11/1/22

MSE-S204

Mechanical Behaviour of Materials

Stress on a plane $\sigma = \frac{F}{A}$ A- area of a plane

depending on orientation of plane

Stress at a point will change.

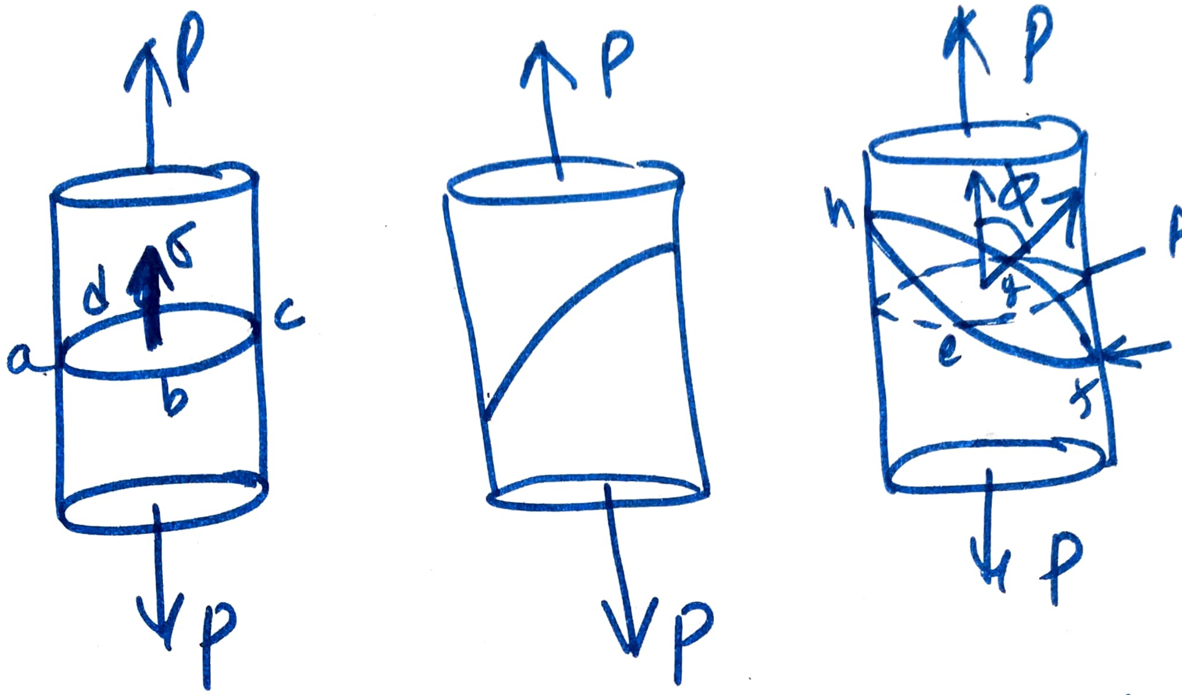
$\boxed{\text{Tensor}}$

Stress → 2^{nd} rank Tensor

Component of a Tensor (3^n)

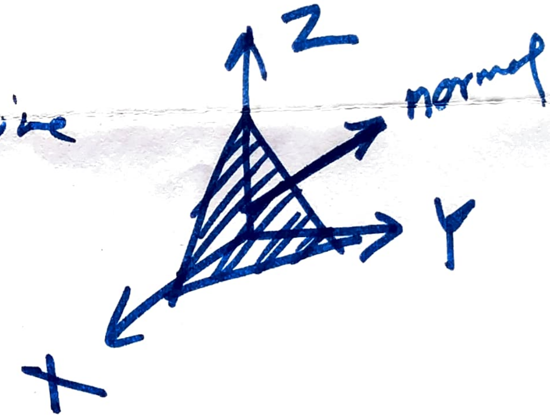
$$3^2 = 9$$

$$\bar{\sigma} = \frac{\bar{P}}{A}$$ load / Area



" Plane is defined by its normal "

