10. Graph Matrices

Since a graph is completely determined by specifying either its adjacency structure or its incidence structure, these specifications provide far more efficient ways of representing a large or complicated graph than a pictorial representation. As computers are more adept at manipulating numbers than at recognising pictures, it is standard practice to communicate the specification of a graph to a computer in matrix form. In this chapter, we study various types of matrices associated with a graph, and our study is based on Narsing Deo [63], Foulds [82], Harary [104] and Parthasarathy [180].

10.1 Incidence Matrix

Let G be a graph with n vertices, m edges and without self-loops. The incidence matrix A of G is an $n \times m$ matrix $A = [a_{ij}]$ whose n rows correspond to the n vertices and the m columns correspond to m edges such that

$$a_{ij} = \begin{cases} 1, & \textit{if jth edge } m_j \textit{ is incident on the ith vertex} \\ 0, & \textit{otherwise.} \end{cases}$$

It is also called *vertex-edge incidence matrix* and is denoted by $A(G)$.

Example Consider the graphs given in Figure 10.1. The incidence matrix of G_1 is

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of G_2 is

$$A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The incidence matrix of G_3 is

$$A(G_3) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

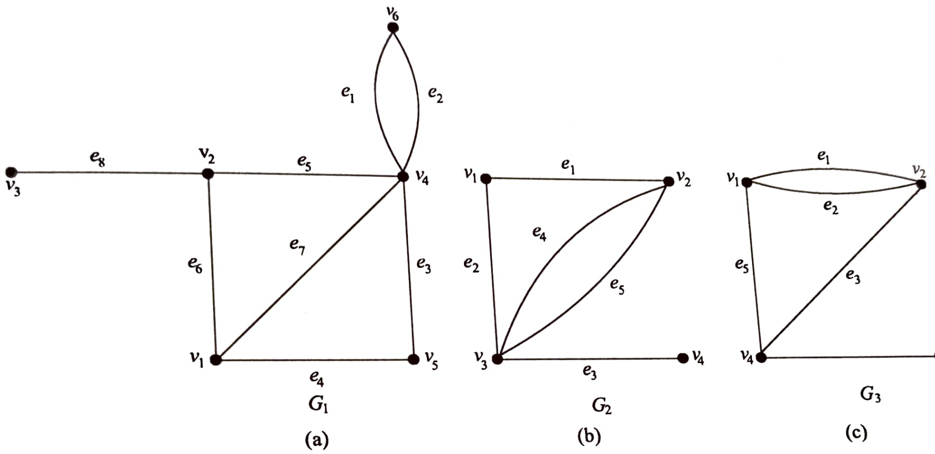

Fig. 10.1

The incidence matrix contains only two types of elements, 0 and 1. This clearly is a binary matrix or a (0, 1)-matrix.

We have the following observations about the incidence matrix A.

1. Since every edge is incident on exactly two vertices, each column of A has exactly two one's.

2. The number of one's in each row equals the degree of the corresponding vertex.

10.3 Cycle Matrix

Let the graph G have m edges and let q be the number of different cycles in G. The cycle matrix $B = [b_{ij}]_{q \times m}$ of G is a $(0, 1)$− matrix of order $q \times m$, with $b_{ij} = 1$, if the ith cycle includes jth edge and $b_{ij} = 0$, otherwise. The cycle matrix B of a graph G is denoted by $B(G)$.

Example Consider the graph G_1 given in Figure 10.3.

