INTRODUCTION TO FREQUENCY RESPONSE

Chapter 15

- The response of first- and second-order systems to sinusoidal forcing functions.
- These frequency responses were derived by using the standard Laplace transform technique.
- Convenient graphical technique will be established for obtaining the frequency response of linear systems.

SUBSTITUTION RULE (A Fortunate Circumstance)

- Consider a simple first-order system with transfer function
- Substituting the quantity *jw for s in Eq.*

$$G(s) = \frac{1}{\tau s + 1} \qquad G(j\omega) = \frac{1}{j\omega\tau + 1}$$

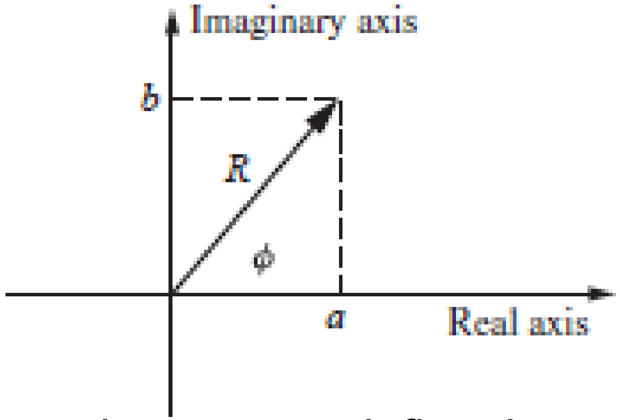
- We may convert this expression to polar form by multiplying numerator and denominator by the conjugate of ($-jw\tau + 1$)

$$G(j\omega) = \frac{-j\omega\tau + 1}{(j\omega\tau + 1)(-j\omega\tau + 1)} = \frac{1}{1+\omega^2\tau^2} - j\frac{\omega\tau}{1+\omega^2\tau^2}$$

To convert a complex number in rectangular form (*a* +*jb*) to polar form $Re^{j\varphi}$, where *R* magnitude and φ = angle, one uses the relationships

$$R = \sqrt{a^2 + b^2} \quad \text{and} \quad \phi = \tan^{-1}\frac{b}{a}$$

For visualization of the polar form



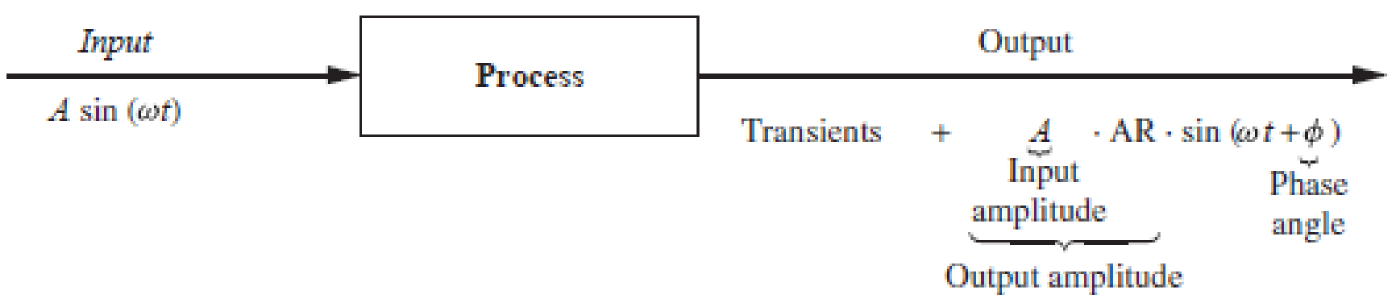
By fixing *R* and φ, we can define the complex number.
Applying these relationships to

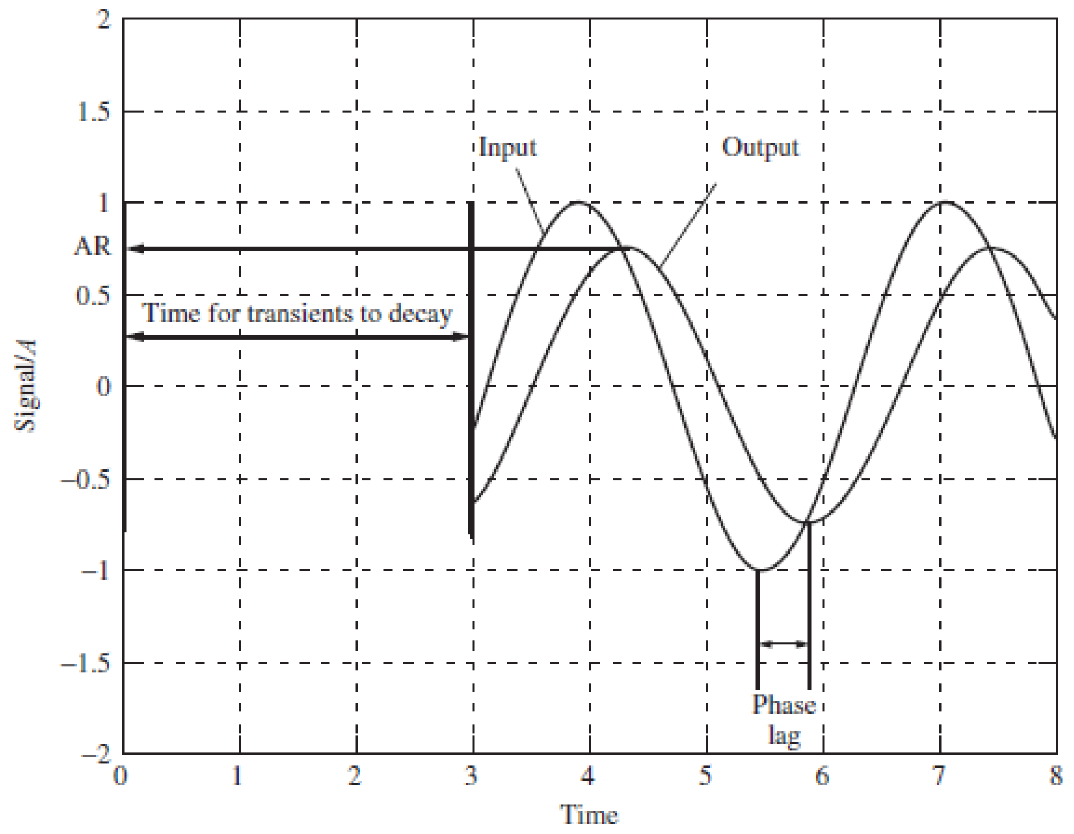
$$G(j\omega) = \underbrace{\frac{1}{\sqrt{\omega^2\tau^2 + 1}}}_{R} \; \underbrace{\measuredangle \tan^{-1}(-\omega\tau)}_{\phi}$$

- So, after sufficient time elapses, the response of a first-order system to a sinusoidal input of frequency ω is also a sinusoid of frequency ω.
- Furthermore, the ratio of the amplitude of the response to that of the input is $1/\sqrt{\omega^2\tau^2+1}$, and the phase difference between output and input is $\tan^{-1}(-\omega\tau)$.

$$\text{Amplitude ratio (AR)} = \frac{\text{output amplitude}}{\text{input amplitude}} = |G(j\omega)| = R$$

$$\text{Phase angle} = \phi = \measuredangle G(j\omega)$$





KEY FEATURES TO NOTE ABOUT THE FREQUENCY RESPONSE OF THE PROCESS

- After transients die out, the output is a sine wave.
- Input frequency output frequency ω.
- In general, the output is attenuated, that is, AR < 1.
- The output is shifted in time (it lags the input by the phase angle φ).
- Amplitude ratio and phase angle are both functions of frequency.

Example

$$G(s) = \frac{1}{0.1s + 1}$$

Frequency of the bath-temperature variation is given as 10/π cycle/min which is equivalent to ω (10 cycles/ π min)(2 π rad/cycle) = 20 rad/min. Hence, let

$$s = \omega j = 20j$$

$$G(20j) = \frac{1}{2j + 1}$$

$$G(20j) = \frac{1}{\sqrt{5}} \measuredangle -1.11 \text{ rad} = \frac{1}{\sqrt{5}} \measuredangle -63.5^\circ$$

$-\tan^{-1}(2)$

Frequency response of the system with the general second-order transfer function, The transfer function is

$$\frac{1}{\tau^2 s^2 + 2\zeta \tau s + 1}$$

Putting $s = j\omega$ yields

$$\frac{1}{1 - \tau^2\omega^2 + j2\zeta\omega\tau}$$

which may be converted to polar form

$$\frac{1}{\sqrt{(1 - \omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}} \measuredangle \tan^{-1}\left(\frac{-2\zeta\omega\tau}{1 - \omega^2\tau^2}\right)$$

$$\text{Amplitude ratio AR} = \frac{1}{\sqrt{\left[1 - (\omega\tau)^2\right]^2 + (2\zeta\omega\tau)^2}}$$

$$\text{Phase angle} = \phi = -\tan^{-1}\frac{2\zeta\omega\tau}{1 - (\omega\tau)^2}$$

Example: Consider a second-order transfer function, with $\tau = 1$ and $\zeta = 0.8$, being disturbed with a sine wave input of 3 sin (0.5t). Determine

- the form of the response after the transients have decayed and steady-state oscillations are established.

