Q- Investigate for consistency of the following equns and if possible, find the solutions:

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21$$

Solution:

$\rho(A) = 2$

Augmented matrix $[A ; B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho[A ; B] = 2$$

Since $\rho(A) = \rho([A ; B]) = 2$ (< number of unknowns).

Hence the given system of equns is consistant and has infinite number of solutions.

Equivalent system of equns is

$$x + y - 3z = -1$$
$$-6y + 18z = 12$$

let $z = k$ then $y = 3k - 2$

$x = 1$

where k is arbitrary constant.

Assignment III

① Find the inverse of

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$ by elementary transform.

② Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

③ Find the rank of the following matrix by reducing it to normal form

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

④ Test the consistency and solve

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0.$$

⑤ For what value of a and b the eqns

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

have (i) no solution (ii) unique solution (iii) Infinite many solutions.

Q- Investigate, for what values of λ and μ do the system of equns

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

have (i) no solution (ii) unique solution (iii) infinite solutions.?

Solution :

Augmented matrix $[A;B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

$R_2 - R_1 \to R_2, \; R_3 - R_1 \to R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3 - R_2 \to R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

Case I : If $\lambda = 3, \; \mu = 10$

Then $\rho[A] = \rho[A;B] < \text{Number of unknowns}$
$\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\quad 2$

Hence the given system has infinite number of solutions

Case II : $\lambda = 3, \; \mu \neq 10$

In this case $\rho(A) = 2, \; \rho[A;B] = 3$

$\because \rho(A) \neq \rho[A;B]$

Hence given system is inconsistent and has no soluti

Case III : $\lambda \neq 3$, μ have any value, then

$\rho(A) = \rho[A;B] = 3 = \text{number of unknowns.}$

Hence the given system has unique solution.

Characteristic equation :-

characteristic eqn of the matrix A is defined by

$$|\lambda I - A| = 0$$

where λ is the eigen value of A.

Eigen vectors :-

$$(\lambda I - A) X = 0$$

where X is the eigen vector corresponding to λ.

Q- Find the eigen values of the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$$

<u>Solution</u> :- we have, characteristic eqn is

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda-1 & 0 & -4 \\ 0 & \lambda-2 & 0 \\ -3 & -1 & \lambda+3 \end{vmatrix} = 0$$

$$(\lambda-1)\left[(\lambda-2)(\lambda+3) - 0\right] - 0 - 4\left[3(\lambda-2)\right] = 0$$

$$(\lambda-1)(\lambda^2+\lambda-6) - 12(\lambda-2) = 0$$

$$\lambda^3 + \cancel{\lambda^2} - 6\lambda - \cancel{\lambda^2} - \lambda + 6 - 12\lambda + 24 = 0$$

$$\lambda^3 - 19\lambda + 30 = 0$$

$$\Rightarrow \lambda = 3, -5, 2$$

Q- Find the eigenvalues and eigen vectors of the

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

<u>Solution</u> :-

$$\lambda = 0, 3, 15$$

Let X_1 is the eigen vector corresponding to $\lambda = 0$

$$(A - \lambda I) X_1 = 0$$

$$A X_1 = 0$$

$$8x - 6y + 2z = 0 \quad \text{——— (1)}$$

$$-6x + 7y - 4z = 0 \quad \text{——— (2)}$$

$$2x - 4y + 3z = 0 \quad \text{——— (3)}$$

Solving eqn (1) & (2) by cross multiplication

$$\frac{x}{24 - 14} = \frac{y}{-(-32+12)} = \frac{z}{56 - 36}$$

$$\Rightarrow \frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{2} = k_1$$

$$\Rightarrow x = k_1, \; y = 2k_1, \; z = 2k_1$$

Hence $X_1 = \begin{bmatrix} k_1 \\ 2k_1 \\ 2k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Let X_2 is the eigen vector corresponding to $\lambda = 3$

$$(A - 3I) X_2 = 0$$

$$5x - 6y - 2z = 0 \quad \text{——— (4)}$$

$$-6x + 4y - 4z = 0 \quad \text{——— (5)}$$

$$2x - 4y = 0 \quad \text{——— (6)}$$

from (4) & (5) $x = 2k_2, \; y = k_2, \; z = -2k_2,$ Hence $X_2 = k_2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Let X_3 is the eigenvector corresponding to $\lambda = 15$,

$$(A - 15I) X_3 = 0, \qquad X_3 = k_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Cayley - Hamilton Theorem :

Every square matrix is the root of its characteristic equ^n.

Q - Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Hence compute A^{-1}. Also evaluate $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$.

<u>Solution</u> : The characteristic equ^n of A is

$$|A - \lambda I| = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Cayley - Hamilton theorem, we have to show that $A^3 - 6A^2 + 9A - 4I = 0$ ——— (1)

Now, taking L.H.S

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = \text{R.H.S.}$$

Hence Cayley Hamilton theorem is verified.

Now, from (1), $A^2 - 6A + 9I - 4A^{-1} = 0$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$$

$$= A^3\left(A^3 - 6A^2 + 9A - 4I + 2I\right) - 12A^2 + 23A - 9I$$

$$= 2A^3 - 12A^2 + 23A - 9I$$

$$= 2A^3 - 12A^2 + 18A - 8I + 5A - I = 0 + 5A - I$$

$$= \begin{bmatrix} 9 & -5 & 5 \\ -5 & 9 & -5 \\ 5 & -5 & 9 \end{bmatrix}$$

Q- Find the characteristic eqn of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ and hence, compute A^{-1}. Also find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.