VECTOR SUBSPACE

Theorem:- Union of two subspaces is a subspace if and only if one is contained in other.

Proof: Suppose W_1 and W_2 are two subspaces of a vector space V over field F.

Let $W_1 \subseteq W_2$ Or $W_2 \subseteq W_1$
then $W_1 \cup W_2 = W_2$ Or W_1

Since W_1 and W_2 are subspaces then so
$W_1 \cup W_2 = W_1$ or $W_1 \cup W_2 = W_2$ is also a subspace

Converse: suppose $W_1 \cup W_2$ is a subspace of V[F]
T.P.T. $W_1 \subseteq W_2$ Or $W_2 \subseteq W_1$

Let us assume that W_1 is not subset of W_2
and W_2 is not subset of W_1
So if W_1 is not subset of $W_2 \Rightarrow \exists\, \omega_1 \in W_1$ and $\omega_1 \notin W_2$ ①
smly W_2 is not subset of $W_1 \Rightarrow \exists\, \omega_2 \in W_2$ and $\omega_2 \notin W_1$ ②
from ① & ⑪ $\omega_1 \in W_1 \cup W_2$
$\omega_2 \in W_1 \cup W_2$
$\Rightarrow \omega_1 + \omega_2 \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is subspace)
$\Rightarrow \underline{\omega_1 + \omega_2 \in W_1}$ or $\underline{\omega_1 + \omega_2 \in W_2}$
Let $\omega_1 + \omega_2 \in W_1$, $\omega_1 \in W_1 \Rightarrow -\omega_1 \in W_1$
$\therefore (\omega_1 + \omega_2) - \omega_1 \in W_1$ ($\because W_1$ is subspace)
$\Rightarrow \omega_2 \in W_1 \Rightarrow\Leftarrow$
W_2 is subspace of W_1
smly if $\omega_1 + \omega_2 \in W_2 \Rightarrow W_1$ is subspace of W_2. Proved

Theorem:- The necessary and sufficient condition for a nonempty subset W of a vector space V(F) to be a subspace of V is

if $a, b \in F$, $u, v \in W \Rightarrow au + bv \in W$

Proof:- Necessary:-

If W is a subspace of V(F) then W is itself a vector space of V[F]

$\Rightarrow$ W is closed under vector addition and scalar multiplication.

$\therefore \left.\begin{array}{l}\forall a, b \in F\\ u, v \in W\end{array}\right\}$ $\begin{array}{l} au \in W \\ bv \in W\end{array}$

and thus $au + bv \in W$.

Sufficient:- suppose W is a non empty subset of V satisfying the given condition i.e.

for $a, b \in F$, $u, v \in W$ s.t $au + bv \in W$

In particular, $a = 1 = b$

(W,+) abelian $\left[\begin{array}{l} \Rightarrow u + v \in W \quad \text{(closure)} \\ u + (v + \omega) = (u + v) + \omega \quad \text{(associative)} \\ a = 0,\ b = 0 \Rightarrow 0 \in W \quad \text{(Identity)} \\ a = -1,\ b = 0 \Rightarrow -u \in W, \forall u \in W \end{array}\right.$

S.m. $b = 0 \Rightarrow au \in W$

Remaining holds true.

so W is subspace of V[F].

Linear Combination of Vectors

Let $V[F]$ be a vector space. If $v_1, v_2, \ldots, v_n \in V$, then any vector $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ is called a linear combination of the vectors $v_1, v_2, \ldots, v_n$.

Ex. $V = \mathbb{R}^2$ $\quad v_1 = (1,2),\ v_2 = (3,4) \in \mathbb{R}^2$
then $v = \alpha_1 v_1 + \alpha_2 v_2$ is L.C. of v_1 and v_2.

Linear Span:- Let $V[F]$ be a v.s. and S be any non empty subset of V. then the linear span of S is the set of all linear combinations of finite sets of elements of S and is denoted by $L(S)$.

$$L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n;\ \alpha_i \in F,\ i=1,\ldots,n\}$$

note: $\quad L(S) = \langle S \rangle$

Prove that $L(S)$ is the smallest subspace of V containing S.