Solution

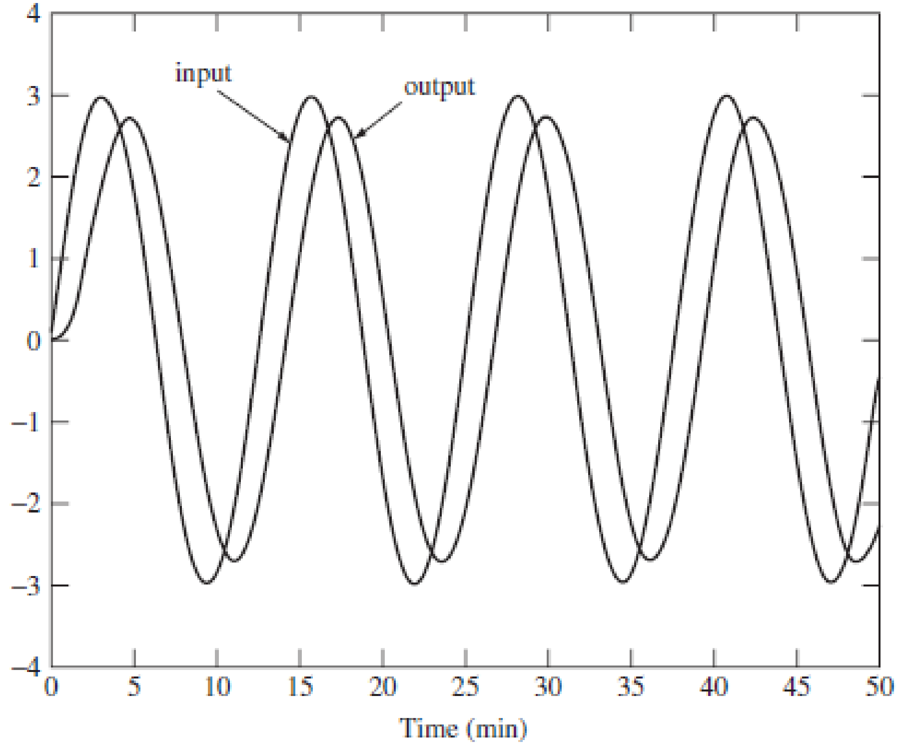
- The steady-state oscillations will have the form 3(AR) sin(0.5t+Φ),
- which is an attenuated (smaller-amplitude) sine wave of the same frequency as the input and shifted by a phase angle Φ.
- Thus, all we need to determine is AR and Φ.

$$\text{AR} = \frac{1}{\sqrt{\left\{1-[0.5(1)]^2\right\}^2 + [(2)(0.8)(0.5)(1)]^2}} = \frac{1}{\sqrt{0.5625+0.64}} = 0.91$$

$$\phi = -\tan^{-1}\left\{\frac{(2)(0.8)(0.5)(1)}{1-[0.5(1)]^2}\right\} = -\tan^{-1}\left(\frac{0.8}{0.75}\right) = -0.818 \text{ rad} = -46.8^\circ$$

The form of the steady-state oscillations is therefore

$$3(\text{AR})\sin(0.5t+\phi) = 2.73\sin(0.5t-0.818\text{ rad})$$
$$= 2.73\sin[0.5(t-1.636)]$$
$$= 2.73\sin(0.5t-46.8^{\circ})$$



Transportation Lag

- The response of a transportation lag is not described by a standard n^{th}-order differential equation (that yields standard transfer functions).
- Rather, a transportation lag is described by the relation

$$Y(t) = X(t - \tau)$$

which states that the output *Y* lags the input *X* by an interval of time *t*.
If *X* is sinusoidal

$$X = A \sin \omega t$$

$$Y = A \sin \omega (t - \tau) = A \sin (\omega t - \omega \tau)$$

It is apparent that the AR is unity and the phase angle is ($\omega\tau$).
To check the substitution rule of the previous section, recall that the transfer function for a transportation lag is given by

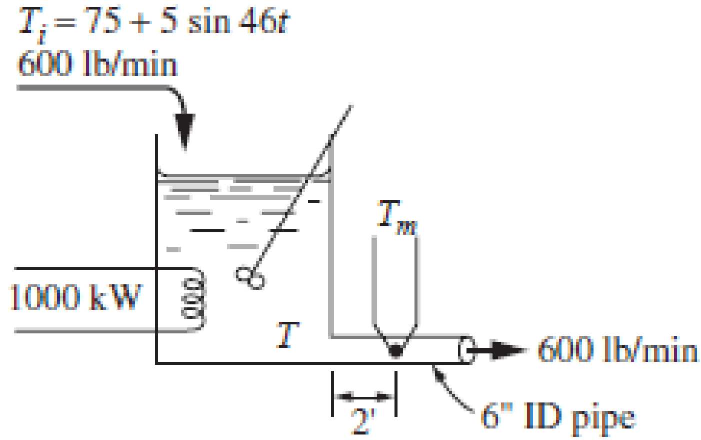
$$G(s) = \frac{Y(s)}{X(s)} = e^{-\tau s}$$

$$G(j\omega) = e^{-j\omega\tau} = \left(\text{form of: } Re^{j\phi},\ R = 1,\ \phi = -\omega\tau\right)$$

$$\text{AR} = \left|e^{-j\omega\tau}\right| = 1$$

$$\text{Phase angle} = \measuredangle e^{-j\omega\tau} = -\omega\tau$$

Example 15.4. Consider a stirred-tank heater with a capacity of 15 gal. Water is entering and leaving the tank at the constant rate of 600 lb/min. The heated water that leaves the tank enters a well-insulated section of 6-in-ID pipe. Two feet from the tank, a thermocouple is placed in this line for recording the tank temperature, as shown in Fig. 15–6. The electrical heat input is held constant at 1000 kW.



- If the inlet temperature is varied according to the relation

  $T_i = 75 + 5 \sin 46t$

- To define a deviation variable for T_m, note that if T_i were held at 75˚F, T_m would come to the steady state satisfying

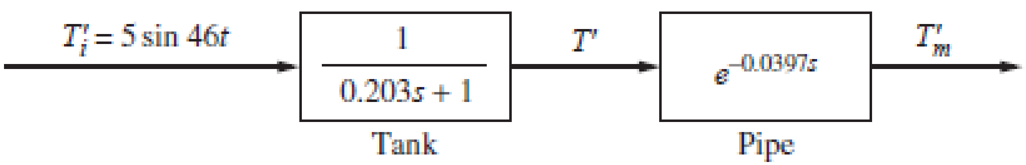
$$q_s = wC(T_{m_s} - T_{i_s})$$

$$T_{m_s} = \frac{q_s}{wC} + T_{i_s} = \frac{(1000 \text{ kW})\left(1000 \frac{\text{W}}{\text{kW}}\right)\left(0.0569 \frac{\text{Btu/min}}{\text{W}}\right)}{\left(600 \frac{\text{lb}}{\text{min}}\right)\left(1.0 \frac{\text{Btu}}{\text{lb} \cdot {}^\circ\text{F}}\right)} + 75^\circ\text{F} = 170^\circ\text{F}$$

Hence, define a deviation variable T'_m as

$$T'_m = T_m - 170$$

Now, the overall system between T_i and T_m is made up of two components in series: the tank and the 2-ft section of pipe.



- The transfer function for the tank is

$$G_1(s) = \frac{1}{\tau_1 s + 1}$$

$$\tau_1 = \frac{\rho V}{w} = \frac{\left(60.8\,\frac{\text{lb}}{\text{ft}^3}\right)(15\text{ gal})}{\left(600\,\frac{\text{lb}}{\text{min}}\right)\left(7.48\,\frac{\text{gal}}{\text{ft}^3}\right)} = 0.203\text{ min}$$

The transfer function of the 2-ft section of pipe, which corresponds to a transportation lag, is

$$G_2(s) = e^{-\tau_2 s}$$

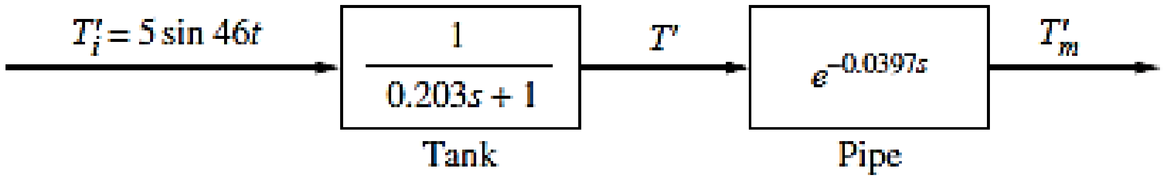
where τ_2 is the length of time required for the fluid to traverse the length of pipe.
This is

$$\tau_2 = \frac{L}{v} = \frac{2\text{ ft}}{\dfrac{600\text{ lb/min}}{\left(60.8\text{ lb/ft}^3\right)\left(0.196\text{ ft}^2\right)}} = 0.0397\text{ min}$$

The factor 0.196 is the cross-sectional area of the pipe in square feet.