Direction cosine



$\cos\alpha = n_1$

$\cos\beta = n_2$

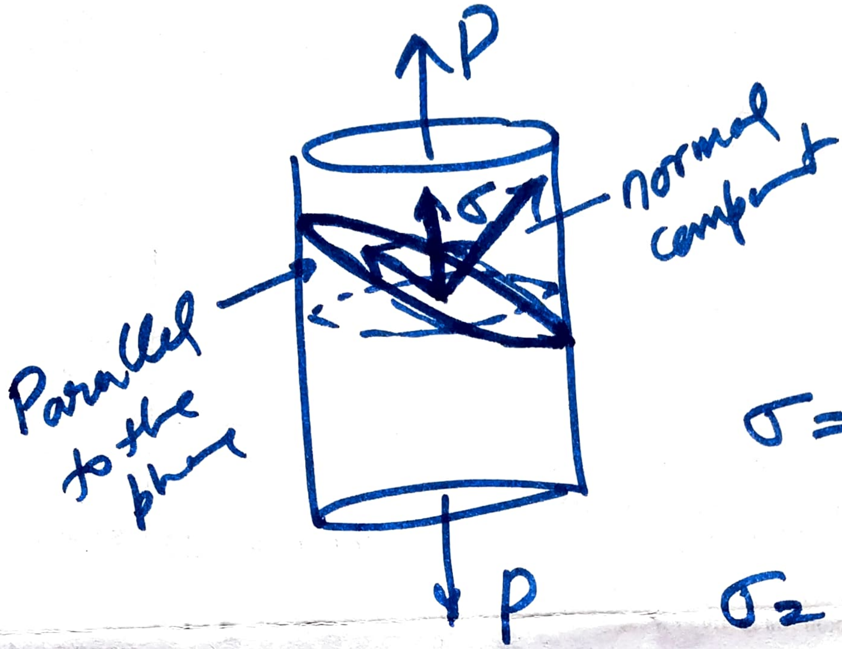
$\cos\gamma = n_3$

$$\boxed{n_1^2 + n_2^2 + n_3^2 = 1}$$

Property of Direction Cosine

abcd $\rightarrow A$

efgh $\rightarrow \dfrac{A}{\cos\phi}$



$$\cos\phi = \frac{A}{A'}$$

$$A' = \frac{A}{\cos\phi}$$

$$\sigma = \frac{P}{A/\cos\phi}$$

$$\sigma = \frac{P}{A}\cos\phi$$

normal component $= \dfrac{P}{A}\cos\phi\cos\phi$

$\Downarrow$

normal stress

(normal to the plane)

Parallel component

$= \dfrac{P}{A}\cos\phi\ \sin\phi$

$\Downarrow$

Shear stress

(lying in the plane)

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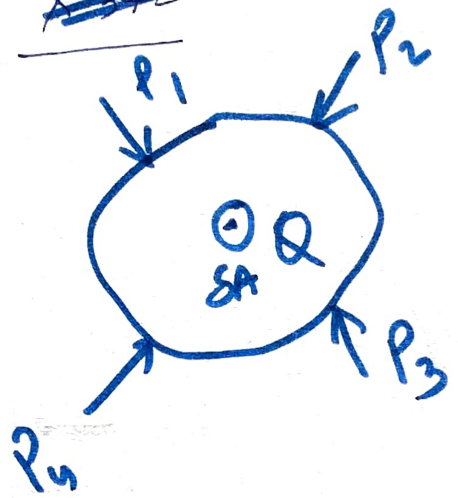


