Introduction to unsteady state heat transfer

Unsteady state heat conduction in various geometries

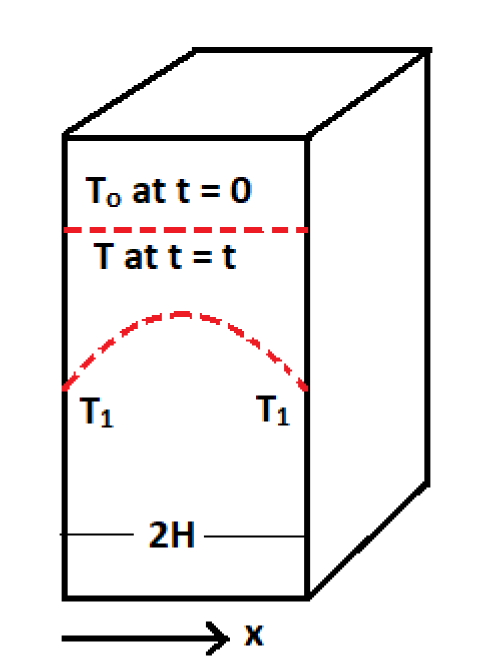
A general situation is considered where the internal resistance is not small (unlike the lumped parameter model) and the temperature in the solid is not constant

- In the first case, the surface convective resistance is negligible compared to the internal resistance
- This is possible due to very large surface heat transfer coefficient or large conductive resistance in the object
- Unsteady state conduction in the $x - direction$ only in a flat plate of thickness $2H$
- The initial temperature profile in the plate at $t = 0$ is uniform at $T = T_o$
- At time $t = 0$, the ambient temperature is suddenly changed to T_1 and held there
- As there is no convective heat transfer resistance, the temperature of the surface is held constant at T_1
- Since conduction is only in the x-direction,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

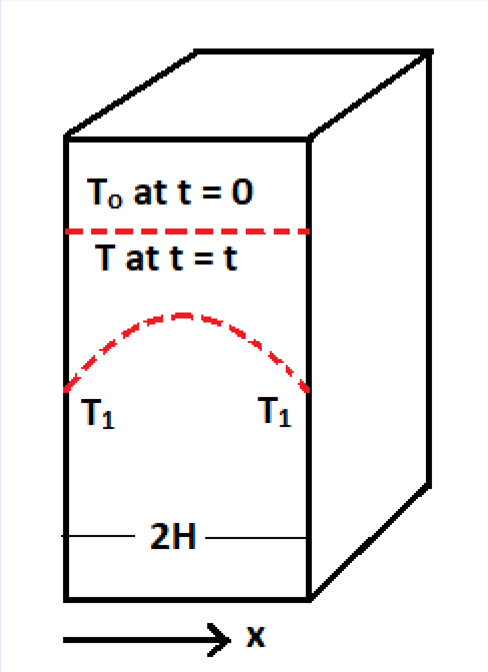
- The initial and boundary conditions are:

$$T = T_o \ , \qquad t = 0 \qquad x = x\ (at\ all\ x)$$
$$T = T_1 \ , \qquad t = t \qquad x = 0$$
$$T = T_1 \ , \qquad t = t \qquad x = 2H$$



It is often more convenient to define a dimensionless temperature Y, so it varies between 0 and 1

Hence,
$$Y = \frac{T_1 - T}{T_1 - T_o}$$



Substituting this we get,

$$\frac{\partial Y}{\partial t} = \alpha \frac{\partial^2 Y}{\partial x^2}$$

Redefining the initial and boundary conditions we get:

$$Y = \frac{T_1 - T_o}{T_1 - T_o} = 1, \qquad t = 0 \qquad x = x \ (at\ all\ x)$$
$$Y = \frac{T_1 - T_1}{T_1 - T_o} = 0, \qquad t = t \qquad x = 0$$
$$Y = \frac{T_1 - T_1}{T_1 - T_o} = 0, \qquad t = t \qquad x = 2H$$

The solution to the above equation is , $Y = e^{-a^2 \alpha t}(A \cos ax + B \sin ax)$ A and B are constants, a is a parameter

Applying the initial and boundary conditions, the final solution to the above equation is an infinite Fourier series