Since the two systems are in series, the overall transfer function between T_i' and T_m' is

$$\frac{T_m'}{T_i'} = \frac{e^{-\tau_2 s}}{\tau_1 s + 1} = \frac{e^{-0.0397s}}{0.203s + 1}$$



$$|G(j\omega)| = |G_1(j\omega)||G_2(j\omega)|\cdots|G_n(j\omega)|$$

$$\measuredangle G(j\omega) = \measuredangle G_1(j\omega) + \measuredangle G_2(j\omega) + \cdots + \measuredangle G_n(j\omega)$$

This rule makes it very convenient to find the frequency response of a number of systems in series

$$\text{AR} = \frac{1}{\sqrt{(46 \times 0.203)^2 + 1}} = \frac{1}{9.39} = 0.107$$

$$\text{Phase angle} = \tan^{-1}[(-46)(0.203)] = -84^\circ$$

For the section of pipe, AR is unity, so that the overall AR is just 0.107. The phase lag due to the pipe may be obtained from

$$\text{Phase angle} = -\omega\tau_2 = -(46)(0.0397) = -1.82 \text{ rad} = -104^\circ$$

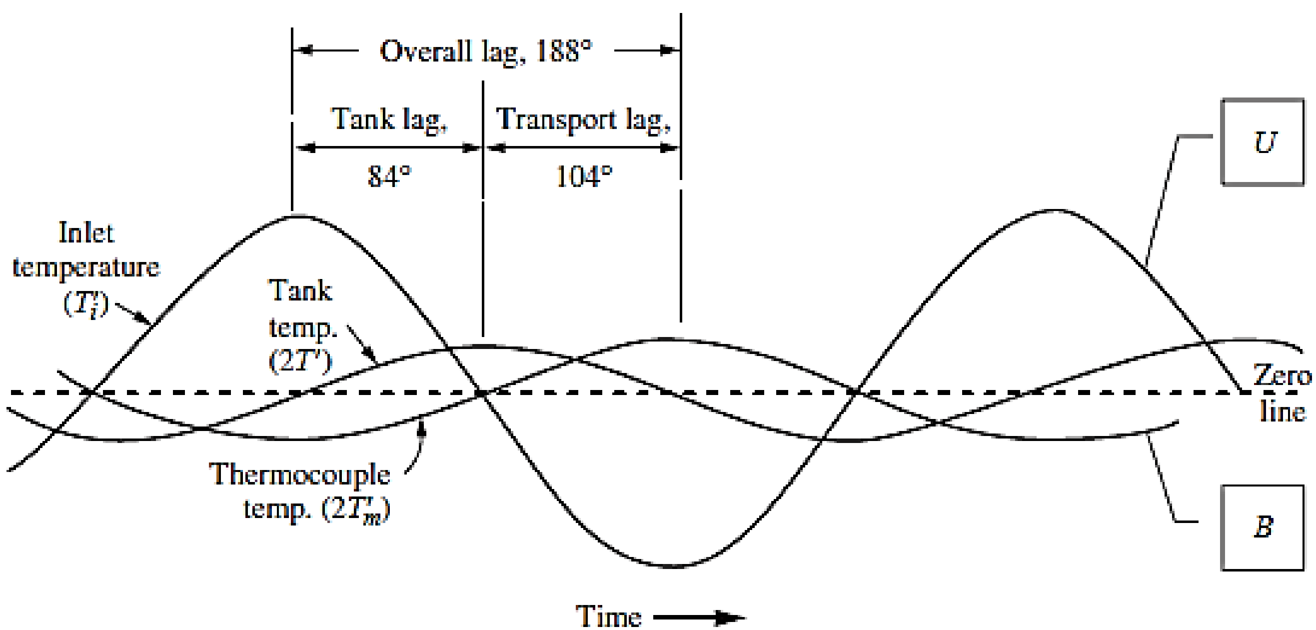
The overall phase lag from T'_i to T'_m is the sum of the individual lags,

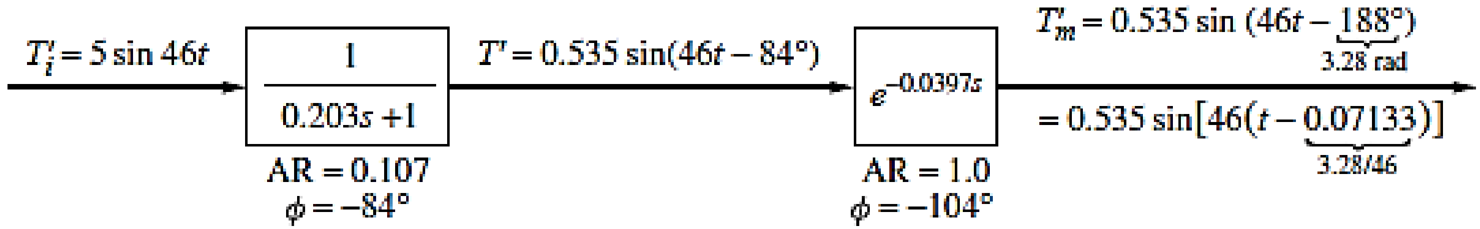
$$\measuredangle \frac{T'_m}{T'_i} = -84 - 104 = -188^\circ$$

$$T_m = 170 + 5(0.107)\sin(46t - 188^\circ)$$

For comparison, a plot of T'_i, T'_m, and T'

$$T' = \text{tank temperature} - 170^\circ\text{F}$$

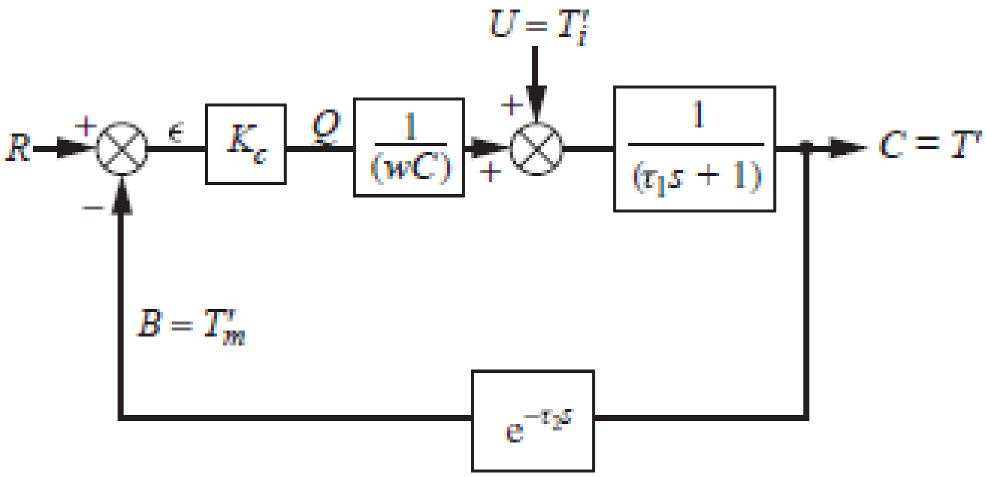




Also, note that, for convenience of scale (so that we easily can see them on the same axes), the tank and thermocouple temperatures have been plotted as $2T'$ and $2T'_m$, respectively.

Therefore, the possibility of an unstable control system exists for this particular sinusoidal variation in frequency. I
The fact that such information may be obtained by study of the *frequency response* (i.e., the particular solution
for a sinusoidal forcing function) justifies further study of this subject

A Control Problem



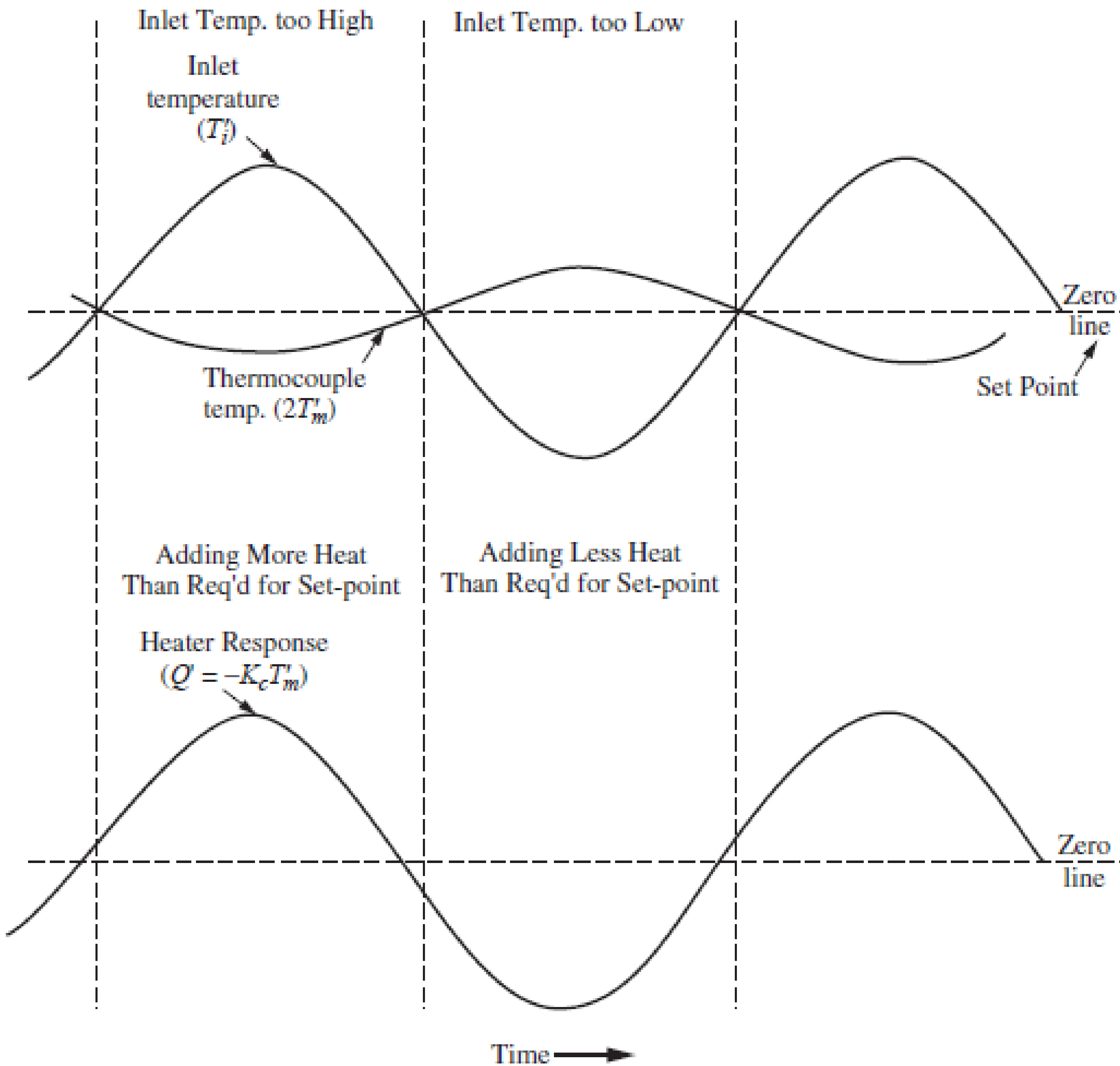
- A block diagram for proportional control might appear as in Fig., where T_i is replaced by *U, T'* by *C,* and T'_m by *B* to conform with our standard block diagram nomenclature.
- The variable *R* denotes the deviation of the set point from 170˚F and is the desired value of the deviation *C.*
- The value of *R* is assumed to be zero in the following analysis (control at 170˚F).

