(13) $\Delta(\lambda) = \lambda^3 - 18\lambda^2 + (5+20+20)\lambda + 0$

$$= \lambda^3 - 18\lambda^2 + 45\lambda$$

$$= \lambda(\lambda^2 - 18\lambda + 45)$$

$$= \lambda\left[\lambda^2 - 15\lambda - 3\lambda + 45\right]$$

$$= \lambda\left[\lambda(\lambda-15) - 3(\lambda-15)\right]$$

$$= \lambda(\lambda-15)(\lambda-3)$$

$\lambda = 0, 3, 15$

$$|A| = 8(5) + 6(-10) + 2(10)$$

$$= 40 - 60 + 20$$

$$= 0$$

(10) $$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & -4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 - R_2 \to R_1$

$$\begin{bmatrix} 1 & 0 & 8 \\ 2 & -3 & -4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1 \to R_2$

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & -3 & -20 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & -1 & 1 \\ 0 & -3 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix}$$

## 7. RANK OF A MATRIX

**Definition:** The rank of a matrix is the order of the largest square submatrix whose determinant is not zero.

**Example:** Find the rank of

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 5 & 4 & 7 \\ -1 & -12 & 5 \end{pmatrix}$$

**Solution:** $|A| = \begin{vmatrix} 1 & -2 & 3 \\ 5 & 4 & 7 \\ -1 & -12 & 5 \end{vmatrix}$

$$= 1(20 + 84) + 2(25 + 7) + 3(-60 + 4) = 0$$

Hence the rank is not 3. Examine the submatrix of order 2.

$$\begin{pmatrix} 1 & -2 \\ 5 & 4 \end{pmatrix}, \begin{vmatrix} 1 & -2 \\ 5 & 4 \end{vmatrix} = 4 + 10 \neq 0$$

$\Rightarrow$ Rank of a matrix $A = 2$.

Tutorial - 1

① Reduce the matrix A to triangular form,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$$

Ans - $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

② Find the inverse of the matrix

$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$, by using elementary transformations.

Ans - $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$.

③ Find the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

Ans - $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

**Example 1.** *Find the rank of the matrix*

(*i*) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

**Sol.** (*i*) Here, $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

Operating $R_{21}$ (2)

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 11 & 15 \end{bmatrix}$$

which is Echelon form.

$\therefore$ $\rho(A)$ = no. of non-zero rows = 2

(*ii*) Here, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$\because$ $|A| = 0 \quad \therefore \quad \rho(A) \neq 3$

A $2 \times 2$ order minor is $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \neq 0$

$\therefore$ $\rho(A) = 2$

(*iii*) Here, $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

Operating $R_{21}$ (– 2)

$$\sim \begin{bmatrix} 2 & 3 & 4 & -1 \\ 1 & -4 & -8 & 1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

Operating $R_{12}$

$$\sim \begin{bmatrix} 1 & -4 & -8 & 1 \\ 2 & 3 & 4 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

Operating $R_{21}$ (– 2), $R_{31}$ (4)

$$\sim \begin{bmatrix} 1 & -4 & -8 & 1 \\ 0 & 11 & 20 & -3 \\ 0 & -11 & -20 & 3 \end{bmatrix}$$

Note: Number of nonzero rows in the Echelon form is called rank of the matrix.

(Q)– Find the Rank of the matrix

$$A=\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$ by Echelon form.

Solution:

Here $$A=\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

$R_2 - 2R_1 \to R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 4 & -1 \\ 1 & -4 & -8 & 1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -4 & -8 & 1 \\ 2 & 3 & 4 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

$R_2 - 2R_1 \to R_2$, $R_3 + 4R_1 \to R_3$

$$A \sim \begin{bmatrix} 1 & -4 & -8 & 1 \\ 0 & 11 & 20 & -3 \\ 0 & -11 & -20 & 3 \end{bmatrix}$$

$R_3 + R_2 \to R_3$

$$A \sim \begin{bmatrix} 1 & -4 & -8 & 1 \\ 0 & 11 & 20 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the Echelon form

$\therefore$ $\text{Rank}(A) = \rho(A) = 2$ //

# Tutorial - 2

① Find the rank of the matrices

(i) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

② Reduce A to Echelon form & find its rank

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

③ Find the rank of the following matrices by reducing it to normal form

(i) $\begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 7 & 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

④ Under what condition, the rank of the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{bmatrix} \text{ is } 3 \text{ ?}$$

⑤ If $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 3 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$. find

rank(A), rank(B), rank(A+B), rank(AB).

(i)
(6) – Under what condition, the rank of the following matrix A is 3? Is it possible for the rank to be 1?

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$$

Ans $x \neq \frac{3}{5}$

(iii) Find the value of $p$ for which rank of the matrix

$$A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$$ is of rank 1.

Ans $p = 3$

Solution of system of linear eqn :-
Solution of system of eqns
consider the system of eqns

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

In matrix notation, these eqns can be written as

$$AX = B$$

If $d_1 = d_2 = d_3 = 0$, then $B = 0$ and the matrix eqn $AX = B$ reduces to $AX = 0$.
such a system of eqns is called a system of homogeneous linear eqns.
If atleast one of $d_1, d_2, d_3$ is non-zero then $B \neq 0$.
Such a system of eqns is called system of non-homogeneous linear eqns.

