Theorem:- A finite Integral domain is a field.

ID : CRU + without zero divisors

Field : CRU + every non zero elements has a multiplicative inverse.

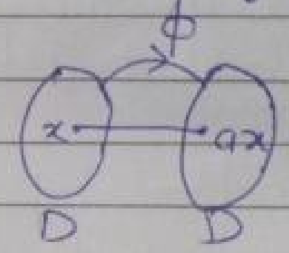
Proof:

suppose D is a finite integral domain.

take $a \neq 0$, $a \in D$

[define $\phi : D \to D$

$\phi(x) = ax$



take $x, y \in D$

$\phi(x) = \phi(y)$

$\Rightarrow ax = ay$

$\Rightarrow a(x-y) = 0$

$\because a \neq 0 \Rightarrow x - y = 0$ ($\because$ D is ID)

$\Rightarrow x = y$

$a(x-y)=0$
$\Rightarrow a=0$ or $x-y=0$
$\because a \neq 0 \Rightarrow x-y=0$

$\therefore \phi$ is one-one map

$\because$ D is finite, $\phi : D \to D$ is one
$\Rightarrow$ D is onto

Result: $f : A \to A$, $|A|$ = finite
$\therefore f$ is one one $\Rightarrow f$ is onto

$\because \phi$ is onto $\Rightarrow \exists\ b \in D$ $(1 \in D)$

$\phi(b) = 1$

$ab = 1$

$a \neq 0$ $ab = 1$

$\Rightarrow b^{-1} = \frac{1}{a}$ exist $\therefore$ D field.

PAGE NO.
DATE: / /

Integral ~~Theorem~~ Domain :-

Embedding Theorem :-

Embedding of a Ring into a ring with unity

Statement :- Every ring can be embedded into a ring with unity.

Proof: Let R be a ring.
we will prove R can be embedded into $R \times Z$
(i.e $\exists$ a 1-1, onto homomorphism from $f: R \longrightarrow R \times Z$)

$f(a+b) = f(a) + f(b)$
$f(ab) = f(a) f(b)$

I :- T.P.T. $R \times Z$ is a ring with unity
$\therefore R \times Z = \{(r, n) ;\ r \in R,\ n \in Z\}$

$\because$ R is ring, Z is ring $\Rightarrow$ $R \times Z$ is also a ring
$(0, 1)$ is unity of $R \times Z$
$(x, y)(1^R, 1) = (1^R, 1) = (x, y)$

$f: R \longrightarrow R \times Z$
Let $(R, +, \cdot)$ be a rings and Z be integers set
now $R \times Z$ is ring with unity $(1^R, 1^Z)$
we have to show that $\exists$ a one-one homomorphism from R to $R \times Z$.

DATE: / /

Define $f : R \longrightarrow R \times Z$
$f(r) = (r, 0)$ 0 is additive identity $r \in R$

T.S.T. f is well defined
Let $r_1 = r_2$
$\Rightarrow (r_1, 0) = (r_2, 0)$
$\Rightarrow f(r_1) = f(r_2)$
$\therefore$ f is well defined.

T.S.T. f is one-one
$f(r_1) = f(r_2)$
$(r_1, 0) = (r_2, 0)$
$\Rightarrow r_1 = r_2 \quad , 0 = 0$
$\Rightarrow r_1 = r_2$
$\Rightarrow$ f is one-one

T.S.T. f is homomorphism

i.e (a) $f(r_1 + r_2) = (r_1 + r_2, 0)$
$= (r_1, 0) + (r_2, 0)$
$= f(r_1) + f(r_2)$

(b) $f(r_1, r_2) = (r_1 r_2, 0)$
$= (r_1, 0)(r_2, 0)$
$= f(r_1) f(r_2)$

$\forall r_1, r_2 \in R$
$\therefore$ f is homomorphism
$\Rightarrow$ f is an embedding mapping : $R \to R \times Z$

$= (x\cdot 1^R, y\cdot 1)$
$= (x, y)$

PAGE NO.
DATE: / /

therefore R can be embedded into $R \times Z$
$\because$ R is an arbitrary.
$\Rightarrow$ every ring can be embedded into a ring with unity.