2D Transformations-

to manipulate object in two dimensional space, we must apply various transformation functions to object. This allows us to change the position, size, and orientation of the objects. Transformations are used to position objects, to shape objects, to change viewing positions, and even to change how something is viewed.

Geometric Transformations-

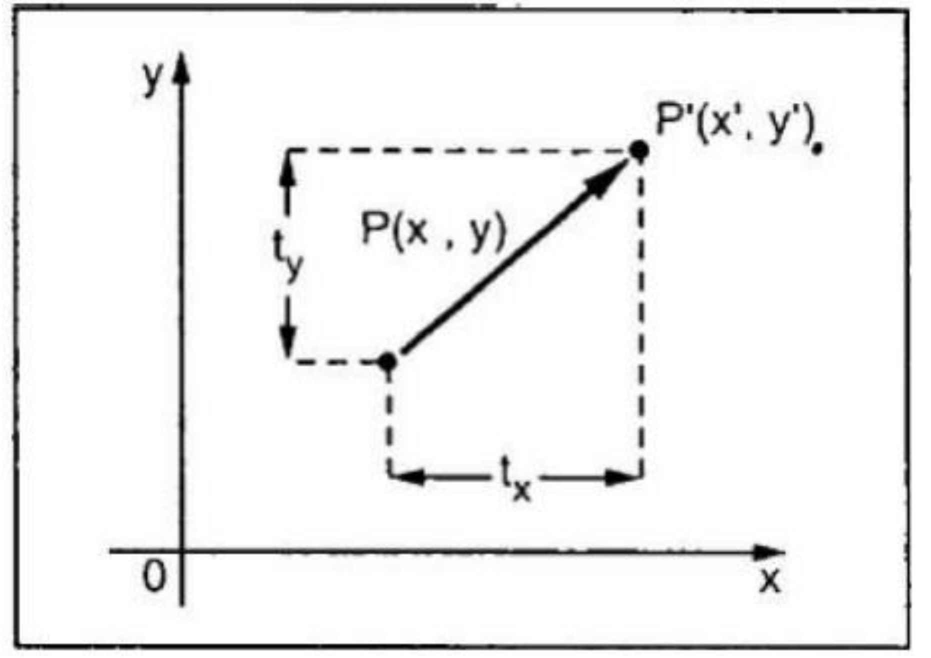
An object Obj in the plane can be considered as a set of points. Every object point P has coordinates (x, y), and so the object is the sum total of all its coordinate points. If the object is moved to a new position, it can be regarded as a new object Obj' , all of whose coordinate point P' can be obtained from the original points P by the application of a geometric transformation.

1 Translation- Translation involves moving the element from one location to another. In the case of a point, the operation would be

$x' = x + m, \; y' = y + n$

where x', y' = coordinates of the translated point

x, y = coordinates of the original point
m, n = movements in the x and y directions, respectively
In matrix notation this can be represented as
(x', y') = (x, y) + T
where T = (m,n), the translation matrix
P’ = P + T



2 Rotation about the origin- the points of an object are rotated about the origin by an angle O. For a positive angle, this rotation is in the counterclockwise direction.

point PX, Y can be represented as − $X = r\cos\phi. \; . \; . \; . \; . \; . \; (1)$