If *R*= 0, then

$$\text{Error } \varepsilon = 0 - B = -B$$

$$\text{Heat input } Q = K_c\varepsilon = -K_cB$$

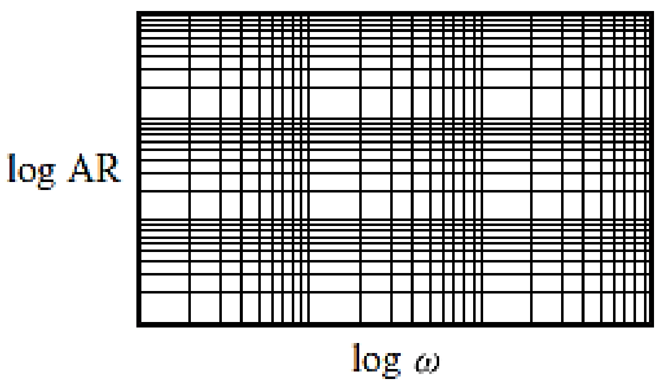
- Thus, the heat being added to the tank is given in deviation variables as *Kc B.*
- it can be seen that the peaks of *U (*which is *Ti)* and *B (*which is *Tm)* are almost exactly opposite because the phase difference is 188˚.
- This means that if the loop were closed, the control system would have a tendency to add *more* heat when the inlet temperature *Ti* was at its high peak, because *B* is then negative and *Kc B* becomes positive. (Recall that the set point *R* is held constant at zero.)
- Conversely, when the inlet temperature is at a low point, the *tendency* will be for the control system to add less heat because *B* is positive.
- This is precisely opposite to the way the heat input should be controlled. Figure clarifies this physical situation.

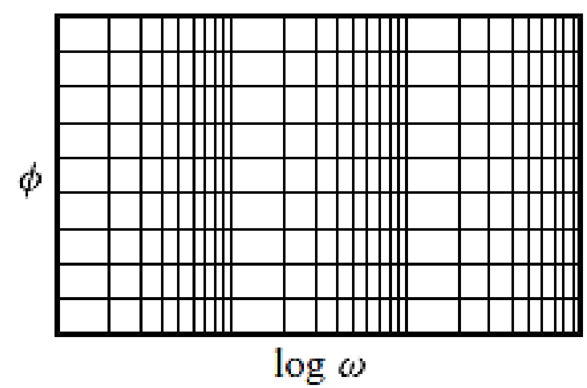


Inlet Temp. too High
Inlet Temp. too Low
Inlet temperature (T_i')
Zero line
Thermocouple temp. ($2T_m'$)
Set Point
Adding More Heat Than Req'd for Set-point
Adding Less Heat Than Req'd for Set-point
Heater Response ($Q' = -K_c T_m'$)
Zero line
Time

BODE DIAGRAMS

- Thus far, it has been necessary to calculate AR and phase lag by direct substitution of *s = jω* into the transfer function for the particular frequency of interest.
- AR and phase lag are functions of frequency.
- There is a convenient graphical representation of their dependence on the frequency that largely eliminates direct calculation.
- *This is called a Bode diagram and consists of two graphs: logarithm of AR versus logarithm of frequency, and phase angle versus logarithm of frequency*



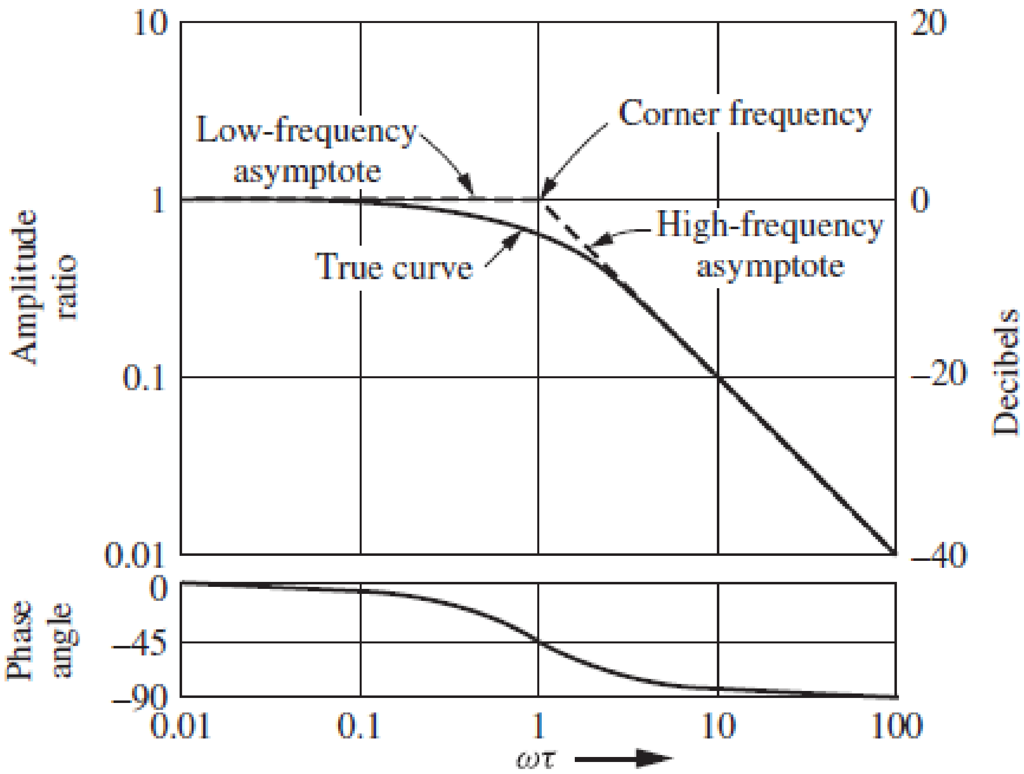


First-Order System

$$\mathrm{AR} = \frac{1}{\sqrt{\tau^2\omega^2 + 1}} \qquad \text{Phase angle} = \tan^{-1}(-\omega\tau)$$

It is convenient to regard these as functions of $\omega\ \tau$ for the purpose of generality

$$\log \mathrm{AR} = -\tfrac{1}{2}\log\left[(\omega\tau)^2 + 1\right]$$



- Some asymptotic considerations can simplify the construction of this plot.
- As $\omega\tau \to 0$, shows that AR $\to$ 1.
- As $\omega\tau \to \infty$, becomes asymptotic to

$$\log \text{AR} = -\log(\omega\tau)$$

which is a line of slope -1, passing through the point

$$\omega\tau = 1 \qquad \text{AR} = 1$$

This line is indicated as the high-frequency asymptote

The frequency $\omega_c = 1/\tau$, where the two asymptotes intersect, is known as the *corner frequency;*

it may be shown that the deviation of the true AR curve from the asymptotes is a maximum at the corner frequency. Using $\omega_c = 1/\tau$, gives

$$\text{AR} = \frac{1}{\sqrt{2}} = 0.707$$

- As the true value, whereas the intersection of the asymptotes occurs at AR = 1.
- Since this is the maximum deviation and is an error of less than 30 percent, for engineering purposes it is often sufficient to represent the curve entirely by the asymptotes.
- Alternately, the asymptotes and the value of 0.707 may be used to sketch the curve if greater accuracy is required.

$$\phi = \tan^{-1}(-\omega\tau) = -\tan^{-1}(\omega\tau)$$

- It is evident that ϕ approaches 0˚ at low frequencies and -90˚ at high frequencies.