load is varying from point to point

To determine stress at any point (Q) of small area δA

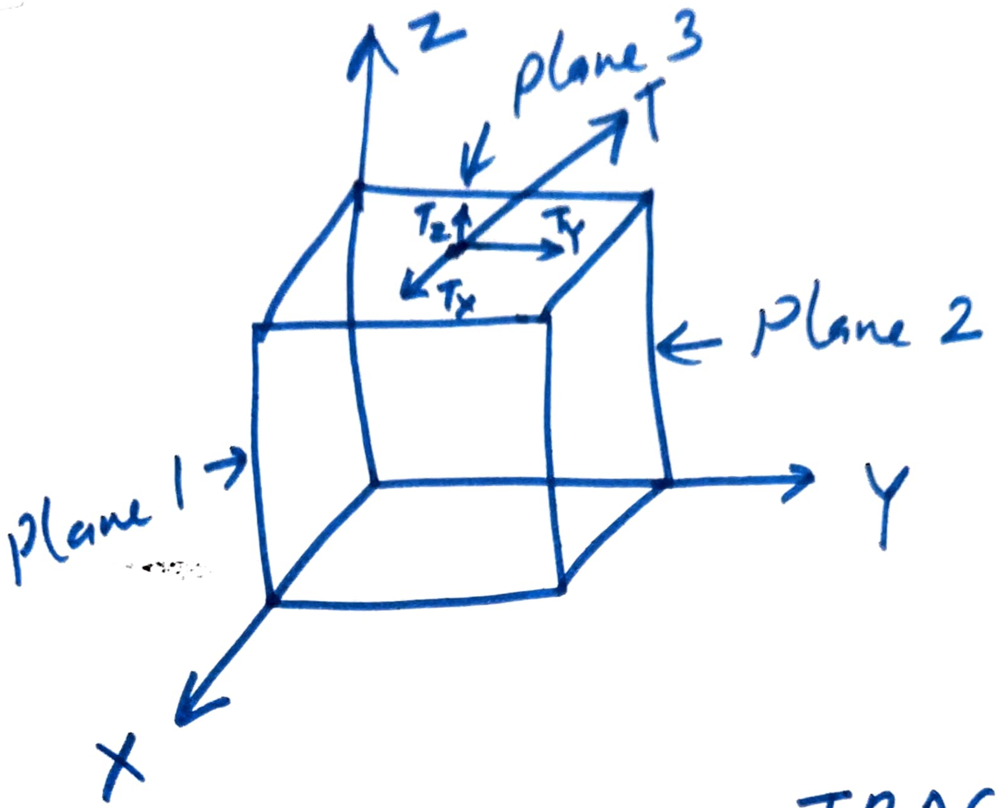
$$\boxed{\sigma = \lim_{\delta A \to 0} \frac{\delta P}{\delta A}}$$