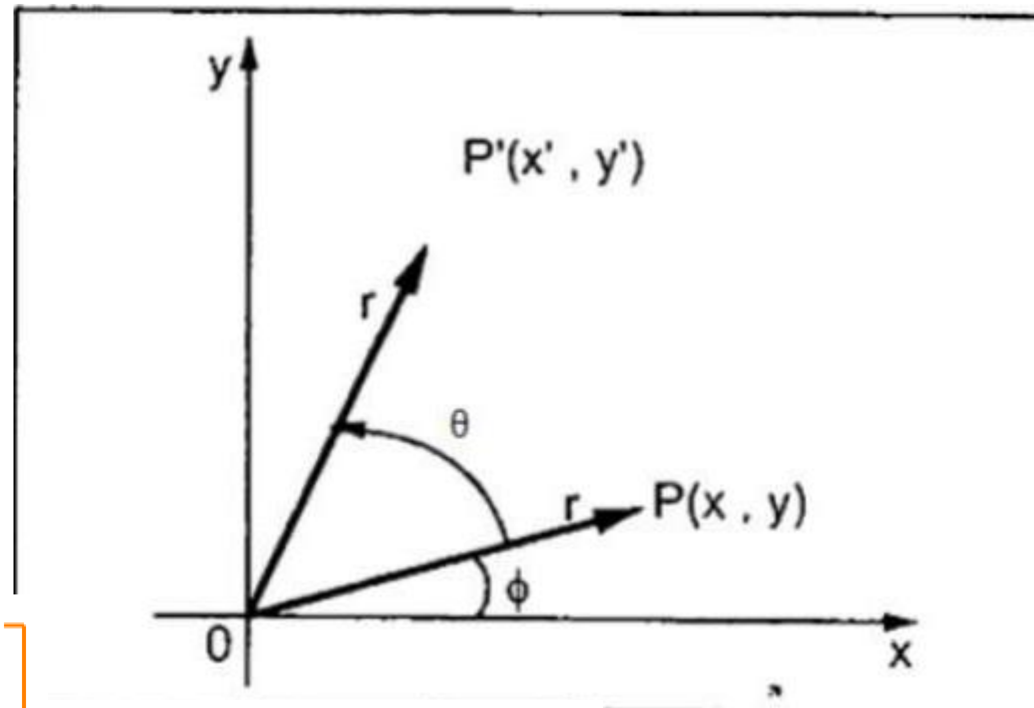
$Y = r\sin\phi. \; . \; . \; . \; . \; . \; (2)$

In matrix notation, the procedure would be as follows:

$(x',y') = (x,y)R$

where $x' = x\cos(\theta) - y\sin(\theta)$

and $y' = x\sin(\theta) + y\cos(\theta)$



$$[X'Y'] = [X'Y']\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} OR$$

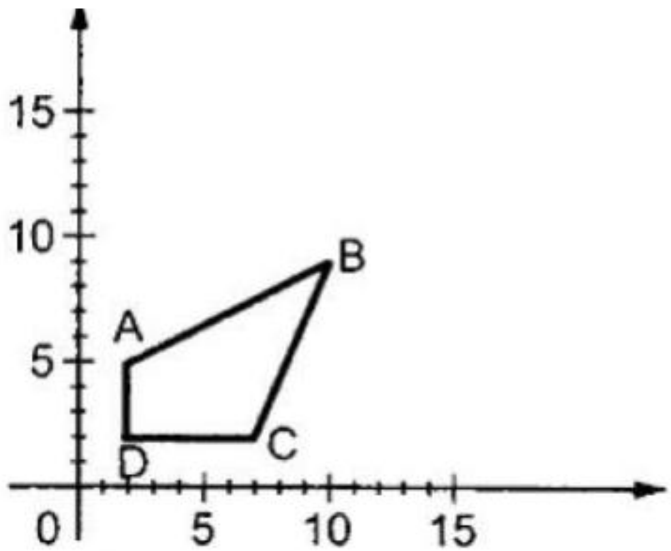
$$P' = P . R$$

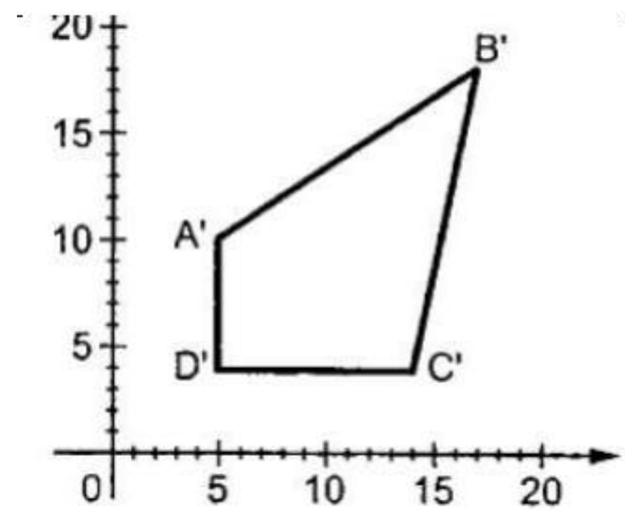
$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

Rotation Matrix

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

3 Scaling- To change the size of an object, scaling transformation is used. In the scaling process, you either expand or compress the dimensions of the object. Scaling can be achieved by multiplying the original coordinates of the object with the scaling factor to get the desired result.





