Eucledian Ring or Eucledian Domain

Let R be an Integral domain. then R is said to be Eucledian ring if for every $a \neq 0, a \in R$, we can assign a non negative integer $d(a)$ s.t.

① $\forall\ a, b \in R, \quad d(ab) \geq d(a) \qquad a \neq 0, b \neq 0$

② for any $a, b \in R$

$b \neq 0, \ \exists\ q, r \in R$

s.t. $a = bq + r$

$\Rightarrow r = 0$ or $d(r) < d(b)$

note:- $d(a)$ = d-value of a and $d(a)$ must be some non negative integer for every non zero element $a \in R$.

Ques:- show that the ring of integers is an Eucledian Ring.

Sol^n:

Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers where $\mathbb{Z} = \{ \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots \}$

Let d be the functions on the non zero elements of $\mathbb{Z}$ be defined by

$d(a) = |a| \quad \forall a \neq 0, a \in \mathbb{Z}$

thus

$d(-5) = |-5| = 5$ .... etc

(I) also $|ab| = |a||b|$ — we know $|b| \geq 1$, $\forall b \neq 0$, $b \in \mathbb{Z}$

(i) $\therefore |ab| = |a||b| \geq |a|$

$\Rightarrow d(ab) \geq d(a)$

(ii) If $a \in \mathbb{Z}$, $0 \neq b \in \mathbb{Z}$,

then $\exists$ q, r s.t. $a = bq + r$

$0 \leq r < |b|$

$\Rightarrow$ either $r = 0$ or $d(r) < d(b)$

$\therefore (\mathbb{Z}, +, \cdot)$ is Eucledian Ring.

Ex (2)

Prove that $(\mathbb{Z}[i], +, \cdot)$ is an Integral domain.

$\because \mathbb{Z}[i]$ is subring of $\mathbb{C}$

$\mathbb{Z}[i] = \{a + ib ; a, b \in \mathbb{Z}\}$

$\mathbb{C}$ is a field.

$\mathbb{Z}[i]$ is field $\Rightarrow \mathbb{Z}[i]$ is Integral domain.

also $1 = 1 + i0 \in \mathbb{Z}[i]$ $\quad (1, 0) \in \mathbb{Z}$

$\therefore \mathbb{Z}[i]$ is an ID with unity

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define $d : Z[i] \longrightarrow Z^+$ by
$$d(a+ib) = a^2+b^2$$

(i) Let $u = a+ib \neq 0 \Rightarrow a \neq 0$ or $b \neq 0$
$$\Rightarrow a^2+b^2 \geq 0$$
$$\Rightarrow d(a+ib) \geq 0$$

(ii) $u = a+ib \neq 0$, $v = c+id \neq 0$

$\Downarrow$

$a \neq 0$ or $b \neq 0$, $c \neq 0$ or $d \neq 0$

$\Downarrow$

$d(a+ib) \geq 0$ $\quad d(c+id) \geq 0$

$$(a+ib)(c+id) = (ac-bd) + i(bc+ad)$$

$$\begin{aligned}
d[(a+ib)(c+id)] &= (ac-bd)^2 + (bc+ad)^2 \\
&= a^2c^2 + b^2d^2 - \cancel{2abcd} \\
&\quad + b^2c^2 + a^2d^2 + \cancel{2abcd} \\
&= (a^2+b^2)c^2 + (a^2+b^2)d^2 \\
&= (a^2+b^2)(c^2+d^2) \\
&= d(a+ib)\,d(c+id) \\
&\geq d(a+ib)
\end{aligned}$$

$\because a^2+b^2 \geq 1$
$d(a,b) \geq 1$

$$\therefore d(uv) \geq d(u)$$

(iii) Let $a+ib \in Z[i]$ and $c+id \neq 0 \in Z[i]$

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{c^2+d^2}$$

$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$

$$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\,\frac{(bc-ad)}{c^2+d^2} \in Q[i]$$