Diffusion Out of a Slab –

As an example of the use of eq (27), consider the loss of material out both sides of a slab of thickness h. The boundary conditions to be assumed are

$$c = c_0 \quad \text{for} \quad 0 < x < h, \quad \text{at} \quad t = 0$$
$$c = 0 \quad \text{for} \quad x = h \quad \text{and} \quad x = 0, \quad \text{at} \quad t > 0$$

By setting all B_n equal to zero, c will be zero at $x = 0$ for all times. To make $c = 0$ at $x = h$, the argument of $\sin \lambda_n x$ must equal zero for $x = h$. This is done by letting $\lambda_n = n\pi/h$, where n is any positive integer. If we substitute $B_n = 0$ and $\lambda_n = n\pi/h$ into eqn (27), the first boundary condition requires that

$$C_0 = \sum_{n=1}^{\infty} A_n \sin \frac{xn\pi}{h} \qquad \text{— (28)}$$

To determine the A_n which will satisfy this eqn, multiply both sides of this eqn by $\sin(xp\pi/h)$, and integrate x over the range $0 \leq x \leq h$. This gives the eqn

$$\int_0^h C_0 \sin \frac{xp\pi}{h}\, dx = \sum_{n=1}^{\infty} A_n \int_0^h \sin \frac{xp\pi}{h} \sin \frac{xn\pi}{h}\, dx$$

Each of the infinity of integrals on the right equals zero, except the one in which $n = p$. This integral is equal to $h/2$. The values of A_n which will satisfy eqn (28) are thus given by the eqn

$$A_n = \frac{2}{h} \int_0^h C_0 \sin \frac{n\pi x}{h}\, dx \qquad \text{— (29)}$$

The integration of this eqn shows that $A_n = 0$ for all even values of n and $A_n = 4C_0/n\pi$ for odd values of n. Changing the summation index so that only odd values of n are summed over gives

$$A_n = A_j = \frac{4C_0}{(2j+1)\pi} \qquad j = 0, 1, 2 \ldots \qquad \text{— (30)}$$

The solution is thus

$$c(x,t) = \frac{4c_0}{\pi} \sum_{j=0}^{\infty} \frac{1}{2j+1} \sin \frac{(2j+1)\pi x}{h} \exp\left[-\left(\frac{(2j+1)\pi}{h}\right)^2 Dt\right] \quad (31)$$

A moment's study of this eqn shows that each successive term is smaller than the preceding one. Also, the percentage decrease between terms increases exponentially with time. Thus after a short time has elapsed, the infinite series can be satisfactorily represented by only a few terms, and for all time beyond some period t', $c(x,t)$ is given by a sine wave. To determine the error involved in using just the first term to represent $c(x,t)$ after some time t', it is easiest to consider the ratio of the maximum values of the first & second terms.

The ratio R is given by the equation

$$R = 3 \exp \frac{8\pi^2 D t'}{h^2}$$

For $h = 2\sqrt{Dt}$, R is about 150, so that for $h^2 \leq 4Dt$ (or $t \geq h^2/4D$) the error in using the first term to represent $c(x,t)$ is less than 1% at all points.

This solution could be applied to the decarburization of a thin sheet of steel, and it is worthwhile to compare the use of this series solution with the error-function solution of eqn (20). For short times the sheet thickness can be considered infinite. The carbon distribution below each surface will be given by the error-function solution as well as by this series solution.

To evaluate $c(x,t)$ in this case using eqn (31) would require the evaluation of many terms, and it is easier to look up the error function in a table. This is true until $h \simeq 3.2\sqrt{Dt}$, at which time $R \simeq 20$, and the error in using the error function is about 2% at the plane $x = h/2$. For times greater than this, the first term of eqn (31) becomes a better approximation & would be used.

One of the most frequent metallurgical application of this type of solution appears in the degassing of metals. Here it is often difficult to determine the concentration at various depths, and what is experimentally determined is the quantity of gas which has been given off or the quantity remaining in the metal. For this purpose of average concentration $\bar{c}$ is needed. This is obtained by integrating eqn (31).

$$\bar{c}(t) = \frac{1}{h}\int_0^h c(x,t)\,dx = \frac{8C_0}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\exp\left[-\left(\frac{(2j+1)\pi}{h}\right)^2 Dt\right] \qquad (32)$$

The ratio of the first & second terms in this series is three times as large as in the case of eqn (31), and for $\bar{c} \leq 0.8\,C_0$ the first term is an excellent approximation to the solution. The solution for $\bar{c}/C_0 \leq 0.8$ can be ~~written as~~ rewritten

$$\frac{\bar{c}}{C_0} = \frac{8}{\pi^2}\exp\left(-\frac{t}{\tau}\right) \qquad (33)$$

where $\tau \equiv h^2/\pi^2 D$ is called the relaxation time, Equation (33) is a type that is met frequently in systems that are relaxing to an equilibrium state. The quantity τ is a measure of how fast the system relaxes, when ~~$t=\tau$~~ $t=\tau$, the system has traveled two thirds of the way from the initial state to the final state. Large values of τ thus characterize slow processes.

Equations similar to (32) & (33) and derived from the degassing of cylinders have been used in the accurate measurements of D.