- This verifies the low- and high-frequency portions of the phase curve. At the corner frequency, $\omega_c = 1/\tau$,

$$\phi_c = -\tan^{-1}(\omega_c\tau) = -\tan^{-1} 1 = -45°$$

- There are asymptotic approximations available for the phase curve, but they are not so accurate or so widely used as those for the AR.

- Instead, it is convenient to note that the curve is symmetric about -45˚.

- It should be stated that, in a great deal of the literature on control theory, amplitude ratios (or gains) are reported in decibels.
- The decibel (dB) is defined by

$$\text{Decibels} = 20 \log (\text{AR})$$

Thus, an AR of unity corresponds to 0 dB, and an amplitude ratio of 0.1 corresponds to 20 dB. The value of the AR in decibels is given on the right-hand ordinate

First-Order Systems in Series

The advantages of the Bode plot become evident when we wish to plot the frequency response of systems in series.

Rules for multiplication of complex numbers indicate that the AR for two first-order systems in series is the product of the individual ARs:

$$\text{AR} = \frac{1}{\sqrt{\omega^2 \tau_1{}^2 + 1}\sqrt{\omega^2 \tau_2{}^2 + 1}}$$

Similarly, the phase angle is the sum of the individual phase angles

$$\phi = \tan^{-1}(-\omega\tau_1) + \tan^{-1}(-\omega\tau_2)$$

Since the AR is plotted on a logarithmic basis, multiplication of the ARs is accomplished by addition of logarithms on the Bode diagram (which, we shall see, is equivalent to adding the slopes of the asymptotes of the individual curves to get the asymptote of the overall curve on log-log coordinates).
The phase angles are added directly. The procedure is best illustrated by an example.

Plot the Bode diagram for the system whose overall transfer function is

$$\frac{1}{(s+1)(s+5)}$$

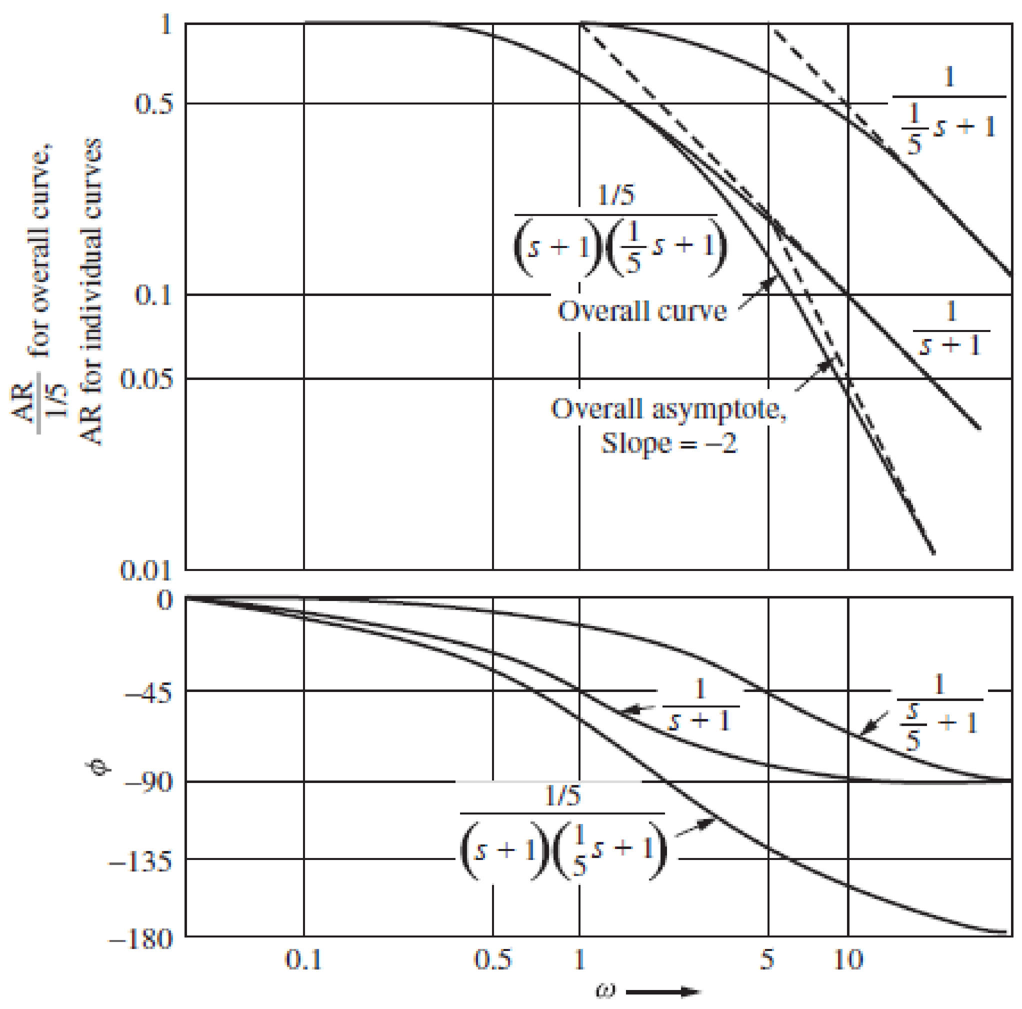
To put this in the form of two first-order systems in series, it is rewritten as

$$\frac{1/5}{(s+1)(\frac{1}{5}s+1)}$$

The time constants are $\tau_1 = 1$ *and* $\tau_2 = 1/5$. The factor 1/5 in the numerator corresponds to the steady-state gain.

$$\mathrm{AR}_{\mathrm{overall}} = \frac{1/5}{\sqrt{\omega^2+1}\sqrt{(\omega/5)^2+1}}$$

$$\log \mathrm{AR}_{\mathrm{overall}} = \log\frac{1}{5} - \frac{1}{2}\log\left(\omega^2+1\right) - \frac{1}{2}\log\left[\left(\frac{\omega}{5}\right)^2+1\right]$$



- Must be plotted as functions of log ω *rather than log (ω τ)* because of the different time constants.
- This is easily done by shifting the curves to the right or left so that the corner frequency falls at ω= *1/ τ.*
- Thus, the individual curves are placed so that the corner frequencies fall at $w_{c1} = 1$ *and* $w_{c2} = 5$.
- *T* these curves are added to obtain the overall curve shown.
- Note that in this case the logarithms are negative and the addition is downward.
- To complete the AR curve, the factor log 1/5 should be added to the overall curve.
- This would have the effect of shifting the entire curve down by a constant amount.
- Instead of doing this, the factor 1/5 is incorporated by plotting the overall curve as $AR_{overall}$ /(1/5) instead of $AR_{overall}$.

- The overall asymptote has a slope (of 0 below *ω= 1), (-1 for ω between 1 and 5), and (2 above ω= 5).*
- *Its* slope is obtained by simply adding the slopes of the individual asymptotes.
- To obtain the phase angle, the individual phase angles are plotted and added.
- The factor 1/5 has no effect on the phase angle, which approaches 180˚ at high frequency.

Graphical Rules for Bode Diagrams

$$\log(\mathrm{AR}_{\text{overall}}) = \log(\mathrm{AR})_1 + \log(\mathrm{AR})_2 + \cdots + \log(\mathrm{AR})_n$$

$$\phi_{\text{overall}} = \phi_1 + \phi_2 + \cdots + \phi_n$$

- Therefore, the following rules apply to the true curves or to the asymptotes on the Bode diagram

1. The overall AR is obtained by adding the individual ARs.

- For this graphical addition, an individual AR that is above unity on the frequency response diagram is taken as positive; an AR that is below unity is taken as negative.
- The asymptote of the overall curve is obtained by adding the slopes of the individual curves.

2. The overall phase angle is obtained by addition of the individual phase angles.
3. The presence of a constant in the overall transfer function shifts the entire AR curve vertically by a constant amount and has no effect on the phase angle.

   It is usually more convenient to include a constant factor in the definition of the ordinate.