Stress at point Q

need six numbers to find out stress of any plane passing that point

State of stress at a point
⇓
get stress on any plane passing through that point

Consider
Very small parrelopiped



TRACTION
= Force / Area
T_x , T_y , T_z

Total six faces
6 X 3
= 18 Component of traction

out of 18 Component
Only 6 will be independent

$18 \xrightarrow[\text{in to}]{\text{reduced}} 6$

X → 1
Y → 2
Z → 3

$\boxed{\sigma_{12}}$

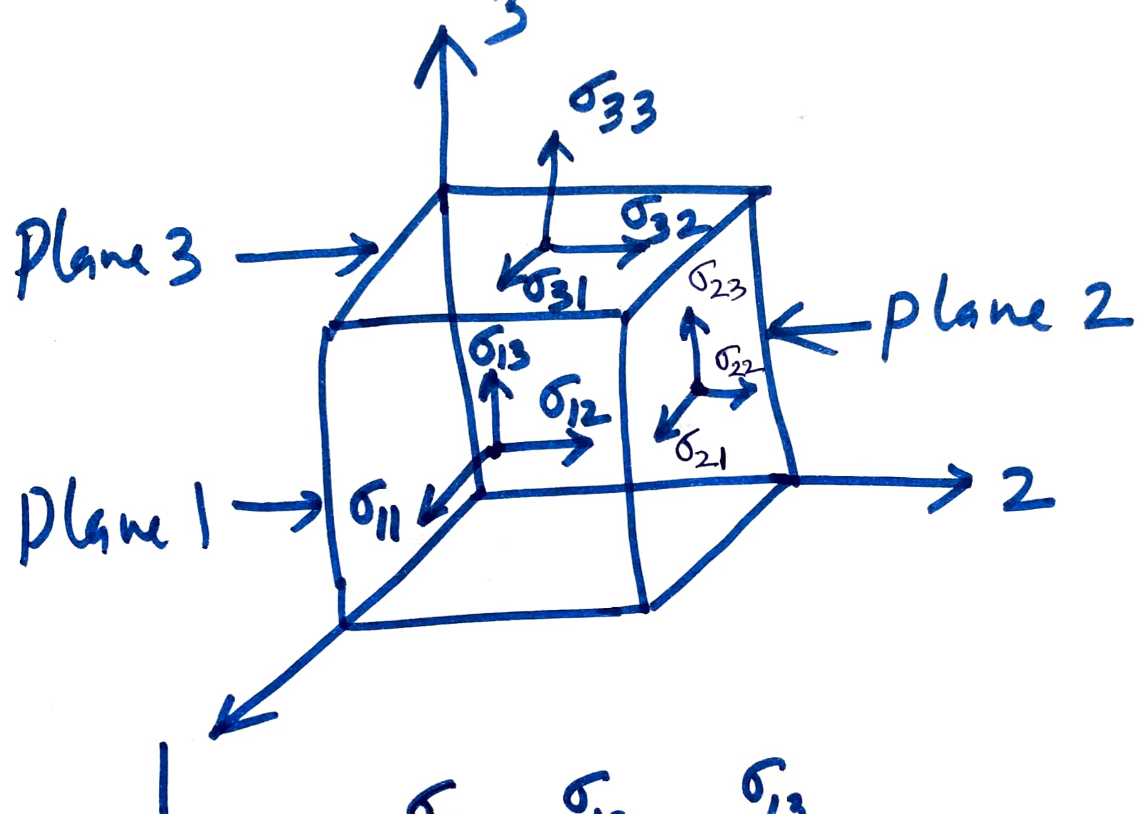
Plane → 1, Direction → 2

In general

$$\sigma_{ij}$$

i → plane (normal)

j → Direction



$$\sigma_{ij} = \begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix}$$

nine stress components

If symmetry exist

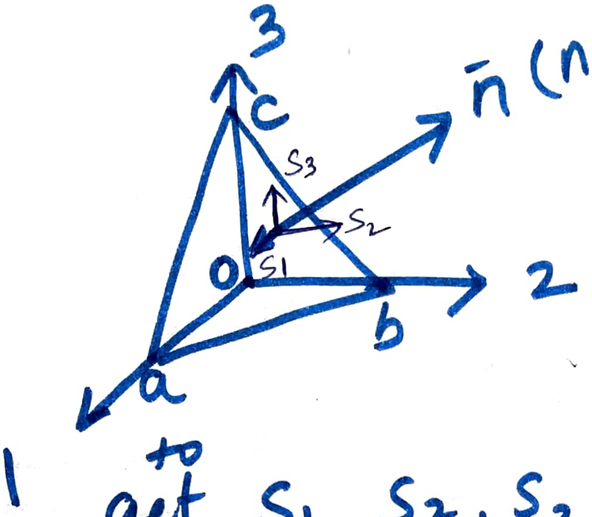
$$\sigma_{ij} = \sigma_{ji}$$

9 component $\xrightarrow[\text{into}]{\text{reduced}}$ 6 component

⇓

Independent

To get stress at a plane (abc)



$\sigma = \frac{F}{A}$

to get S_1, S_2, S_3

S_1 (area of abc) =

$= \sigma_{31}(oab) + \sigma_{11}(obc) + \sigma_{21}(oac)$

$S_1(\cancel{abc}) = \sigma_{31}(\cancel{abc} \cdot n_3) + \sigma_{11}(\cancel{abc} \cdot n_1) + \sigma_{21}(\cancel{abc} \cdot n_2)$

$S_1 = \sigma_{11} \cdot n_1 + \sigma_{21} \cdot n_2 + \sigma_{31} \cdot n_3$

$$S_i = \sum_{j=1}^{3} \sigma_{ij} n_j \qquad (i, 3)$$

Einstein convention

Sum over repeated index is automatically understood.

$$S_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$

$$S_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$

$$S_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

Total stress

$$\overline{S}_{total} = \sqrt{S_1^2 + S_2^2 + S_3^2}$$

Normal stress

$$S_n = S_1 n_1 + S_2 n_2 + S_3 n_3$$

Shear stress

$$\overline{S}_{shear} = \sqrt{S_{total}^2 - S_n^2}$$

Direction of S (total stress)

$$n_1 = \frac{S_1}{S}$$

$$n_2 = \frac{S_2}{S}$$

$$n_3 = \frac{S_3}{S}$$

- Stress at a point

$$\sigma_{ij} = \begin{bmatrix} 300 & 150 & 250 \\ 150 & 200 & 190 \\ 250 & 190 & 100 \end{bmatrix} MPa$$

To calculate

Traction S

acting on any plane passing through

that point

Plane Direction cosines

$$n_1 = \frac{1}{\sqrt{3}} , \quad n_2 = \frac{1}{\sqrt{2}}$$

also calculate

normal stress

&

Shear stress.