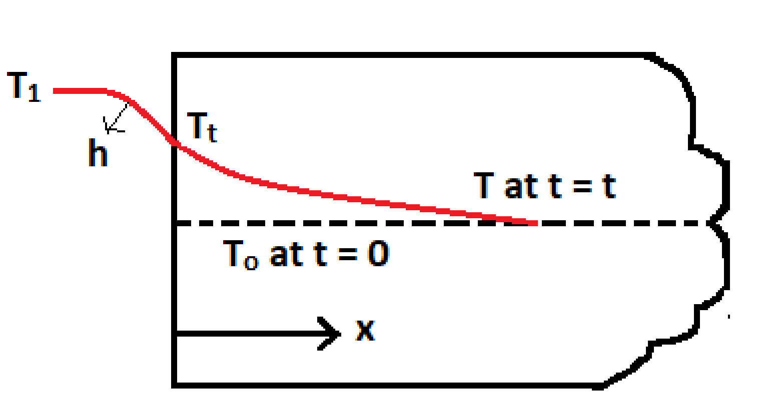
$$\frac{T_1 - T}{T_1 - T_o} = \frac{4}{\pi}\left(\frac{1}{1} exp \frac{-1^2\pi^2\alpha t}{4H^2} sin \frac{1\pi x}{2H} + \frac{1}{3} exp \frac{-3^2\pi^2\alpha t}{4H^2} sin \frac{3\pi x}{2H} + \frac{1}{5} exp \frac{-5^2\pi^2\alpha t}{4H^2} sin \frac{5\pi x}{2H} + \cdots\right)$$

The temperature T at any position x and time t can be determined

These equations are very time-consuming to use and convenient charts have been prepared for different applications

Unsteady state heat conduction in a semi-infinite solid

- A semi-infinite solid extends to ∞ in the $x - direction$ and heat conduction occurs only in the $x - direction$
- The temperature in the solid is uniform at T_o
- At time $t = 0$, the solid is suddenly exposed to or immersed in a large mass of ambient fluid at a temperature of T_1



- The solution to the differential equation under these conditions is

$$\frac{T - T_o}{T_1 - T_o} = 1 - Y = erfc\frac{x}{2\sqrt{\alpha t}} - exp\left[\frac{h\sqrt{\alpha t}}{k}\left(\frac{x}{\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]erfc\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

where x is the distance into the solid from the surface, $erfc$ is the complimentary error function

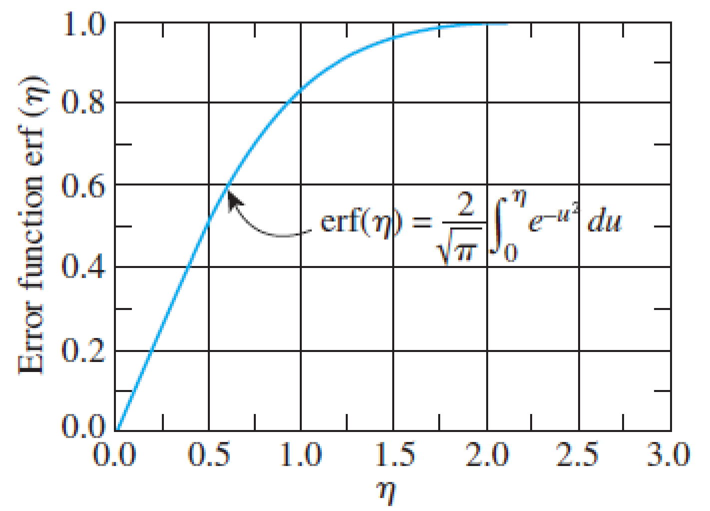
The function $erfc = (1 - erf)$ where erf is the error function

Numerical values of the error function are available in standard text books

(samples are shown in the next slide)

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

Error function is a standard mathematical function, just like the sinus and tangent functions, whose value varies between 0 and 1.