Let us assume that the original coordinates are X, Y, the scaling factors are (SX, SY), and the produced coordinates are X ′ , Y ′ . This can be mathematically represented as shown below –

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \times \begin{bmatrix} X \\ Y \end{bmatrix}$$

Scaling Matrix

$$X' = X \,.\, SX \text{ and } Y' = Y \,.\, SY$$

$$P' = P \,.\, S$$

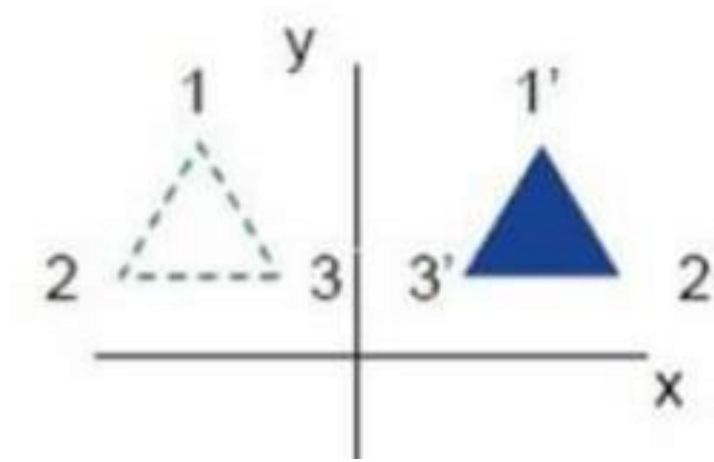
4 Reflection - Reflection is the mirror image of original object. In other words, we can say that it is a rotation operation with 180°. In reflection transformation, the size of the object does not change.

the mirror reflection about the y-axis is

$$x' = -x \text{ and } y' = y \,.$$

mirror reflection transformation x M about the x-axis is given by

$$x' = x \text{ and } y' = -y \,.$$



$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

Reflection Matrix

(Reflection Along X Axis)

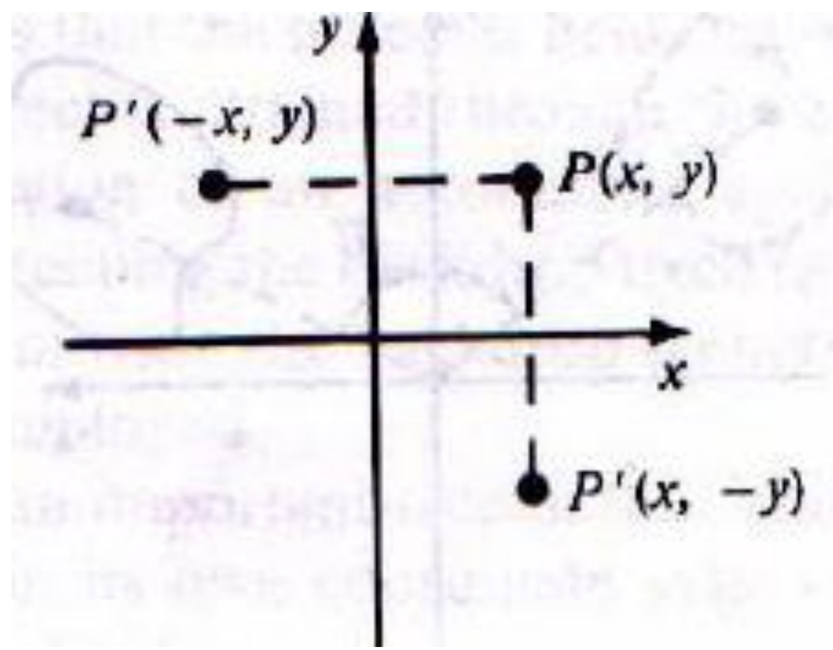
Reflection M

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

Reflection Matrix

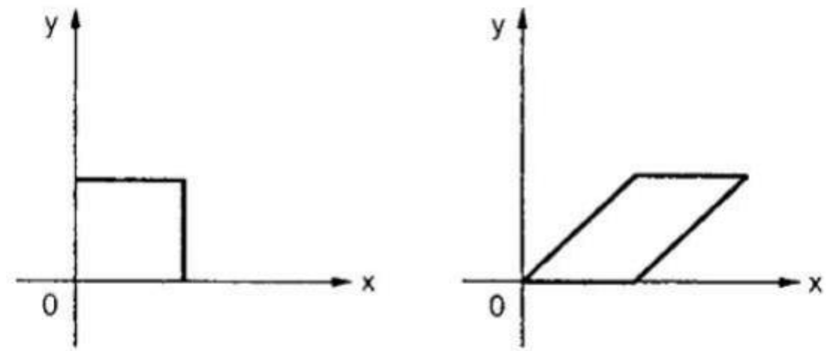
(Refl

Reflection Matrix in Computer Graphics | Reflection .



