Curves representation: A curve is an integral part of any design and an engineer needs to draw one or other type of curves (curved surfaces) applicable to many engineering components used in automotive, aerospace industries.

- Different types of shape constraints (e.g., continuity and/or curvature) are imposed to accomplish specific shapes of the curve or curved surfaces.
- When a curve is two-dimensional, it lies entirely in a plane known as planar curve.
- However, three-dimensional curve lies in space called space curve.
- in general, engineering applications require smooth curves. A smooth curve can be generated by reducing the spacing between the data points.
- The data points may be equally spaced or non-evenly spaced.
- Increasing the density of data points in the region where radius of the curvature is small improves the quality, as regards to the smoothness, of the curve.
- The advantage of non-evenly spaced data points can be visualized for generating the smooth curve.

(a) Evenly Spaced data points (b) Non-evenly spaced data points

There are two techniques for the curve representation.

1-Analytic Curves :-Analytic curves can be represented by the analytical (mathematical) equations such as lines, circles and conics. This type of curve representation has the following advantages:

- Precise and easy evaluation of the intermediate points
- Mathematical representation of curve is computer friendly, i.e., compact storage of curve
- Curve properties such as slope and radius of curvature can be easily evaluated
- Drawing of curves is easy from the storage data
- Alteration / manipulation of curve is easy to meet the modified design criteria

2-Synthetic Curves Unfortunately, it is not possible to represent all types of curves required in engineering applications analytically; therefore, the method based on the data points (synthetic curves) is very useful in designing the objects with curved shapes such as ship hull, car body, aerofoil section, automobile components, etc.

- Synthetic curves such as Bezier curves and splines are described by a set of data points known as control points.
- Parametric polynomials usually fit the control points.
- Synthetic curves provide greater flexibility to the designer just by changing the positions of the control points.
- Moreover, it is possible to achieve a local control and global control of the shape of the curve.

Synthetic Curves The data (control) point representation of curve suffers from the following disadvantages:

- Slope of the curve is obtained using numerical differentiation, a well-known inaccurate procedure.
- A good quality circle requires a minimum of 32 points on its circumference; therefore, a huge storage is required as compared to the analytical representation of circle in which centre and radius is sufficient to represent the circle.
- Intermediate points are obtained using the interpolation techniques. The resulting intermediate points do not actually lie on the curve.
- It is not possible to calculate the exact property of the curve because exact shape of the curve is not known.
- Difficult to handle the transformations of curve due to the large number of data points.

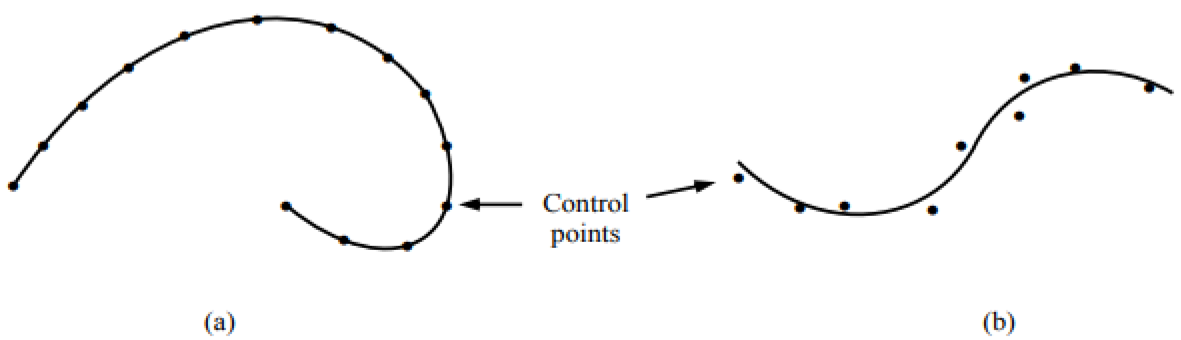
INTERPOLATION AND APPROXIMATION:

Interpolation

- The interpolation is a technique by which a curve, represented with known set of data points, can be defined analytically.
- The data points may be obtained through the experimental measurements or from some known function.
- When curve passes through all the data points, it is said to fit the data.
- ‘Piecewise polynomial approximation’ technique of curve fitting is used to determine the coefficient of polynomials of some degree.
- The curve shape between the data points depends upon the degree of polynomial and the associated boundary conditions.

