left[\frac{35 - 29}{200 - 29}\right] = -0.083794 \times h$$

$h = 39.98$ W/m^{2o}C

Problem

A solid cube of side 30 cm at an initial temperature of 1000 K is kept in vacuum at absolute zero temperature. Calculate the time required to cool the cube to 500 K.

The material has the following properties:

$\rho = 2700$ kg/m^3, C_p = 0.9 kJ/kg^{o}C, $\varepsilon = 0.1$ Stefan-Boltzman constant, σ = 5.669 x 10^{-8} W/m^{2}K^4

Since the atmosphere is vacuum, the heat transfer is only by radiation.

$$\varepsilon\sigma A(T^4 - T_a^4) = -\rho V C_p \frac{dT}{dt}$$

Area $(A) = 6a^2 = 6 \times (0.3)^2 = 0.54$ m^2

$$0.1 \times 5.669 \times 10^{-8} \times 0.54(T^4 - 0) = -(0.3)^3 \times 2700 \times 0.9 \times 1000 \frac{dT}{dt}$$

$$3.06126 \times 10^{-9}(T^4) = -65610 \frac{dT}{dt}$$

$$-21.432 \times 10^{11} \int_{1000}^{500} \frac{dT}{T^4}(T^4) = \int_0^t dt$$

$$21.432 \times 10^{11} \left[\frac{T^{-3}}{3}\right]_{1000}^{500} = t$$

$$t = \frac{21.432 \times 10^{11}}{3}\left[\frac{1}{500^3} - \frac{1}{1000^3}\right] \quad \Rightarrow \quad t = 50008\, sec = 13.89\, h$$

Application of lumped parameter model

- The lumped-capacity type of analysis or lumped parameter model assumes a uniform temperature distribution throughout the solid body
- This is equivalent to saying that the **surface-convection resistance is large compared with the internal-conduction resistance**
- The lumped parameter model is expected to yield reasonable estimates when the following condition is met:

$$\frac{hx_L}{k} < 0.1$$

x_L is the characteristic dimension of the body

- The term on the left hand side is called the Biot Number – a ratio of the internal conduction resistance to surface convective resistance to heat transfer

- For sphere, $x_L = \frac{V}{A} = \frac{4\pi r^3/3}{4\pi r^2} = \frac{r}{3}$
- For long cylinder, $x_L = \frac{V}{A} = \frac{\pi D^2/4 \times L}{\pi D L} = \frac{D}{4} = \frac{r}{2}$
- For long square rod, $x_L = \frac{V}{A} = \frac{(2x)^2 \times L}{4(2x)L} = \frac{x}{2}$

  $(x = \frac{1}{2}\ thickness)$

- Values of Biot No must be calculated to ensure that the criterion is satisfied

Examples of lumped-capacity systems

Physical situation	k, W/m · °C	Approximate value of h, W/m^2 · °C	$\frac{h(V/A)}{k}$
1. 3.0-cm steel cube cooling in room air	40	7.0	8.75×10^{-4}
2. 5.0-cm glass cylinder cooled by a 50-m/s airstream	0.8	180	2.81
3. Same as situation 2 but a copper cylinder	380	180	0.006
4. 3.0-cm hot copper cube submerged in water such that boiling occurs	380	10,000	0.132