5 Shear - A transformation that slants the shape of an object is called the shear transformation. There are two shear transformations X-Shear and Y-Shear. One shifts X coordinates values and other shifts Y coordinate values. However; in both the cases only one coordinate changes its coordinates and other preserves its values. Shearing is also termed as Skewing.

X-Shear The X-Shear preserves the Y coordinate and changes are made to X coordinates, which causes the vertical lines to tilt right or left as shown in below figure.



(a) Original object

(b) Object after x shear

The transformation matrix for X-Shear can be represented as –

$$X_{sh} = \begin{bmatrix} 1 & 0 & 0 \\ shx & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 1 & Sh_x \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

Shearing Matrix

(In X axis)

Shearing Matrix in Computer Graphi

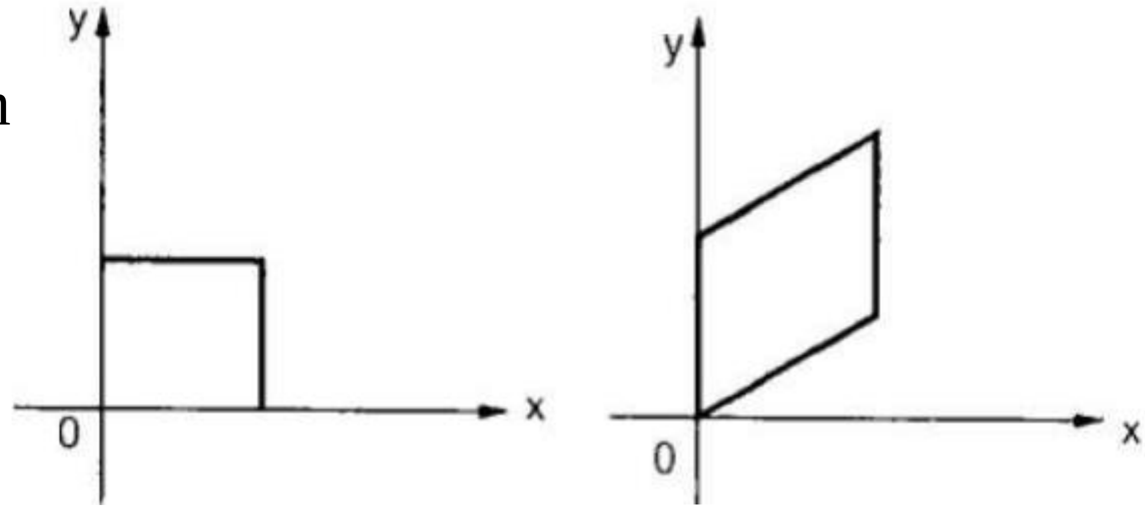
$X' = X + Shx \,.\, Y$

$Y' = Y$

Y-Shear The Y-Shear preserves the X coordinates and changes the Y coordinates which causes the horizontal lines to transform into lines which slopes up or down as shown in the following figure.

The Y-Shear can be represented in matrix from as −

$$Y_{sh} \begin{bmatrix} 1 & shy & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



(a) Original object

(b) Object after y shear

$$Y' = Y + Shy . X$$

$$X' = X$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Sh_y & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

Shearing Matrix

(In Y axis)

5 Inverse Geometric Transformation

Each geometric transformation has an inverse, which is described by the opposite operation performed by the transformation.

