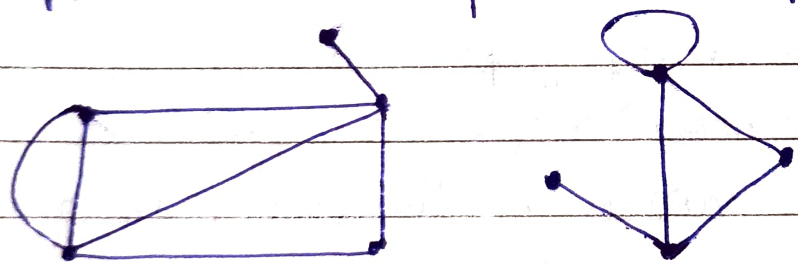
CONNECTED GRAPHS, DISCONNECTED AND COMPONENTS :-

→ A graph G is said to be connected if there is atleast one path between every pair of vertices in G.,
Otherwise, G is disconnected.

→ A null graph of ~~on~~ more than one vertex is disconnected.

→ A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component. A component itself is a graph.



A disconnected graph with two components.

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Theorem ① :- A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 & V_2 such that there exists no edge in G whose one end vertex is in subset V_1 & other in subset V_2.

Proof :- Suppose that such a partitioning exists. Consider two arbitrary vertices a & b of G, such that $a \in V_1$ & $b \in V_2$. No path can exist b/w vertices a & b; otherwise, there would be atleast one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

~~Contra~~ Conversely, let G be a disconnected graph. Consider a vertex a in G, let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition.

Theorem ②:- If a graph (connected or disconneted) has exactly two vertices of odd degrees, there must be a path joining these two vertices.

Proof → Let G be graph with all even vertices (even degree) except vertices v_1 & v_2, which are odd.

We know that for every graph & therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, v_1 & v_2 must belong to the same component, & hence must have a path between them.

Theorem (3):- A simple graph (a graph without parallel edges or self loops) with n vertices & k components can have at most $(n-k)(n-k+1)/2$ edges.

Proof → Let the no. of vertices in each of the k components of a graph be $n_1, n_2, \ldots, n_k$. Thus we have

$$n_1 + n_2 + \ldots + n_k = n$$

$$n_i \geq 1,$$

The proof of the theorem depends on a algebric inequality :

$$\sum_{i=1}^{k} (n_i - 1) = n - k \quad , \text{ Squaring both sides}$$

$$\left(\sum_{i=1}^{k} n_i - 1\right)^2 = n^2 + k^2 - 2nk$$

or $$\sum_{i=1}^{k} (n_i^2 - 2n_i) + k + \text{nonegative crossterms} = n^2 + k^2 - 2nk$$

because $(n_i - 1) \geq 0$,

for all i,

therefore

$$\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n$$

$$= n^2 - (k-1)(2n-k).$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$$

Now the maximum no. of edges in the ith component of (simple connected graph) is $\frac{1}{2} n_i(n_i - 1)$. Therefore, maximum no. of edges in G is

$$\frac{1}{2} \sum_{i=1}^{k} (n_i - 1) n_i = \frac{1}{2} \left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2}$$

$$\leq \frac{1}{2}\left[n^2 - (k-1)(2n-k)\right] - \frac{n}{2}$$

$$= \frac{1}{2}\cdot (n-k)(n-k+1)$$