Second-Order System

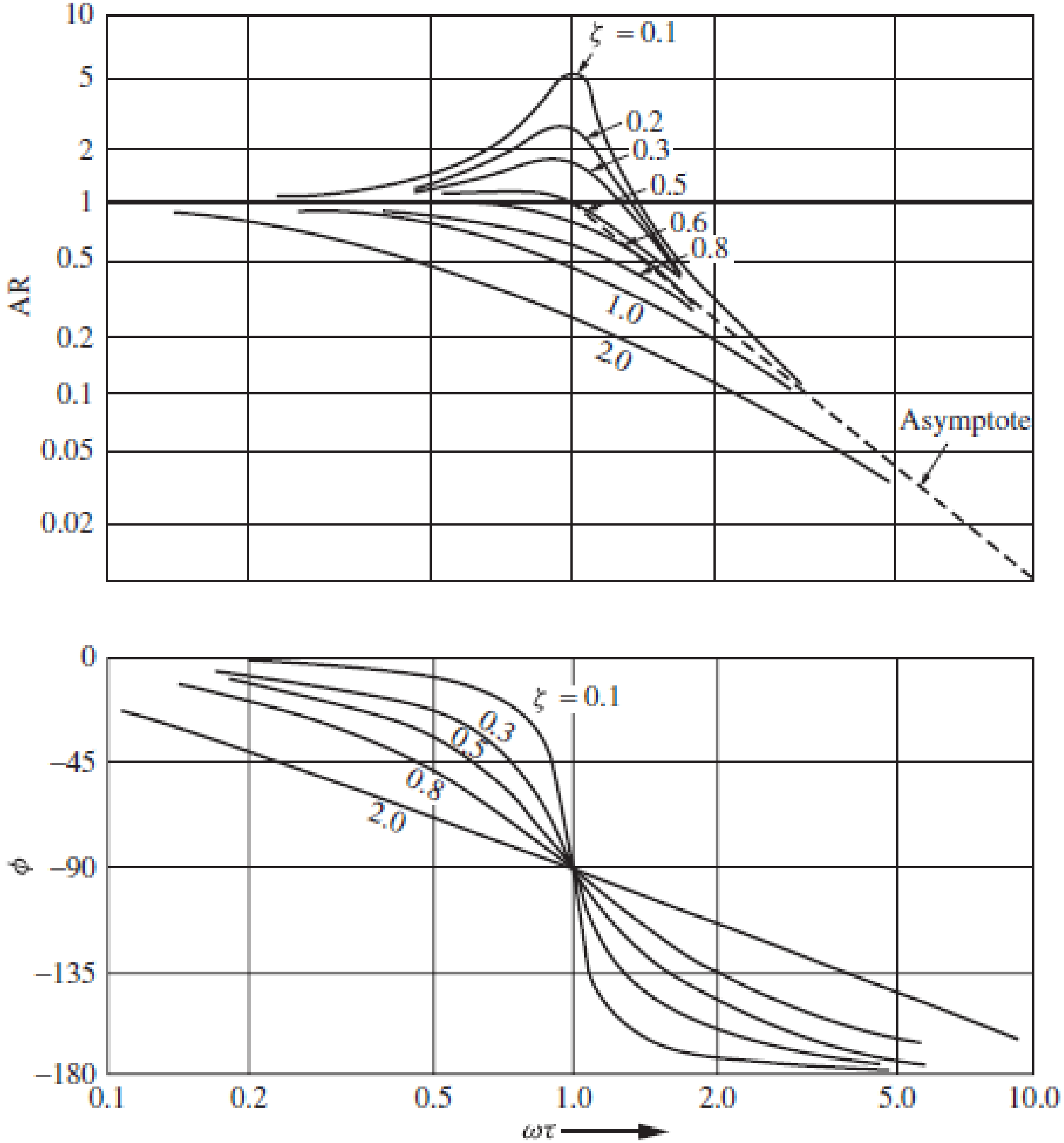
- The frequency response of a system with a second-order transfer function

$$G(s) = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \qquad \text{AR} = \frac{1}{\sqrt{(1-\omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}}$$

$$\text{Phase angle} = \tan^{-1}\frac{-2\zeta\omega\tau}{1-(\omega\tau)^2}$$

If $\omega\tau$ is used as the abscissa for the general Bode diagram, it is clear that ζ will be a parameter. That is, there is a different curve for each value of ζ.

The calculation can be done most clearly with the aid ofζ a plot of $\tan^{-1} x$



Notice that for $\omega\tau = 1$, the phase angle is 90˚, independent of ζ.

For $\omega\tau << 1$, the AR, or gain, approaches unity. For $\omega\tau \rightarrow \infty$, the only significant term in the denominator is $\sqrt{(\omega\tau)^4}$, and the AR becomes asymptotic to the line

$$\mathrm{AR} = \frac{1}{(\omega\tau)^2}$$

This asymptote has slope -2 and intersects the line AR = 1 at $\omega\tau = 1$.

For $\zeta \geq 1$, we have shown that the second order system is equivalent to two first-order systems in series.

The fact that the AR for $\zeta \geq 1$ (as well as for $\zeta < 1$) attains a slope of -2 and phase of 180° is, therefore, consistent.

for $\zeta < 0.707$, the AR curves attain maxima in the vicinity of $\omega\tau = 1$.

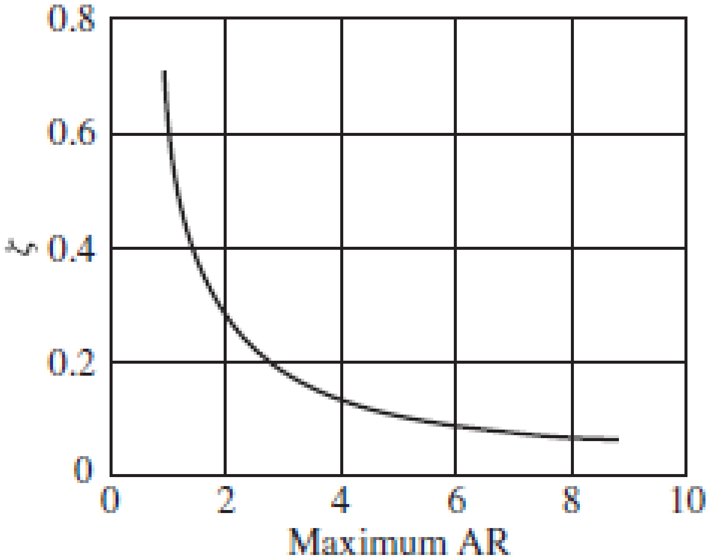
This can be checked by differentiating the expression for the AR with respect to $\omega\tau$ and setting the derivative to zero.

$$(\omega\tau)_{max} = \sqrt{1 - 2\zeta^2} \qquad \zeta < 0.707$$

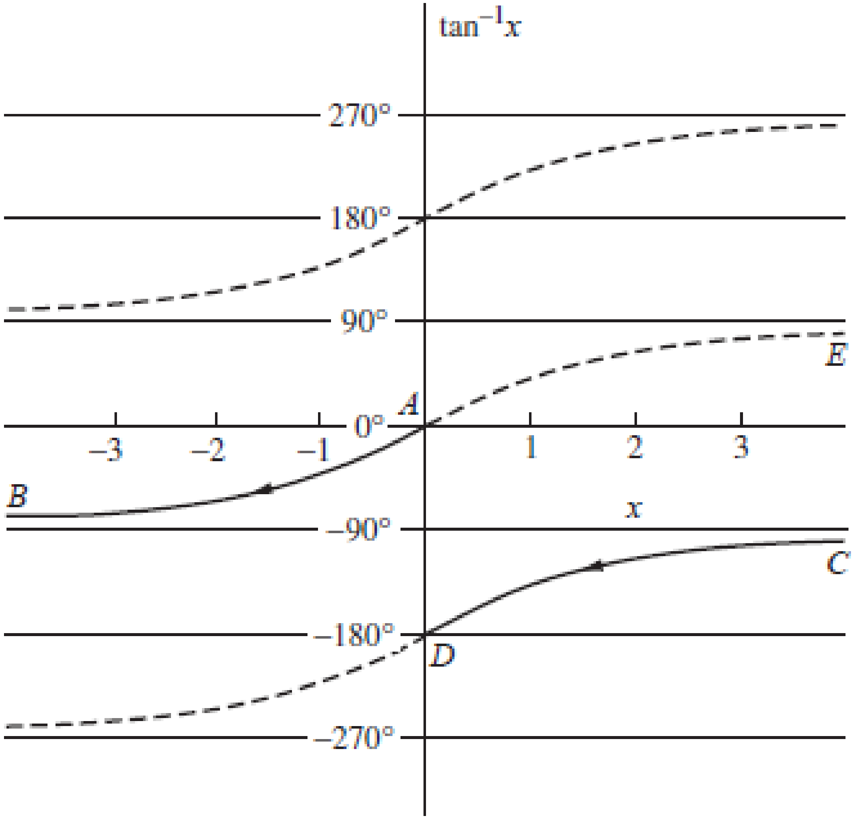
for the value of $\omega\tau$ at which the maximum AR occurs. The value of the maximum AR, obtained by substituting $(\omega\tau)_{max}$ into Eq.

$$(\text{AR})_{max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \qquad \zeta < 0.707$$

A plot of the maximum AR versus ζ is given in Fig.
The frequency at which the maximum AR is attained is called the resonant frequency and is obtained from



It may be seen that AR values exceeding unity are attained by systems for which $\zeta < 0.707$.
This is in sharp contrast to the first-order system, for which the AR is always less than unity.