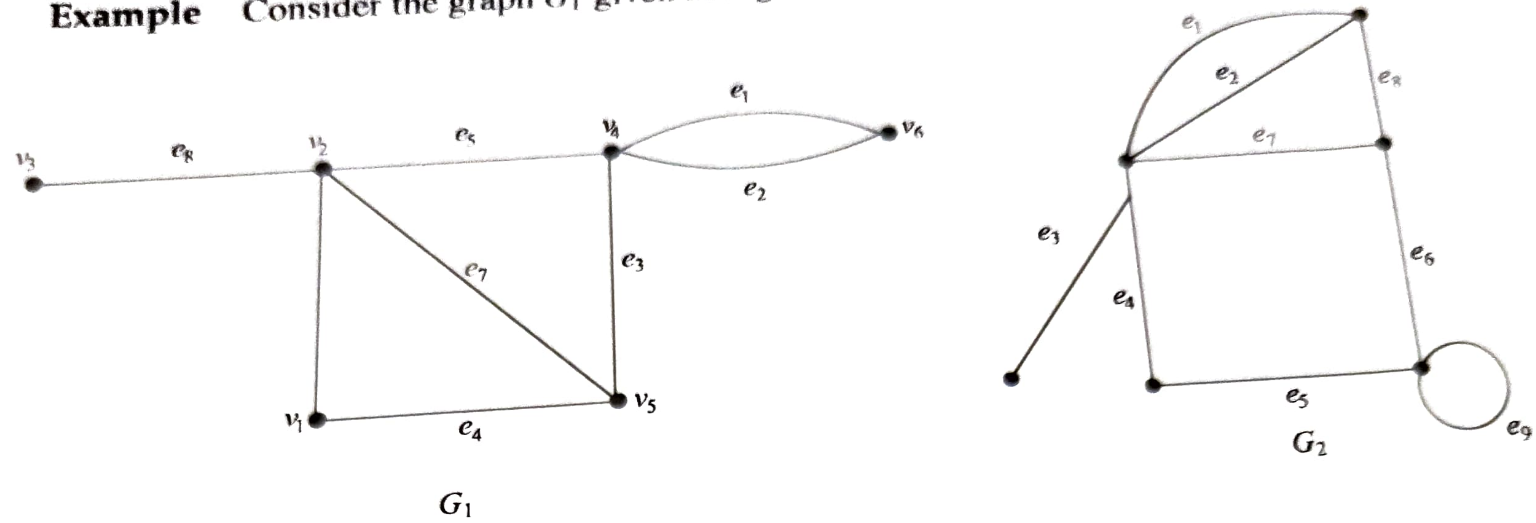

Fig. 10.3

The graph G_1 has four different cycles $Z_1 = \{e_1, e_2\}$, $Z_2 = \{e_3, e_5, e_7\}$, $Z_3 = \{e_4, e_6, e_7\}$ and $Z_4 = \{e_3, e_4, e_6, e_5\}$.

The cycle matrix is

$$B(G_1) = \begin{array}{cc} & \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The graph G_2 of Figure 10.3 has seven different cycles, namely, $Z_1 = \{e_1, e_2\}$, $Z_2 = \{e_2, e_7, e_8\}$, $Z_3 = \{e_1, e_7, e_8\}$, $Z_4 = \{e_4, e_5, e_6, e_7\}$, $Z_5 = \{e_2, e_4, e_5, e_6, e_8\}$, $Z_6 = \{e_1, e_4, e_5, e_6, e_8\}$ and $Z_7 = \{e_9\}$. The cycle matrix is given by

$$B(G_2) = \begin{array}{cc} & \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

We have the following observations regarding the cycle matrix $B(G)$ of a graph G.

1. A column of all zeros corresponds to a non cycle edge, that is, an edge which does not belong to any cycle.

2. Each row of $B(G)$ is a cycle vector.

3. A cycle matrix has the property of representing a self-loop and the corresponding row has a single one.

4. The number of ones in a row is equal to the number of edges in the corresponding cycle.

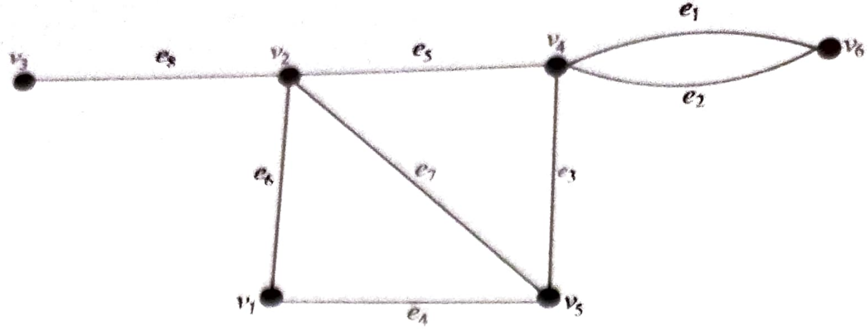

Fig. 10.5

Clearly,

$$AB^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \equiv 0(\text{mod}2).$$

Fundamental cycle matrix

We know that a set of fundamental cycles (or basic cycles) with respect to any spanning tree in a connected graph are the only independent cycles in a graph. The remaining cycles can be obtained as ring sums (i.e., linear combinations) of these cycles. Thus, in a cycle matrix, if we take only those rows that correspond to a set of fundamental cycles and remove all other rows, we do not lose any information. The removed rows can be formed from the rows corresponding to the set of fundamental cycles. For example, in the cycle matrix of the graph given in Figure 10.6, the fourth row is simply the mod 2 sum of the second and the third rows. Fundamental cycles are

$$C_{ij} = \begin{cases} 1, & \text{iff } i\text{th circuit includes } j\text{th edge of spanning tree} \\ 0, & \text{otherwise} \end{cases}$$

$Z_1 = \{e_1, e_2, e_4, e_7\}$
$Z_2 = \{e_3, e_4, e_7\}$
$Z_3 = \{e_5, e_6, e_7\}$