Error function

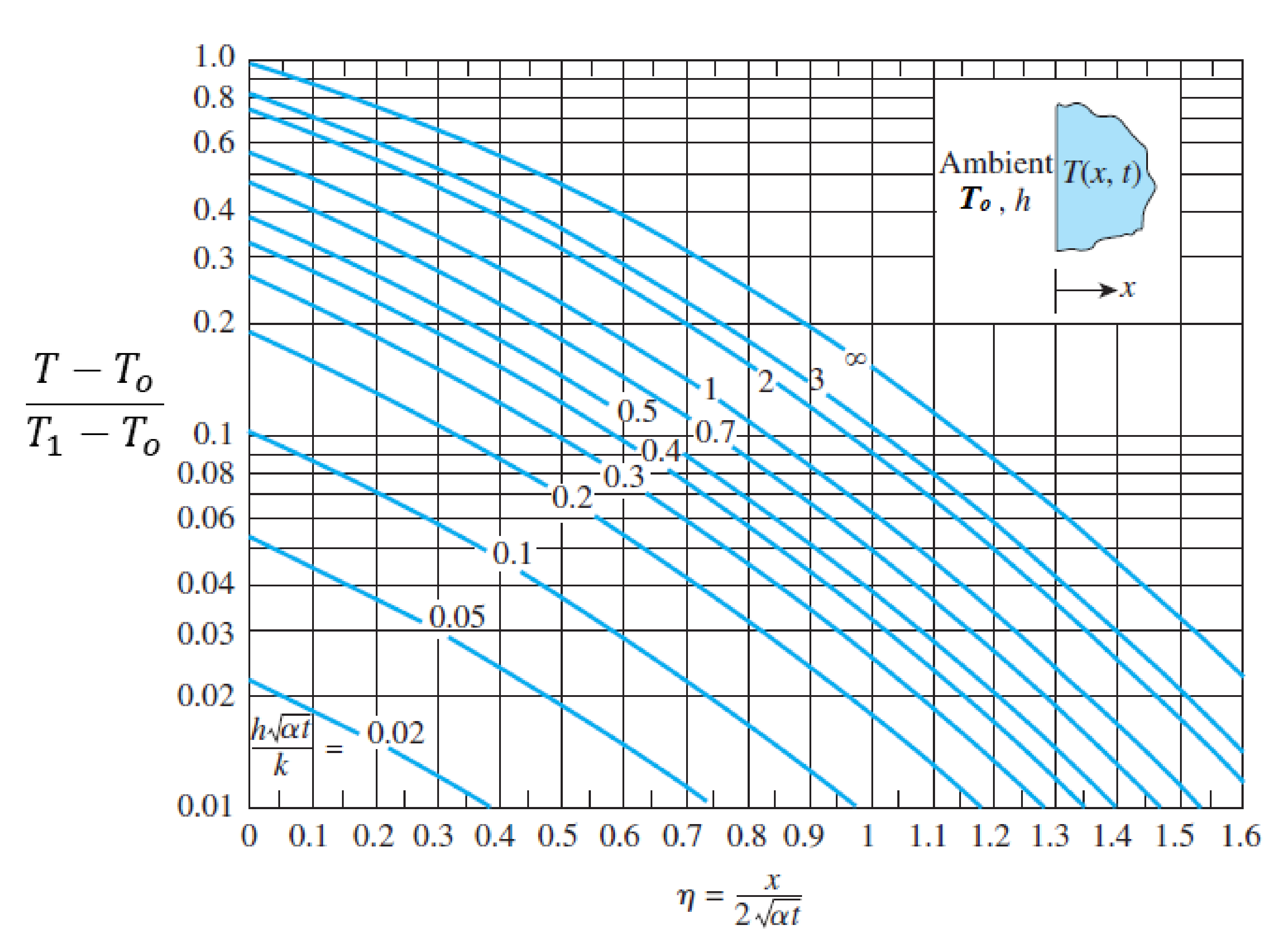
Numerical Values of Error Function $\mathrm{erf}(z) = \dfrac{2}{\sqrt{\pi}} \int_{\xi=0}^{z} e^{-\xi^2} d\xi$

z	erf z	z	erf z	z	erf z	z	erf z	z	erf z
0.00	0.00000	0.50	0.52049	1.00	0.84270	1.50	0.96610	2.00	0.99532
0.01	0.01128	0.51	0.52924	1.01	0.84681	1.51	0.96727	2.20	0.99814
0.02	0.02256	0.52	0.53789	1.02	0.85083	1.52	0.96841	2.40	0.99931
0.03	0.03384	0.53	0.54646	1.03	0.85478	1.53	0.96951	2.60	0.99976
0.04	0.04511	0.54	0.55493	1.04	0.85864	1.54	0.97058	2.80	0.99992
0.05	0.05637	0.55	0.56332	1.05	0.86243	1.55	0.97162	3.00	0.99998
0.06	0.06762	0.56	0.57161	1.06	0.86614	1.56	0.97262		
0.07	0.07885	0.57	0.57981	1.07	0.86977	1.57	0.97360		
0.08	0.09007	0.58	0.58792	1.08	0.87332	1.58	0.97454		
0.09	0.10128	0.59	0.59593	1.09	0.87680	1.59	0.97546		
0.10	0.11246	0.60	0.60385	1.10	0.88020	1.60	0.97634		
0.11	0.12362	0.61	0.61168	1.11	0.88353	1.61	0.97720		
0.12	0.13475	0.62	0.61941	1.12	0.88678	1.62	0.97803		
0.13	0.14586	0.63	0.62704	1.13	0.88997	1.63	0.97884		
0.14	0.15694	0.64	0.63458	1.14	0.89308	1.64	0.97962		
0.15	0.16799	0.65	0.64202	1.15	0.89612	1.65	0.98037		
0.16	0.17901	0.66	0.64937	1.16	0.89909	1.66	0.98110		
0.17	0.18999	0.67	0.65662	1.17	0.90200	1.67	0.98181		
0.18	0.20093	0.68	0.66378	1.18	0.90483	1.68	0.98249		
0.19	0.21183	0.69	0.67084	1.19	0.90760	1.69	0.98315		