Translation: $T_v^{-1} = T_{-v}$ or translation in the opposite direction

Rotation: $\mathrm{R}_\theta^{-1} = \mathrm{R}_{-\theta}$ or rotation in the opposite direction

Scaling: $S_{s_x, s_y}^{-1} = S_{1/s_x, 1/s_y}$

Mirror reflection: $\mathrm{M}_x^{-1} = \mathrm{M}_x$ and $\mathrm{M}_y^{-1} = \mathrm{M}_y$

Transformations play an important role in computer graphics to reposition the graphics on the screen and change their size or orientation

Homogeneous Coordinates for Two Dimensions-

To perform a sequence of transformation such as translation followed by rotation and scaling, we need to follow a sequential process −

- Translate the coordinates,
- Rotate the translated coordinates, and then
- Scale the rotated coordinates to complete the composite transformation

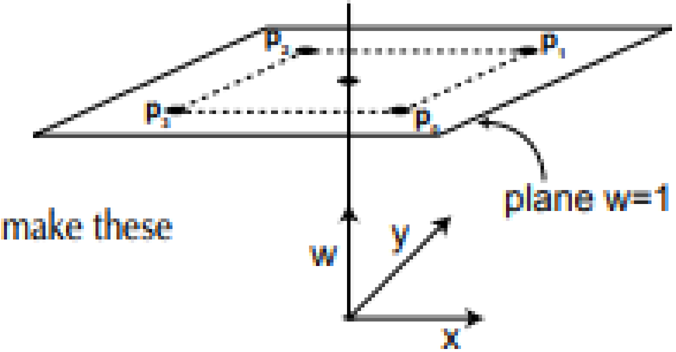
To shorten this process, we have to use 3×3 transformation matrix instead of 2×2 transformation matrix. To convert a 2×2 matrix to 3×3 matrix, we have to add an extra dummy coordinate W.

In this way, we can represent the point by 3 numbers instead of 2 numbers, which is called **Homogenous Coordinate** system. In this system, we can represent all the transformation equations in matrix multiplication. Any Cartesian point PX,Y can be converted to homogenous coordinates by P' (X_h, Y_h, h).

An important, practical aspect of the homogeneous coordinate system is its unification of the translation, scaling and rotation of geometric objects.

ultimately simpler result is obtained by introducing a third column to the matrix and a third (homogeneous) coordinate to the result-

We first take all of our points $\mathbf{x} = (x, y)$, express them as 2D vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and make these



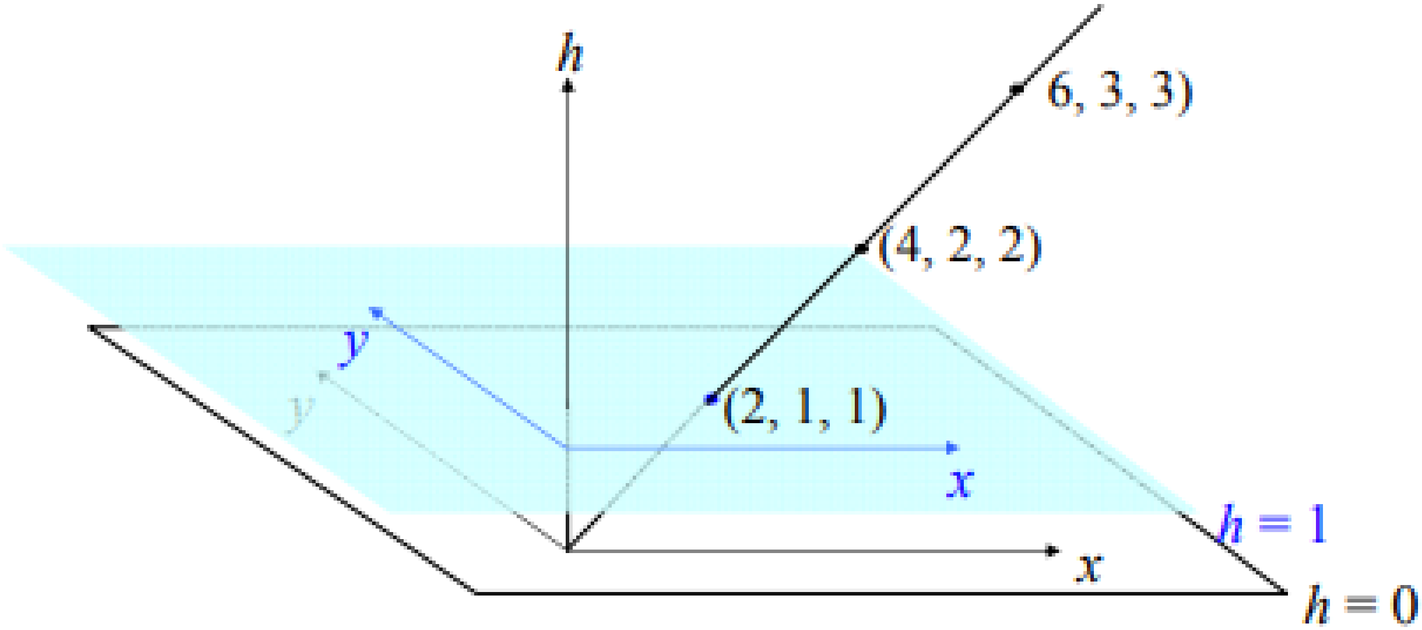
into 3D vectors with identical (thus the term homogeneous) 3rd coordinates set to 1:

$$\begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

By convention, we call this third coordinate the w coordinate, to distinguish it from the usual 3D z coordinate. We also extend our 2D matrices to 3D homogeneous form by appending an extra row and column, giving

$$\text{Scale:} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{Rotate:} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{Shear:} \begin{bmatrix} 1 & h_x & 0 \\ h_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Conversely, the point (2, 1) of the Cartesian corresponds to (2, 1, 1), (4, 2, 2) or (6, 3, 3) of the homogeneous system



$$\text{Translate: } \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix},$$

$$[p_x' \;\; p_y' \;\; 1] = [p_x \; p_y \; 1] \begin{bmatrix} a_{00} & a_{01} & 0 \\ a_{10} & a_{11} & 0 \\ c_1 & c_2 & 1 \end{bmatrix}.$$

Composite Transformation-

If a transformation of the plane T1 is followed by a second plane transformation T2, then the result itself may be represented by a single transformation T which is the composition of T1 and T2 taken in that order. This is written as

$$T = T1 \cdot T2.$$

Composite transformation can be achieved by concatenation of transformation matrices to obtain a combined transformation matrix.

A combined matrix −

[T] = [T1] [T2] [T3] [T4] …. [Tn]

P' = P*T

There are two conventions for vectors and matrices where vectors are written as columns:

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices multiply on the left:

$$\begin{aligned} \mathbf{Mv} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \end{aligned}$$

However, watch out for other forms where vectors are written as rows:

$$\mathbf{v} = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}^{\mathrm{T}}$$

Matrices multiply on the right:

$$\begin{aligned} \mathbf{vM} &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix} \end{aligned}$$

[P*] = [P] [T]where,

[P*] is the new coordinates matrix in row vector

[P] is the original coordinates matrix, or points matrix in row vector

[T] is the transformation matrix

Example 1: If the triangle A(1,1), B(2,1), C(1,3) is scaled by a factor 2, find the new coordinates of the triangle.

Solution:

Writing the points matrix in homogeneous coordinates, we have-

$$[P] = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 \end{pmatrix}$$

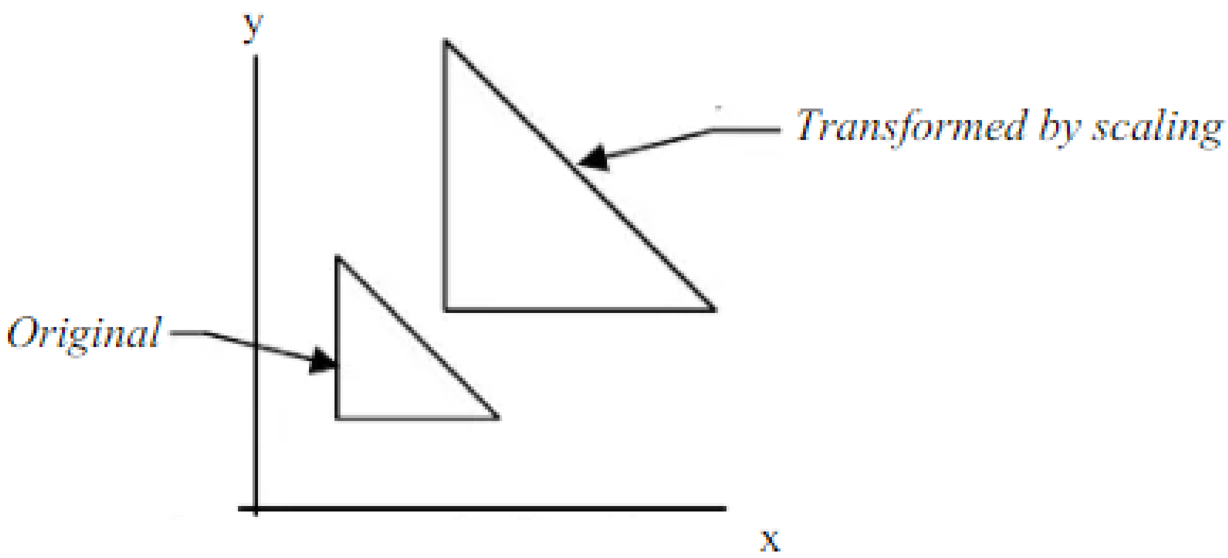
and the scaling transformation matrix is,

$$[T_s] = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The new points matrix can be evaluated by the equation-