15.1.2022

$$\boxed{S_i = \sum_{j=1}^{3} \sigma_{ij} n_j}$$

$i = 1 - 3$

$j = 1 - 3$

As per Einstein convention

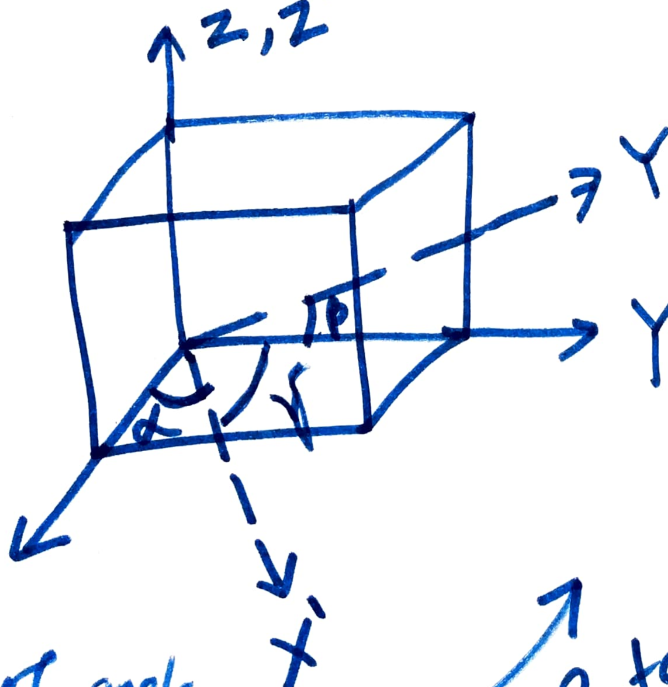
Sum over repeated index is automatically understood

$$\boxed{S_i = \sigma_{ij} n_j}$$

$\boxed{a_{ij}}$

new axis

old axis



$a_{11} = \cos\alpha$

$a_{22} = \cos\beta$

$a_{12} = \cos\gamma$

a_{ij} = cos of angle between new i & old j

Rotated

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

in coordinate system $\Rightarrow X, Y, Z$

if change coordinate axis $\Rightarrow X', Y', Z'$

then new values of $\sigma_{ij} \Rightarrow \sigma'_{ij}$

$$\boxed{\sigma'_{ij} = a_{ik}\, a_{jl}\, \sigma_{kl}}$$

$k \to 1-3$

$l \to 1-3$

$$\sigma'_{11} = a_{1k}\, a_{1l}\, \sigma_{kl}$$

$$= \left[\bar{a}_{11}\, a_{1l}\, \sigma_{1l} + a_{12}\, a_{1l}\, \sigma_{2l} + a_{13} a_{1l}\, \sigma_{3l}\right]$$

$$= [\, l=1\,] + [\, l=2\,] + [\, l=3\,]$$

$$\sigma'_{ij} = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix}$$

(a) A coordinate axes is rotated clockwise by 30° about the z-axis. Find the values of a_{ij}. (12)

(b)

Find the new values of the components of the following stress tensor after the above change of axes.

$$\begin{matrix} 5 & 4 & 3 \\ 4 & 6 & 2 \\ 3 & 2 & 7 \end{matrix} \quad \text{MPa}$$

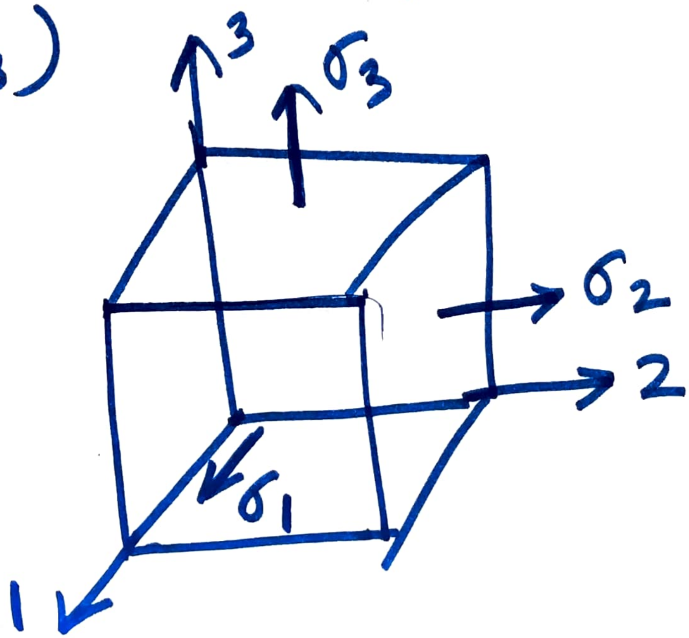
$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \xrightarrow[\text{change}]{\text{axes}} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