The complementary error function

η	erfc (η)	η	erfc (η)	η	erfc (η)
0.00	1.00000	0.38	0.5910	0.76	0.2825
0.02	0.9774	0.40	0.5716	0.78	0.2700
0.04	0.9549	0.42	0.5525	0.80	0.2579
0.06	0.9324	0.44	0.5338	0.82	0.2462
0.08	0.9099	0.46	0.5153	0.84	0.2349
0.10	0.8875	0.48	0.4973	0.86	0.2239
0.12	0.8652	0.50	0.4795	0.88	0.2133
0.14	0.8431	0.52	0.4621	0.90	0.2031
0.16	0.8210	0.54	0.4451	0.92	0.1932
0.18	0.7991	0.56	0.4284	0.94	0.1837
0.20	0.7773	0.58	0.4121	0.96	0.1746
0.22	0.7557	0.60	0.3961	0.98	0.1658
0.24	0.7343	0.62	0.3806	1.00	0.1573
0.26	0.7131	0.64	0.3654	1.02	0.1492
0.28	0.6921	0.66	0.3506	1.04	0.1413
0.30	0.6714	0.68	0.3362	1.06	0.1339
0.32	0.6509	0.70	0.3222	1.08	0.1267
0.34	0.6306	0.72	0.3086	1.10	0.1198
0.36	0.6107	0.74	0.2953	1.12	0.1132

- The equation for solving the temperature at a particular x and t in case of unsteady state heat conduction for a semi-infinite solid is plotted in the form of a figure

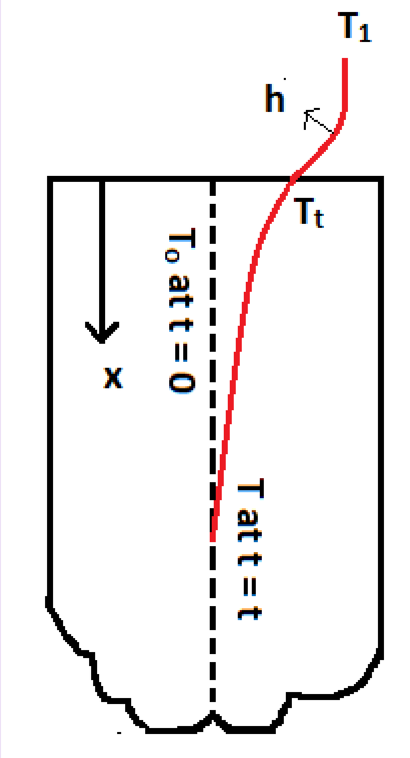


Problem

The depth in the soil of the earth at which freezing temperatures penetrate is often of importance in agriculture and construction. During a certain fall day, the temperature in the earth is constant at 15.6°C to a depth of several metres. A cold wave suddenly reduces the air temperature from 15.6°C to -17.8°C. The convective coefficient above the soil is 11.36 W/m^{2}K. The soil properties can be assumed as α = 4.65 x 10^{-7} m^2/s and k = 0.865 W/mK

(a) What is the surface temperature after 5 h?

(b) To what depth in the soil will the freezing temperature of 0°C penetrate in 5 h?



This is a case of unsteady state condition in a semi-infinite solid

For part (a) the value of x which is the distance from the surface, $x = 0$

The value of $\frac{x}{2\sqrt{\alpha t}} = \frac{0}{2\times\sqrt{4.65\times10^{-7}\times5\times3600}} = 0$ and $\frac{h\sqrt{\alpha t}}{k} = \frac{11.36\sqrt{4.65\times10^{-7}\times5\times3600}}{0.865} = 1.2$

From the Figure at $\frac{x}{2\sqrt{\alpha t}} = 0$ and $\frac{h\sqrt{\alpha t}}{k} = 1.2,$ $\frac{T-T_o}{T_1-T_o} = 0.63$

$T_o = 15.6 + 273 = 288.6\ K$

and $T_1 = -17.8 + 273 = 255.2\ K$

$$T = 288.6 + 0.63(255.2 - 288.6) = 267.76\ K \quad or\ -5.44°C$$

For part (b) the temperature at a certain depth of x is $T = 273\ K \quad or\ 0°C$

$$\frac{T - T_o}{T_1 - T_o} = \frac{273 - 288.6}{255.2 - 288.6} = 0.467$$

Now $\frac{h\sqrt{\alpha t}}{k} = \frac{11.36\sqrt{4.65\times10^{-7}\times5\times3600}}{0.865} = 1.2$

From the Figure at $\frac{T-T_o}{T_1-T_o} = 0.467$

and $\frac{h\sqrt{\alpha t}}{k} = 1.2,$ it is seen that $\frac{x}{2\sqrt{\alpha t}} = 0.16$

Therefore,

$$x = 0.16 \times 2\sqrt{4.65 \times 10^{-7} \times 5 \times 3600}$$

$$x = 0.0293 \text{ m}$$