⑥ – Solve

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

<u>Ans</u> : $x = 2, \ y = 1, \ z = 0.$

<u>Note</u> :- If rank of the given matrix is equal to number of variables, then the system is consistent otherwise inconsistent. $\rho[A:B]$ → rank of augmented mat.

<u>Solution</u> :- coefficient matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

Since $|A| \neq 0$

$\therefore \ \rho(A) = 3$

Augmented matrix $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\therefore \ \rho[A:B] = 3$$

Since, $\rho[A:B] = \rho(A) = 3$ (number of unknown)

Hence the given system of eqns is consistent and has unique solⁿ.

Equivalent system of eqns is

$$x + y + z = 3$$
$$y + 2z = 1$$
$$2z = 0$$

$$\Rightarrow x = 2, \ y = 1, \ z = 0.$$

<u>Ans</u>

— Using matrix method, show that the eqns

$3x+3y+2z=1$, $x+2y=4$, $10y+3z=-2$,

$2x-3y-z=5$

are consistent and hence obtain the solutions for $x, y$ and $z$.

Sol: $\rho(A)=3$, $\rho[A:B]=3$

$x=2$, $y=1$, $z=-4$.

Q— Test the consistency of following system of linear eqns and hence find the solution if exists.

(i) $4x_1-x_2=12$
$-x_1+5x_2-2x_3=0$
$-2x_2+4x_3=-8$

(ii) $7x_1+2x_2+3x_3=16$
$2x_1+11x_2+5x_3=25$
$x_1+3x_2+4x_3=13$

Ans (i) consistent & unique soln
$x_1=\frac{44}{15}$, $x_2=-\frac{4}{15}$, $x_3=-\frac{32}{15}$

(ii) consistent & unique soln
$x_1=\frac{95}{91}$, $x_2=\frac{100}{91}$,
$x_3=\frac{197}{91}$

Q— Show that the system of eqns $2x+4y+7z=7$, $x+y+z=-3$, $3x+y-2z=-2$, is not consistent.

Ans $\rho(A)=2$
$\rho[A:B]=3$
Hence the given system is inconsistent and has no solution

Q— Investigate for consistency of the following eqns and if possible, find the soln:
$15x-3y+9z=21$, $4x-2y+6z=8$, $x+y-3z=-1$,

Ans: $\rho(A)=\rho[A:B]=2$ (< no. of variable)
Hence the given system of eqns is consistent but has infinite number of soln.
let $z=k$ then $y=3k-2$ & $x=1$, where $k$ is arbitrary constant.

$$[A : B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$R_1 + R_2 \to R_1$

$$\begin{bmatrix} 1 & 1 & 4 & 12 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

$R_2 + R_1 \to R_2$, $R_3 - 3R_1 \to R_3$

$$\begin{bmatrix} 1 & 1 & 4 & 12 \\ 0 & 3 & 5 & 16 \\ 0 & -2 & -16 & -36 \end{bmatrix}$$

~~$R_2 \leftrightarrow R_3$~~

$$\begin{bmatrix} 1 & 1 & 4 & 12 \\ 0 & -2 & -16 & -36 \\ 0 & 3 & 5 & 16 \end{bmatrix}$$

$R_2 + R_3 \to R_2$

$$\begin{bmatrix} 1 & 1 & 4 & 12 \\ 0 & 1 & -11 & -20 \\ 0 & -2 & -16 & -36 \end{bmatrix}$$

$R_3 + 2R_2 \to R_3$

$$\begin{bmatrix} 1 & 1 & 4 & 12 \\ 0 & 1 & -11 & -20 \\ 0 & 0 & -38 & -76 \end{bmatrix}$$

$x + y + 4z = 12$

$y - 11z = -20$

$-38z = -76$

$z = 2$

$y - 22 = -20$

$y = 2$

$x + 2 + 8 = 12$

$x = 2$

Hence,

$x = y = z = 2$

Since $\rho(A) = \rho[A ; B] = 3 =$ number of unknown

Hence the given system is consistent & has unique solution

Augmented matrix is denoted by $[A ; B]$.

Note :-

If $\rho[A] = \rho[A ; B]$ then the system is consistent, otherwise inconsistent.

If the system is consistent then

(i) has a unique solution. If $\rho(A) = \rho[A ; B] =$ number of unknowns.

(ii) has infinite many solutions If $\rho(A) = \rho[A ; B] \neq$ number of unknowns.

If the system is inconsistent then there is no solution.

Rank Nullity theorem :-

Rank + Nullity = Number of columns in the matrix

Q - Solve

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

Q-3(ii) Let

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$C_1 \leftrightarrow C_2$

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$C_3 - 2C_1 \rightarrow C_3, \quad C_4 + 2C_1 \rightarrow C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

$R_3 - R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$R_2 - 2R_3 \rightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 - 2C_2 \rightarrow C_3, \quad C_4 - 3C_2 \rightarrow C_4$

$$\left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

which is Normal form. Hence $\rho(A) = 2$