- only diagonal components
- off diagonal elements are zero
- have only three elements (diagonal)

axes $\Rightarrow$ called Principal axes

Stresses $\Rightarrow$ Principal Stresses

$(\sigma_1, \sigma_2, \sigma_3)$



$\sigma_1 \rightarrow \sigma_{11}$

$\sigma_2 \rightarrow \sigma_{22}$

$\sigma_3 \rightarrow \sigma_{33}$

Those planes which are at right angle to the principal axes are called <u>principal plane</u>.

&

there will be only normal stress

&

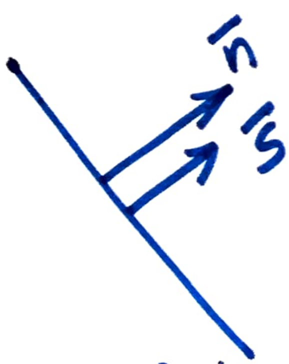
no shear stress.

<u>To find principal axes</u>

In a general case

$\sigma_{ij} \rightarrow$ Six component of stress (arbitrary axis)

$\sigma_{ij} \xrightarrow[\text{rotation}]{\text{after}} S_1, S_2, S_3$ principal stress



$$\left.\begin{aligned} S_1 &= n_1 S \\ S_2 &= n_2 S \\ S_3 &= n_3 S \end{aligned}\right]$$

$$\left. S_i = \sigma_{ij} n_j \right]$$

$$S_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = n_1 S$$

$$\left.\begin{aligned} (\sigma_{11} - S) n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 &= 0 \\ \sigma_{12} n_1 + (\sigma_{22} - S) n_2 + \sigma_{23} n_3 &= 0 \\ \sigma_{13} n_1 + \sigma_{23} n_2 + (\sigma_{33} - S) n_3 &= 0 \end{aligned}\right]$$

Determinant of coefficient $= 0$

$$\begin{vmatrix} (\sigma_{11} - S_1) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & (\sigma_{22} - S) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & (\sigma_{33} - S) \end{vmatrix} = 0$$

$$S^3 - \overbrace{(\sigma_{11} + \sigma_{22} + \sigma_{33})}^{I_1} S^2 + \underbrace{(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{23}^2 - \sigma_{31}^2 - \sigma_{12}^2)}_{I_2} S - \underbrace{(\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{23}\sigma_{13}\sigma_{12} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2)}_{I_3} = 0$$

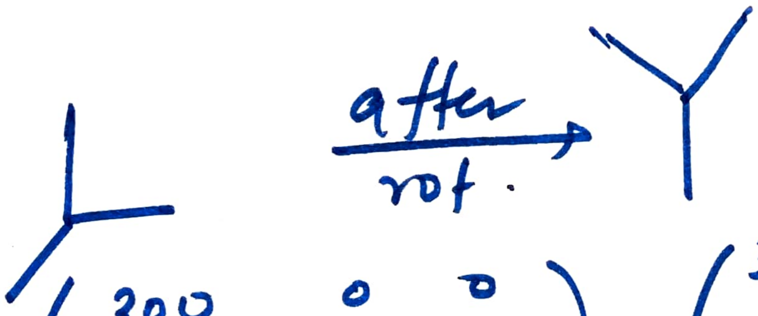
Solve it

get S_1, S_2, S_3

get orientation of plane.

$$S^3 - I_1 S^2 + I_2 S - I_3 = 0$$

I_1, I_2 & I_3 $\rightarrow$ Invariants of stress.



$$\begin{pmatrix} 300 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 100 \end{pmatrix} \qquad \begin{pmatrix} 325 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 75 \end{pmatrix}$$

$= 600$ $\qquad$ $= 600$