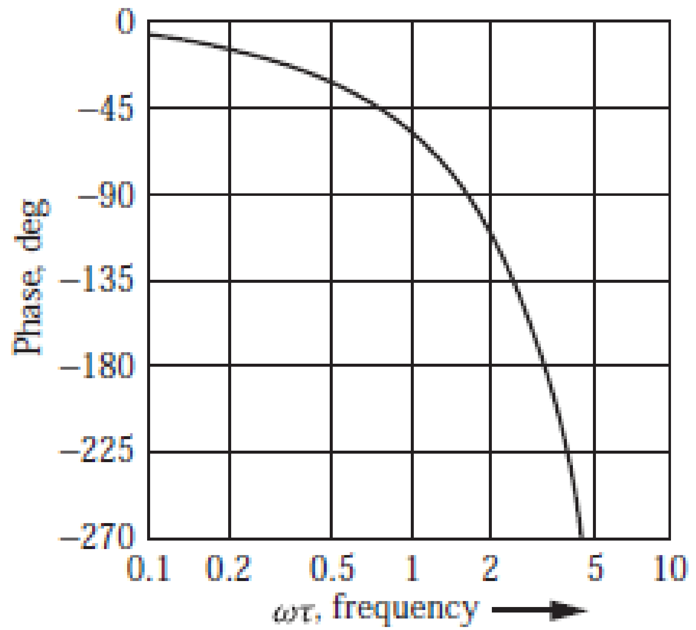
As $\omega\tau$ goes from 0 to 1, that the argument of the arctan function goes from 0 to ∞ and the phase angle goes from 0 to 90˚

As $\omega\tau$ crosses unity from a value less than unity to a value greater than unity, the sign of the argument of the arctan function shifts from negative to positive. To preserve continuity in angle as $\omega\tau$ crosses unity, the phase angle must go from 90 to 180˚ as $\omega\tau$ goes from 1 to ∞

Transportation Lag

- Frequency response for $G(s) = e^{-\tau s}$ is

  AR =1 $\phi = -\omega\tau$ rad or $\phi = -57.2958\omega\tau$ deg



Proportional Controller

- A proportional controller with transfer function K_c has amplitude ratio K_c and phase angle zero at all frequencies.
- No Bode diagram is necessary for this component.

Proportional-Integral Controller

This component has the ideal transfer function

$$G(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

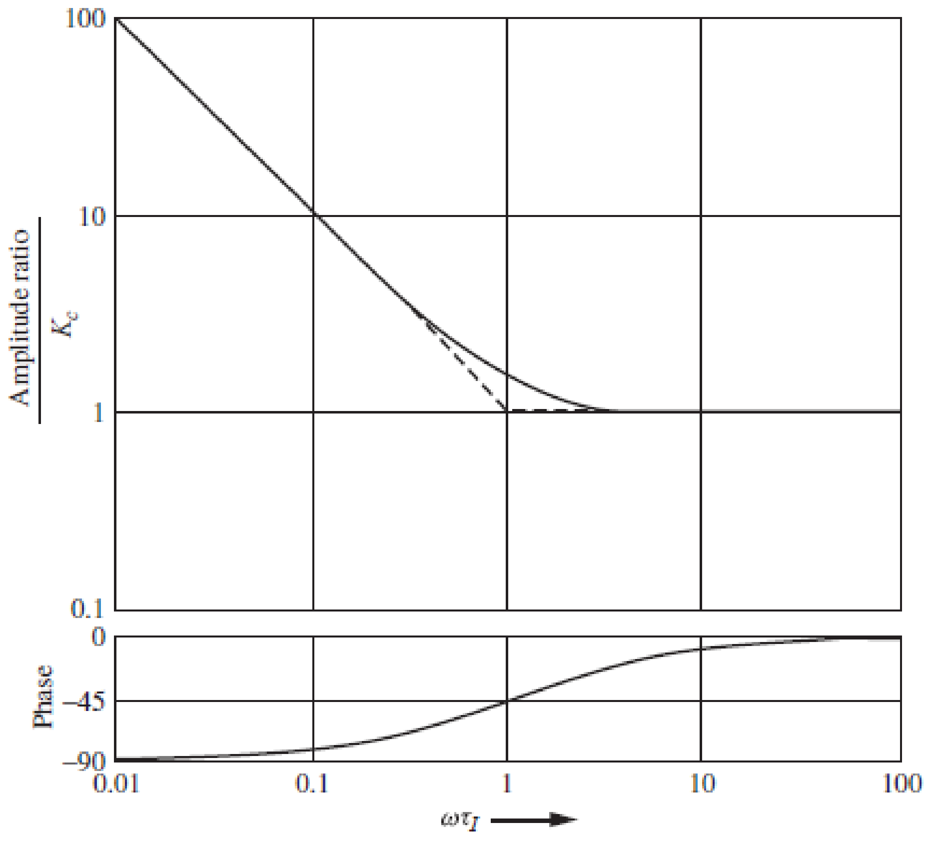
Accordingly, the frequency response is given by

$$\text{AR} = |G(j\omega)| = K_c\left|1 + \frac{1}{\tau_I j\omega}\right| = K_c\sqrt{1 + \frac{1}{(\omega\tau_I)^2}}$$

$$\text{Phase} = \measuredangle G(j\omega) = \measuredangle\left(1 + \frac{1}{\tau_I j\omega}\right) = \tan^{-1}\left(-\frac{1}{\omega\tau_I}\right)$$

- The Bode plot of uses $\omega\tau_I$ as the abscissa.
- The constant factor K_c is included in the ordinate for convenience.
- Asymptotes with a corner frequency of $\omega_c = 1/\tau_I$ are indicated.

Bode diagram for PI controller.



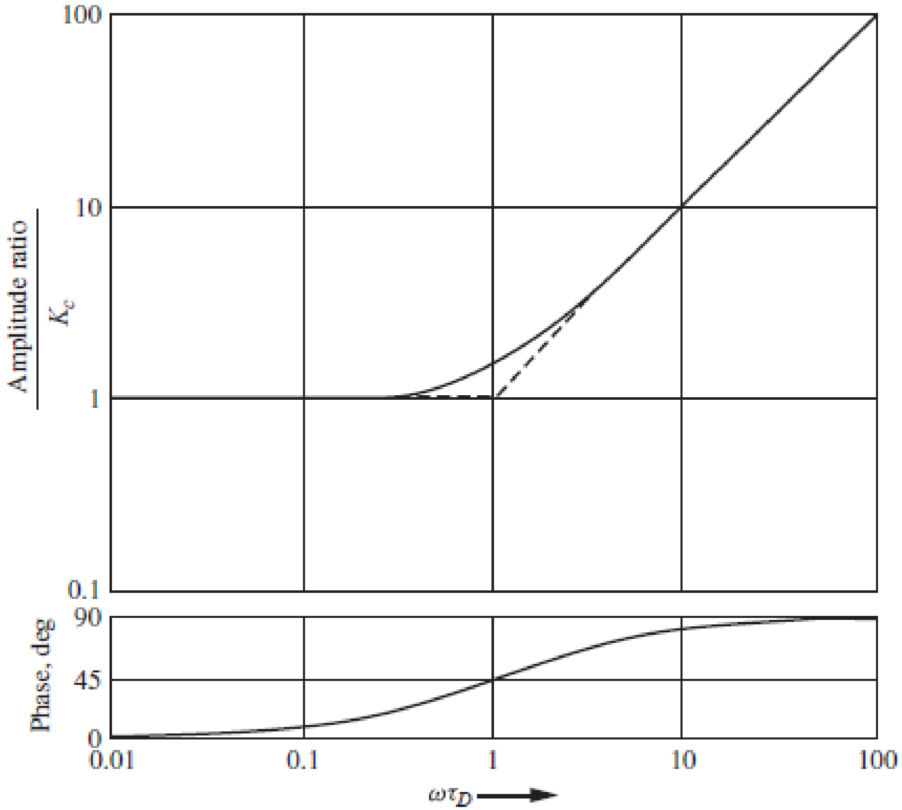
Proportional-Derivative Controller

$$G(s) = K_c(1 + \tau_D s)$$

The reader should show that this has amplitude and phase behavior that are just the inverse of the first-order system

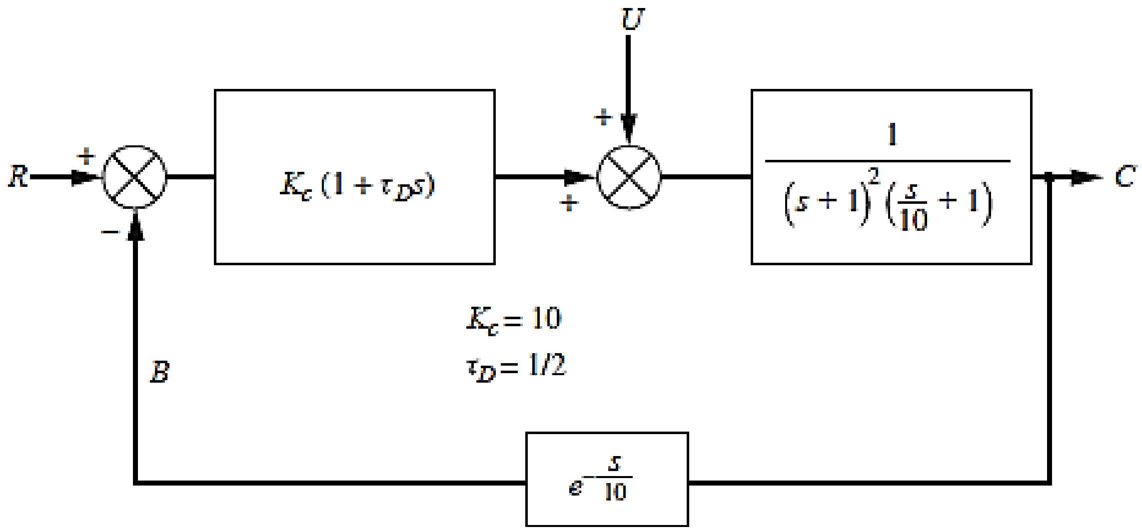
$$\frac{1}{\tau s + 1}$$

The corner frequency is $\omega_c = 1/\tau_D$.