Approximation

- If data points (control points) are only approximation to some true values (e.g., measurement points, etc.) then the curve does not necessarily pass through the data points rather than it approximates or fairs the data points.
- The curve depicts the trend of data points.
- Least square approximation is a common curve fairing technique, which produces the curve of the form y = f (x) , which minimizes the sum of y squared deviations between the data and the derived curve.
- Depending upon the information about the phenomenon that produces the data points, the curve y = f (x) may be Power functions, □ Exponential functions, Polynomial functions , Trigonometric functions, and Probability distributions, etc.



Interpolation and Approximation of data points (a) curve fitting (b) curve fairing

CLASSICAL REPRESENTATION OF CURVES

Mathematically, non-parametric and parametric equations are used for the representation of planar curves / space curves.

Non-parametric Curves

- A non-parametric curve representation may be explicit (y=f(x)) or implicit (y-f(x)=0). In explicit form, coordinate(s) of a point y and/or z are explicitly represented as function of x.
- Explicit, non-parametric space curve is represented as x = x, y = f (x) and z = f (x)
- The above equation has one-to-one relationship. Therefore, this is not suitable for the representation of closed or multivalued curves.
- Closed or multiple valued planar curves, e.g., a circle, parabola, ellipse, etc. gives two values of y for each value of x.
- This form of curve representation is known as implicit non-parametric form of the curve.
- a general implicit non-parametric planar curve can be represented as f (x, y) = 0

- For example, a general second-degree implicit non-parametric equation is written as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

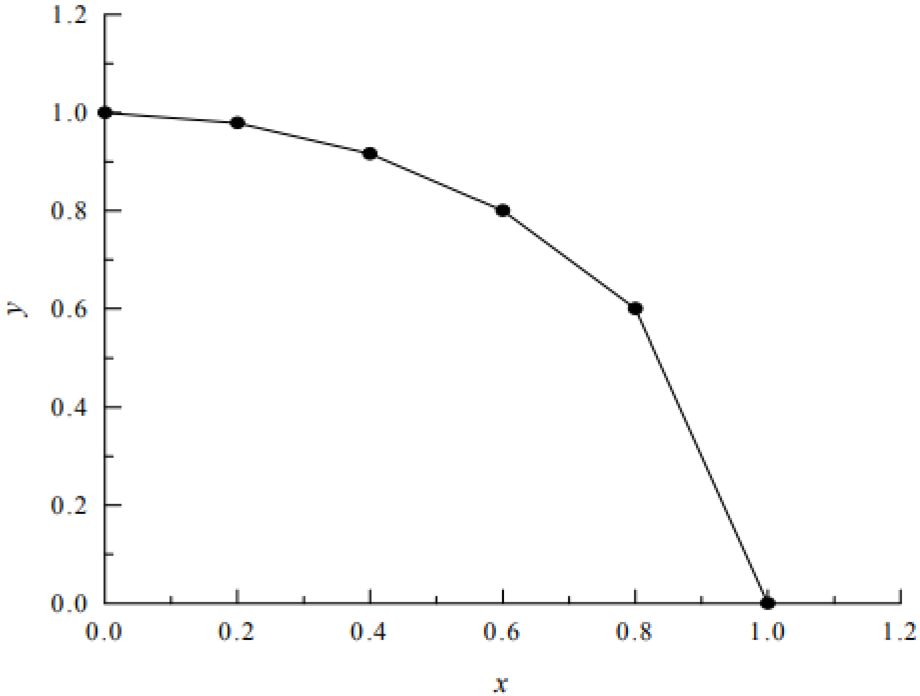
- Above equation gives a variety of two-dimensional (planar) curves called conic sections.
- Depending upon the values of coefficients in the equation, a planar curve may be described by specifying the following conditions:
    - Positions and slopes at the two endpoints of the curve segment
    - Positions of the two endpoints and slopes either at the beginning or at the end of curve segment

Properties of Non-parametric Curves-

I. Explicit and implicit non-parametric curve representations are axis dependent.

II. A non-parametric representation of curve results into unequal distribution of points on the curve, which in turn, affects the quality and accuracy of the curve.