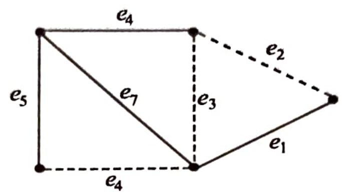

Fig. 10.6

graph & its fundamental circuit matrix (with respect to the spanning tree)

Find out the fundamental circuit

$$\begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_2 & e_3 & e_6 & & e_1 & e_4 & e_5 & e_7 \end{array} \\ \left[\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}.$$

Corollary 10.2 We know, rank A + nullity $A = n$, and using this in (10.13.4), we get

$$n - \text{rank } AB \leq n - \text{rank } A + n - \text{rank } B.$$

Therefore, rank $AB \geq$ rank $A +$ rank $B - n$.

If in above, $AB = 0$, then rank A + rank $B \leq n$.

10.4 Cut-Set Matrix

Let G be a graph with m edges and q cutsets. The cut-set matrix $C = [c_{ij}]_{q \times m}$ of G is a (0, 1)-matrix with

$$c_{ij} = \begin{cases} 1, & \text{if } i\text{th cutset contains } j\text{th edge}, \\ 0, & \text{otherwise}. \end{cases}$$

Example Consider the graphs shown in Figure 10. 7.

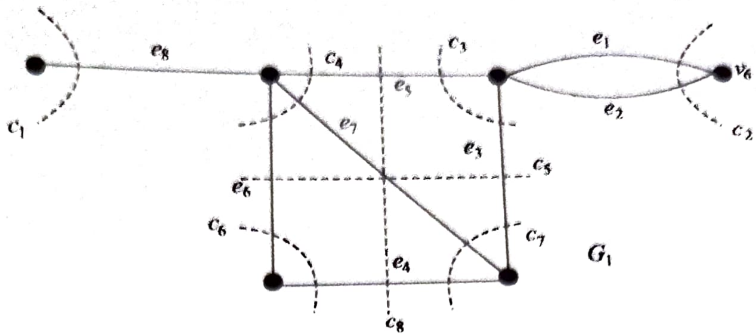

Fig. 10.7(a)

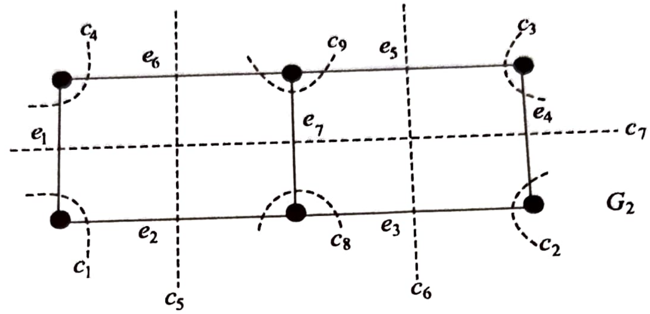

Fig. 10.7(b)

In the graph G_1, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

The cut-sets are $c_1 = \{e_8\}$, $c_2 = \{e_1, e_2\}$, $c_3 = \{e_3, e_5\}$, $c_4 = \{e_5, e_6, e_7\}$, $c_5 = \{e_3, e_6, e_7\}$, $c_6 = \{e_4, e_6\}$, $c_7 = \{e_3, e_4, e_7\}$ and $c_8 = \{e_4, e_5, e_7\}$.

The cut-sets for the graph G_2 are $c_1 = \{e_1, e_2\}$, $c_2 = \{e_3, e_4\}$, $c_3 = \{e_4, e_5\}$, $c_4 = \{e_1, e_6\}$, $c_5 = \{e_2, e_6\}$, $c_6 = \{e_3, e_5\}$, $c_7 = \{e_1, e_4, c_7\}$, $c_8 = \{e_2, e_3, e_7\}$ and $c_9 = \{e_5, e_6, e_7\}$.

Thus the cut-set matrices are given by

$$C(G_1) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}, \text{ and}$$

$$
C(G_2) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\end{array}.
$$

We have the following observations about the cut-set matrix $C(G)$ of a graph G.

1. The permutation of rows or columns in a cut-set matrix corresponds simply to renaming of the cut-sets and edges respectively.
2. Each row in $C(G)$ is a cut-set vector.
3. A column with all zeros corresponds to an edge forming a self-loop.
4. Parallel edges form identical columns in the cut-set matrix.
5. In a non-separable graph, since every set of edges incident on a vertex is a cut-set, therefore every row of incidence matrix $A(G)$ is included as a row in the cut-set matrix $C(G)$. That is, for a non-separable graph G, $C(G)$ contains $A(G)$. For a separable graph, the incidence matrix of each block is contained in the cut-set matrix. For example, in the graph G_1 of Figure 10.7, the incidence matrix of the block $\{e_3, e_4, e_5, e_6, e_7\}$ is the 4×5 submatrix of C, left after deleting rows c_1, c_2, c_5, c_8 and columns e_1, e_2, e_8.
6. It follows from observation 5, that rank $C(G) \geq$ rank $A(G)$. Therefore, for a connected graph with n vertices, rank $C(G) \geq n-1$.