[P*] = [P] [T]

$$P^* = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 4 & 2 & 0 & 1 \\ 2 & 6 & 0 & 1 \end{pmatrix}$$



***Example 2* :** Translate the rectangle (2,2), (2,8), (10,8), (10,2) 2 units along x-axis and 3units along y-axis.

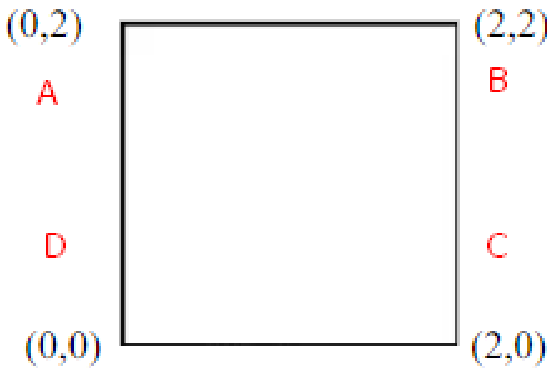
***Solution* :** Using the matrix equation for translation, we have

$$[P^*] = [P]\,[T]$$

$$[P^*] = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 8 & 0 & 1 \\ 10 & 8 & 0 & 1 \\ 10 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 & 0 & 1 \\ 4 & 11 & 0 & 1 \\ 12 & 11 & 0 & 1 \\ 12 & 5 & 0 & 1 \end{pmatrix}$$

Example 3 *:* Rotate the rectangle (0,0), (2,0), (2, 2), (0, 2) shown below, 30 degree ccw about its centroid and find the new coordinates of the rectangle.



Solution **:** Centroid of the rectangle is at point (1, 1).

We will first translate the centroid to the origin, then rotate the rectangle, and finally, translate the rectangle so that the centroid is restored to its original position.

1. ***Translate the centroid to the origin***: The matrix equation for this step is-

$$[P^*]_1 = [P]\,[T_t], \text{ where } \quad [P] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

$$[T_t] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}$$

2. ***Rotate the Rectangle 30 degree ccw About the z-axis*** **:**

The matrix equation for this step is given as[P*2] = [P1*] [Tr],

where, [P1*] is the resultant points matrix obtained in step 1, and [Tr] is the rotation transformation, where θ = 30degree ccw. The transformation matrix is,

$$[T_r]_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .866 & .5 & 0 & 0 \\ -.5 & .866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. ***Translate the Rectangle so that the Centroid Lies at its Original Position*****:**

The matrix equation for this step is

[P3*] = [P2*] [T-t],

where [T-t] is the reverse translation matrix, given as

$$[T_{-t}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Now we can write the entire matrix equation that combines all the three steps outlined above. The equation is, [P*] = [P] [Tt] [Tr] [T-t]

$$[P^*] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 30^0 & \sin 30^0 & 0 & 0 \\ -\sin 30^0 & \cos 30^0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \text{x}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.6340 & -0.3660 & 0 & 1 \\ -0.3660 & 1.3660 & 0 & 1 \\ 1.3660 & 2.3660 & 0 & 1 \\ -0.3660 & 1.3660 & 0 & 1 \end{pmatrix}$$

The first two columns represent the new coordinates of the rotated rectangle.

Example 4: Given the triangle, described by the homogeneous points matrix below, scale it by afactor 3/4, keeping the centroid in the same location. Use (a) separate matrix operation.

$$[P] = \begin{array}{c} A \\ B \\ C \end{array} \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 5 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{pmatrix}$$

Solution

(a) The centroid of the triangle is at,

x = (2+2+5)/3 = 3, and y = (2+5+5)/3 = 4 or the centroid is C(3,4).