III. If a curve is to be displayed as a series of points or straight-line segments, the computations involved could be expensive.



Explicit non-parametric representation of a unit radius origin centered circle

PARAMETRIC CURVES:

- Parametric representations of closed or multivalued curves overcome the difficulties associated with the non-parametric representations.
- Parametric representations for commonly used curves such as conic sections employ polynomials in place of equations involving the square root calculations.
- Thus, parametric representations for the curves are more general and suitable for the CAD applications due to the ease in computations.
- In parametric form, each point on the curve is expressed as a function of single parameter.
- Thus, position vector of a point on the curve is fixed by a single parameter. For two-dimensional (planar) curve with t as a parameter, the Cartesian coordinates of a point on the curve is expressed as

$$x = x(t) \text{ and } y = y(t) \quad \text{or} \quad P(t) = \begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix}$$

- A single non-parametric curve equation (in terms of x and y) may be obtained from two parameter equations by eliminating the parameter t.
- The tangent vector (or derivative) for a parametric curve is defined as-

$$P'(t) = \begin{Bmatrix} x'(t) \\ y'(t) \end{Bmatrix}$$

- Therefore, the slope of the parametric curve is given by-

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

- With parametric representations, the infinite slope condition (dy/dx=∞) can easily be obtained by substituting x'(t) = 0
- Since a point on the curve is specified by a single parameter t; therefore, the parametric curves are axis independent.

  Mostly 0 =< t <=1
- The position and slope at the endpoints of the curve is specified by the parameter t, which is fixed within the parameter range.

Lines

The parametric representation of a straight line connecting the two position vectors P1 and P2 is given by $P(t) = P_1 + (P_2 - P_1).t$, $0 \le t \le 1$

The position vector P (t) has a parametric representation x (t) and y (t) ; therefore $x(t) = x_1 + (x_2 - x_1).t \quad 0 \le t \le 1$

$$y(t) = y_1 + (y_2 - y_1).t \quad 0 \le t \le 1$$

Moreover, tangent vector of the line is given as

$$P'(t) = P_2 - P_1$$

$$x'(t) = x_2 - x_1$$

$$y'(t) = y_2 - y_1$$

Thus, the tangent vector of the line is independent of the parameter t. The infinite slope (vertical line) condition and zero slope (horizontal line) condition can be obtained from above eqn.

SPACE CURVES (3D)

- Space (three-dimensional) curves and surfaces are mostly used in the design of automobile bodies, aerospace wings, ship hulls, propeller blades, shoes, bottles, etc.
- These applications require curves and surfaces as basic entities.
- Curve is the collection of points and they form basic entities of the surfaces. Surfaces can be obtained by digitizing the physical model or a drawing, followed by curve fitting through the set of data points.
- Mathematically, curve fitting (interpolation) and curve fairing (approximation) techniques are used for generating the curves in computer graphics.
- The analytical form of planar curves is not suitable for designing the complex three dimensional curves and surfaces used for designing the complex shaped objects.
- The designer prefers the synthetic curve, which passes through the set of data points, because designer has full control on its shape as per the new design requirements.

PROPERTIES FOR CURVE DESIGN

- In computer graphics, a curve is represented in such a manner that it must be mathematically tractable and computationally convenient.
- The experiences of designer suggest that the curve must possess the following important properties for the design and representation in computer graphics:

1 Control points

* The control points govern the shape of the curve in a predictable manner.

* It is possible to control the shape of the curve interactively through proper location of the control points.

* A curve must interpolate (pass) the control points.