- This system is important because it introduces *phase lead.*
- Thus, it can be seen that using PD control for the tank temperature control system of decrease the phase lag at all frequencies.
- In particular, 180˚ of phase lag would not occur until a higher frequency.
- This may exert a stabilizing influence on the control system.

Plot the Bode diagram for the open-loop transfer function of the control system of Fig. This system might represent PD control of three tanks in series, with a transportation lag in the measuring element.

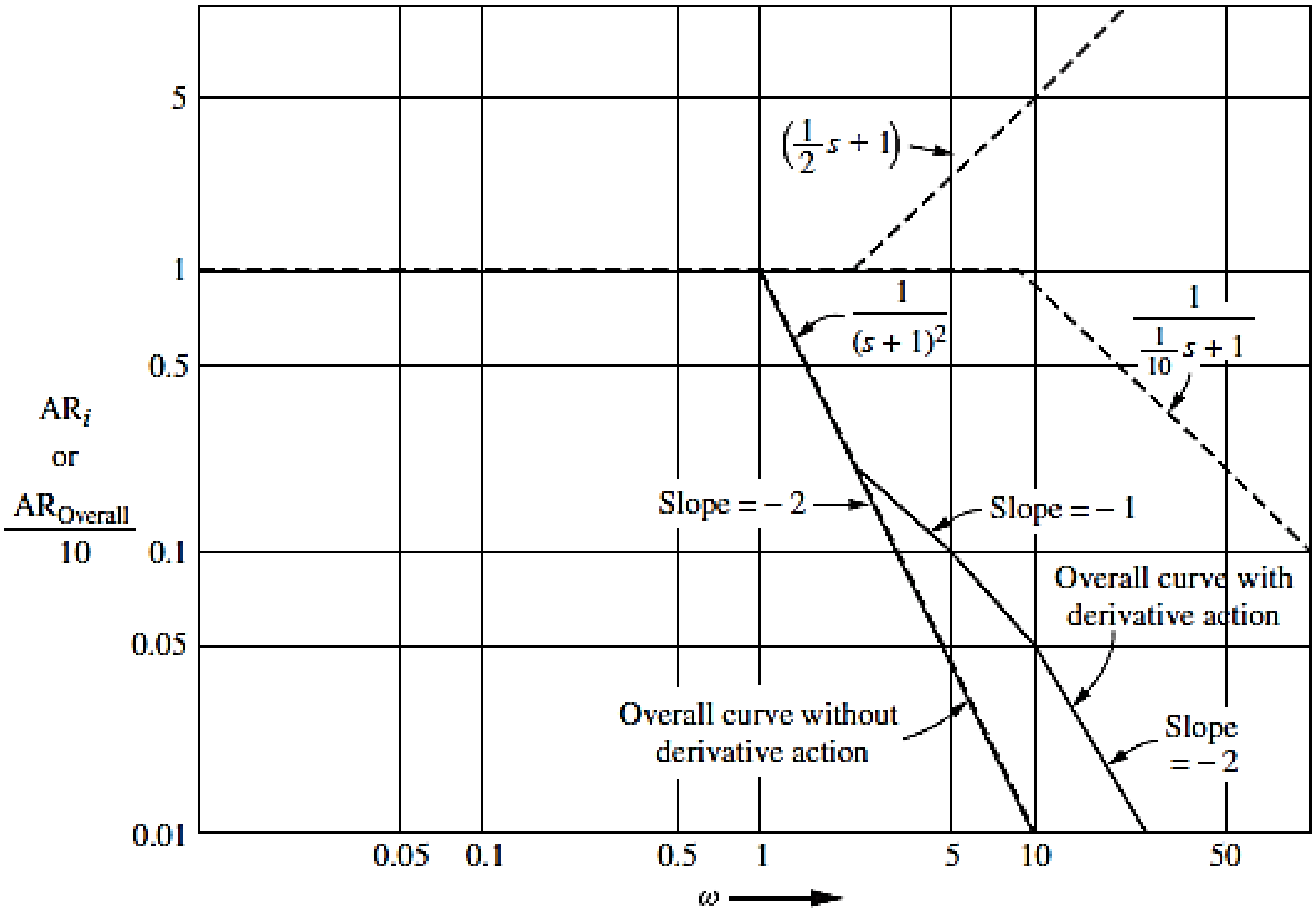


The open-loop transfer function is

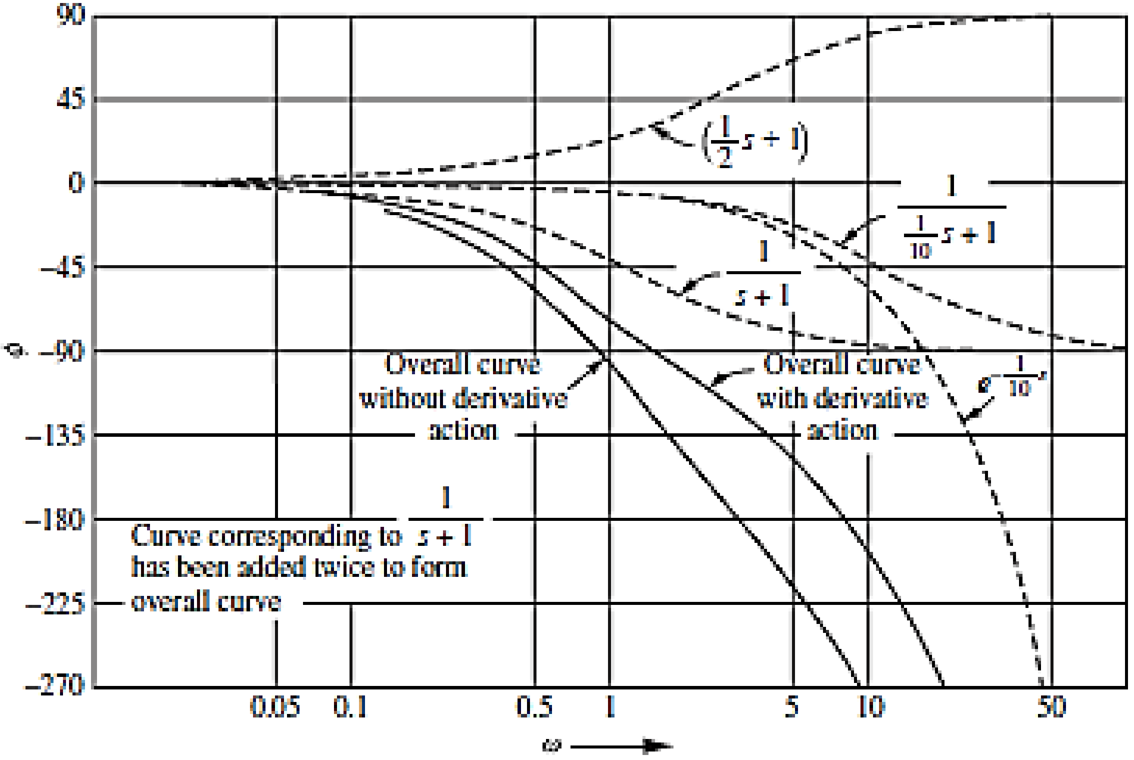
$$G(s) = \frac{10(0.5s + 1)e^{-s/10}}{(s + 1)^2(0.1s + 1)}$$

- The individual components are plotted as dashed lines in Fig.
- Only the asymptotes are used on the AR portion of the graph.
- Here it is easiest to plot the factor $(s + 1)^{-2}$ as a line of slope -2 through the corner frequency of 1.
- For the phase-angle graph, the factor $(s+1)^{-1}$ is plotted and added in twice to form the overall curve.
- For comparison, the overall curves obtained without derivative action (i.e., by not adding in the curves corresponding to $0.5s+ 1$) are also shown.
- Note that on the asymptotic AR diagram, the slopes of the individual curves are added to obtain the slope of the overall curve.

Block diagram for Amplitude ratio;



Block diagram for phase angle



Characteristics of Bode plots for some common transfer functions

Transfer function	Amplitude ratio AR	Phase angle (deg) ϕ	Corner frequency ω_c	Slope on Bode plot $\omega < \omega_c$	Slope on Bode plot $\omega > \omega_c$
$\frac{K}{1+\tau s}$ First-order system	$K\frac{1}{\sqrt{1+\omega^2\tau^2}}$	$\tan^{-1}(-\omega\tau)$ $\phi \to -90^\circ$ as $\omega \to \infty$	$\frac{1}{\tau}$	0	−1
$K(1+\tau s)$ First-order lead	$K\sqrt{1+\omega^2\tau^2}$	$\tan^{-1}(\omega\tau)$ $\phi \to +90^\circ$ as $\omega \to \infty$	$\frac{1}{\tau}$	0	1
$\frac{1}{\tau s}$ Integral	$\frac{1}{\omega\tau}$	−90	0	−1	−1
τs Derivative	$\omega\tau$	+90	0	1	1
$\frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$ Second-order system	$K\frac{1}{\sqrt{(1-\omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}}$	$\tan^{-1}\left(\frac{-2\zeta\omega\tau}{1-\omega^2\tau^2}\right)$ $\phi \to -180^\circ$ as $\omega \to \infty$	$\frac{1}{\tau}$	0	−2
$\exp(-\tau s)$ Transportation lag	1	$-\omega\tau\left(\frac{180}{\pi}\right)$	0	—	0