10.7 Path Matrix

Let G be a graph with m edges, and u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u, v) = [p_{ij}]_{q \times m}$, where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & \text{if } j\text{th edge lies in the } i\text{th path}, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, a path matrix is defined for a particular pair of vertices, the rows in $P(u, v)$ correspond to different paths between u and v, and the columns correspond to different edges in G. For example, consider the graph in Figure 10.10.

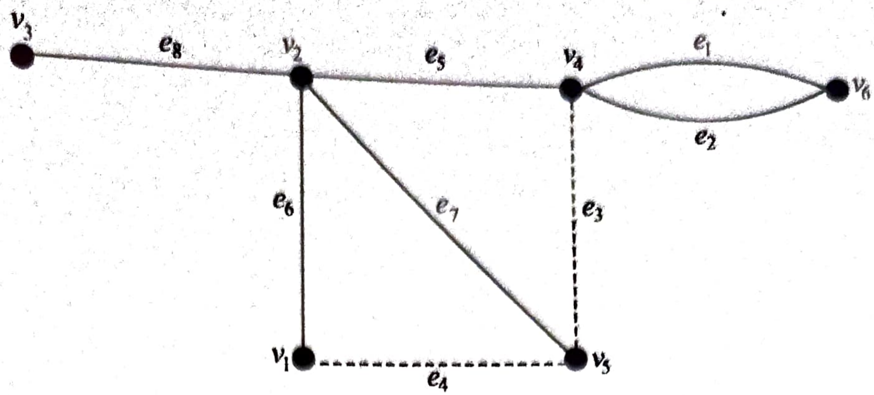

Fig. 10.10

The different paths between the vertices v_3 and v_4 are

$$p_1 = \{e_8, e_5\}, \ p_2 = \{e_8, e_7, e_3\} \text{ and } p_3 = \{e_8, e_6, e_4, e_3\}.$$

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

We have the following observations about the path matrix.

1. A column of all zeros corresponds to an edge that does not lie in any path between u and v.
2. A column of all ones corresponds to an edge that lies in every path between u and v.
3. There is no row with all zeros.
4. The ring sum of any two rows in $P(u, v)$ corresponds to a cycle or an edge-disjoint union of cycles.

10.8 Adjacency Matrix

Let $V = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ and without parallel edges. The adjacency matrix of G is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E, \\ 0, & \text{otherwise}. \end{cases}$$

Example Consider the graph G given in Figure 10.12.

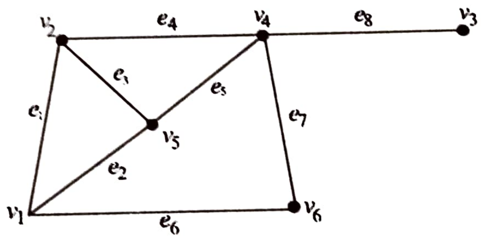

Fig. 10.12

The adjacency matrix of G is given by

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

We have the following observations about the adjacency matrix X of a graph G.

1. The entries along the principal diagonal of X are all zeros if and only if the graph has no self-loops. However, a self-loop at the ith vertex corresponds to $x_{ii} = 1$.

2. If the graph has no self-loops, the degree of a vertex equals the number of ones in the corresponding row or column of X.

3. Permutation of rows and the corresponding columns imply reordering the vertices. We note that the rows and columns are arranged in the same order. Therefore, when two rows are interchanged in X, the corresponding columns are also interchanged. Thus two graphs G_1 and G_2 without parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related by

$$X(G_2) = R^{-1} X(G_1) R,$$

where R is a permutation matrix.

4. A graph G is disconnected having components G_1 and G_2 if and only if the adjacency matrix $X(G)$ is partitioned as

$$X(G) = \begin{bmatrix} X(G_1) & \vdots & O \\ \cdot\cdot & \vdots & \cdot\cdot \\ O & \vdots & X(G_2) \end{bmatrix},$$

where $X(G_1)$ and $X(G_2)$ are respectively the adjacency matrices of the components G_1 and G_2. Obviously, the above partitioning implies that there are no edges between vertices in G_1 and vertices in G_2.

5. If any square, symmetric and binary matrix Q of order n is given, then there exists a graph G with n vertices and without parallel edges whose adjacency matrix is Q.