2 Axis Independence

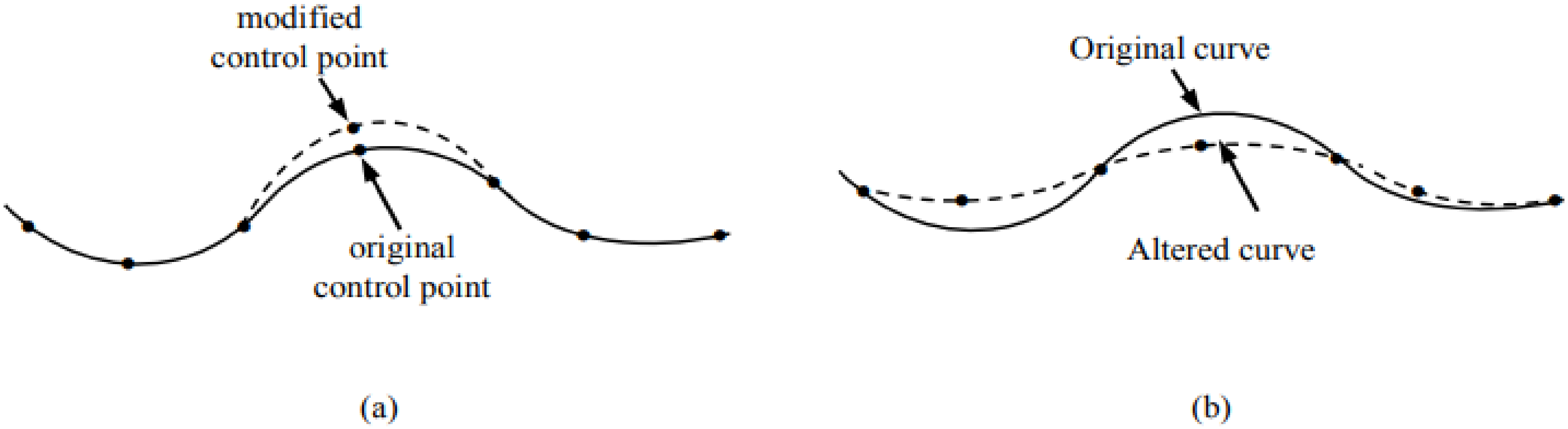
* The shape of curve must not change if control points are measured in different coordinate system.

* For example, if control points rotate by 30o , the entire curve must rotate by 30o , keeping the shape unchanged.

* Due to its axis independent nature, it is possible to transform a parametric curve into a curve of the same shape but with different orientations.

3 Local Control and Global Control

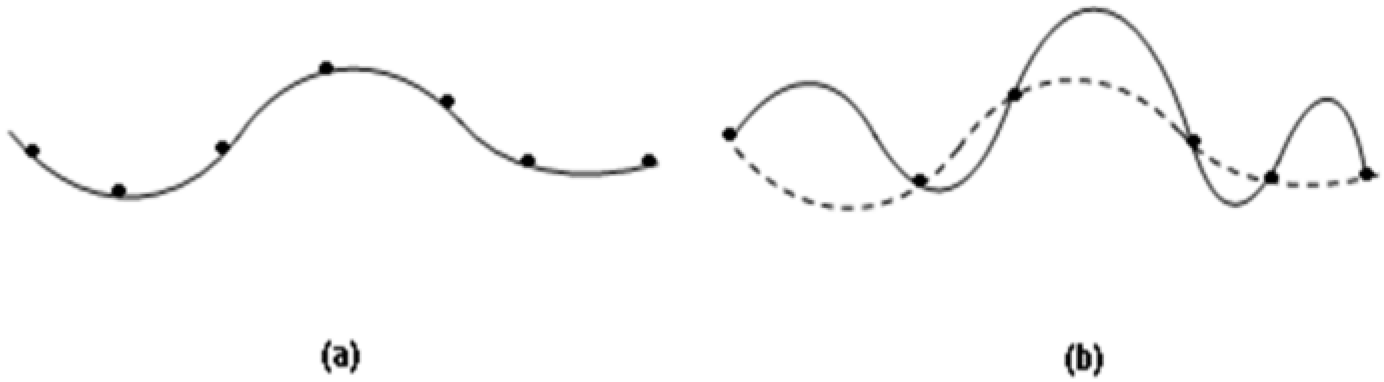
- In computer graphics, it is frequently required to modify the portion of the curve.
- The curve may change its shape only in the portion near to the control point or the shape of entire curve may change.
- The first modification in the curve shape is termed local control whereas second modification as global control.
- A designer is always interested in local control because altering the position of control point does not propagate the change in the remaining portion of the curve.