We will first translate the centroid to the origin, then scale the triangle, and finally translate it back to the centroid. Translation of triangle to the origin will give,

$$[P^*]_1 = [P]\,[T_t] = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 5 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

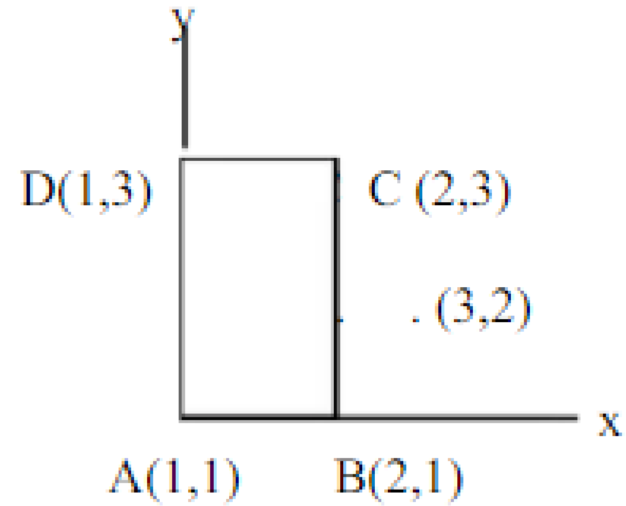
Scaling the triangle, we get,

$$[P^*]_2 = [P^*]_1\,[T_s] = \begin{pmatrix} -1 & -2 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} .75 & 0 & 0 & 0 \\ 0 & .75 & 0 & 0 \\ 0 & 0 & .75 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -0.75 & -1.5 & 0 & 1 \\ -0.75 & 0.75 & 0 & 1 \\ 1.5 & 0.75 & 0 & 1 \end{pmatrix}$$

Translating the triangle so that the centroid is positioned at (3, 4), we get

$$[P^*] = [P^*]_2 \, [T_{-t}] = \begin{pmatrix} -.75 & -1.5 & 0 & 1 \\ -.75 & .75 & 0 & 1 \\ 1.5 & .75 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2.25 & 2.5 & 0 & 1 \\ 2.25 & 4.75 & 0 & 1 \\ 4.5 & 4.75 & 0 & 1 \end{pmatrix}$$

Example 5 : Rotate the rectangle formed by points A(1,1), B(2,1), C(2,3), and D(1,3) 30 degree ccwabout the point (3,2).



***Solution*:** We will first translate the point (3,2) to the origin, then rotate the rectangle about the origin, and finally, translate the rectangle back so that the original point is restores to its original position (3,2).

The new coordinates of the rectangle are found as follows.

$$[P^*] = [P]\,[Tt]\,[Tr]\,[T-t]$$

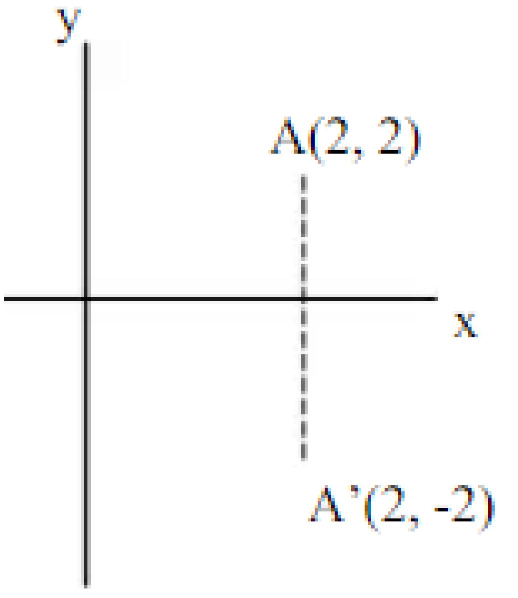
$$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 1 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} .866 & .5 & 0 & 0 \\ -.5 & .866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1.77 & .13 & 0 & 1 \\ 0.77 & 1.87 & 0 & 1 \\ 1.63 & 2.37 & 0 & 1 \\ 2.63 & 0.63 & 0 & 1 \end{pmatrix}$$

These are the new coordinates of the rectangle after the rotation.

Example 6: we want to mirror the point A(2,2) about the x-axis(i.e., xz-plane), as shown in the figure.

Solutions:- can be obtained with the matrix transformation given below



$$[P^*] = [2 \quad 2 \quad 0 \quad 1]\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= [2 \quad -2 \quad 0 \quad 1]$$