Shape control of curve (a) Local (b) Global

4 Variation Diminishing Property

- A curve is said to be smooth if it has a tendency to pass through the control points smoothly.
- If curve oscillates about the control points it is usually not desirable.
- Thus, the curve which passes through the control points smoothly and does not show the tendency to amplify the irregularities in the form of oscillations, is said to possess the variation diminishing property.



(a) Variation diminishing property

(b) Curve with undesirable oscillations about the control points

5 Versatility

- The mathematical model for curve representation should allow the designer to change its shape by either adding or removing the control points.
- This implies the versatility of the curve, i.e., addition of the control points defining the curve gives additional shapes to the curve depending upon the position of additional control points.

6 Order of Continuity

- It is difficult to achieve the complex shape of object with a single curve.
- Usually, several curves are joined together end to end to accomplish the complex shape.
- The order of continuity decides the exact shape of the joint.
- The parametric continuity results by matching the parametric positions and the parametric derivatives of adjoining curves at their common boundary.

There are three types of order of continuity:

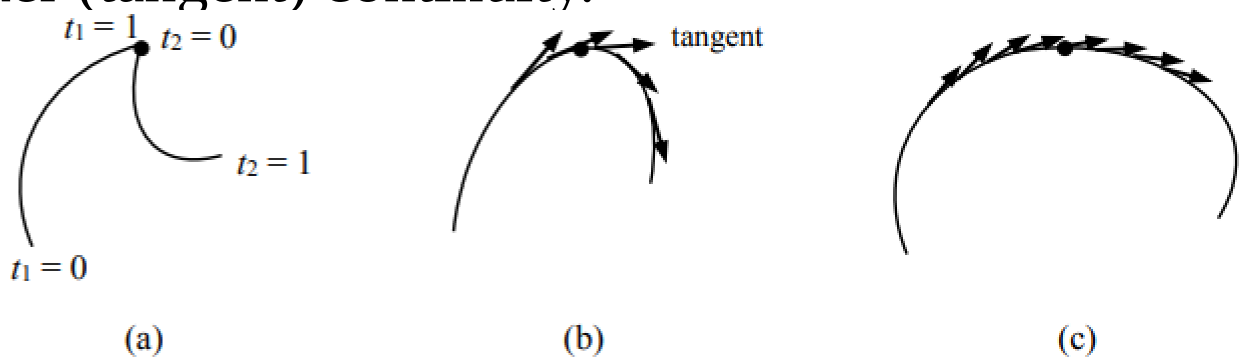
Zero Order Continuity or Continuity C^0

Zero order continuity exists when adjoining curves simply meet to form a joint, i.e., parameter t at the joint, for the two adjoining curves are same. Figure (a) shows the zero order (position) continuity.

First Order Continuity or Continuity C^1

- First order continuity exists when first order derivatives (i.e., tangent) for two adjoining curves, at their joining point, are same.
- The rate of change of the tangent vectors (second derivatives) can be quite different so that general shapes of the two adjacent sections can change abruptly.
- A joint with continuity also possesses C^0 and C^1 continuity. Figure (b) shows the first order (tangent) continuity.

(a) Zero Order Continuity (b) First Order continuity (c) Second Order Continuity



Parametric continuities at the junction point of two curves

Second Order Continuity or Continuity C^2

- First order continuity is generally sufficient for digitizing the drawings whereas second order continuity is required for setting up animation paths for the camera motion.
- A camera moving in C^1 continuity or tangent continuity with equal steps in parameter t experiences acceleration at the joint of two adjoining curve sections, leading to the discontinuity in motion in the form of jerks.
- Therefore, C^2 continuity or curvature continuity is desirable for camera motion during the animation.

SYNTHETIC CURVES