Chapter 3

PRESSURE AND FLUID STATICS

Objectives

- Determine the variation of pressure in a fluid at rest
- Calculate pressure using various kinds of manometers
- Calculate the forces exerted by a fluid at rest on plane or curved submerged surfaces
- Analyze the stability of floating and submerged bodies
- Analyze the rigid-body motion of fluids in containers during linear acceleration or rotation

3–1 ■ PRESSURE

Pressure: A normal force exerted by a fluid per unit area

$$1\ \text{Pa} = 1\ \text{N/m}^2$$

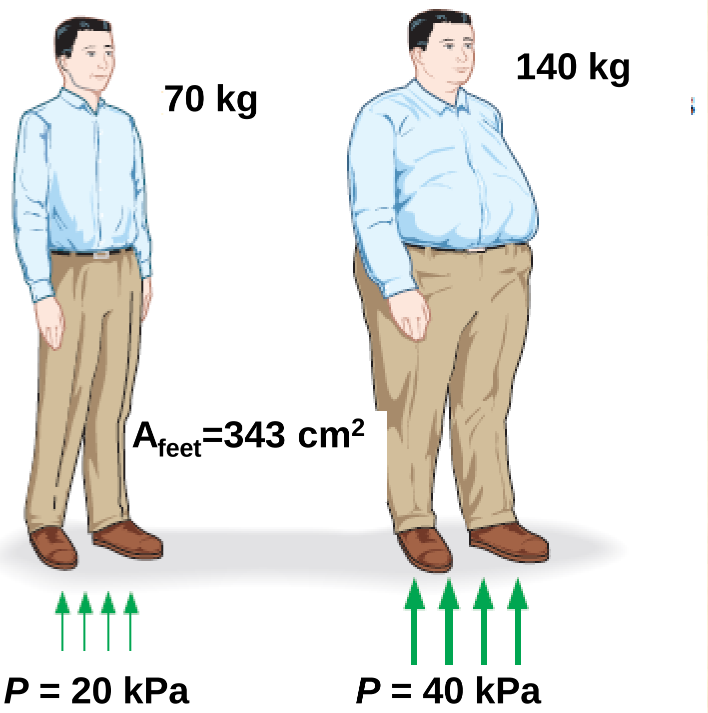
$$1\ \text{bar} = 10^5\ \text{Pa} = 0.1\ \text{MPa} = 100\ \text{kPa}$$

$$1\ \text{atm} = 101{,}325\ \text{Pa} = 101.325\ \text{kPa} = 1.01325\ \text{bars}$$

$$1\ \text{kgf/cm}^2 = 9.807\ \text{N/cm}^2 = 9.807 \times 10^4\ \text{N/m}^2 = 9.807 \times 10^4\ \text{Pa}$$

$$= 0.9807\ \text{bar}$$

$$= 0.9679\ \text{atm}$$



P = (70×9.81/1000) kN / 0.0343 m² = 20 kPa

The normal stress (or "pressure") on the feet of a chubby person is much greater than on the feet of a slim person.

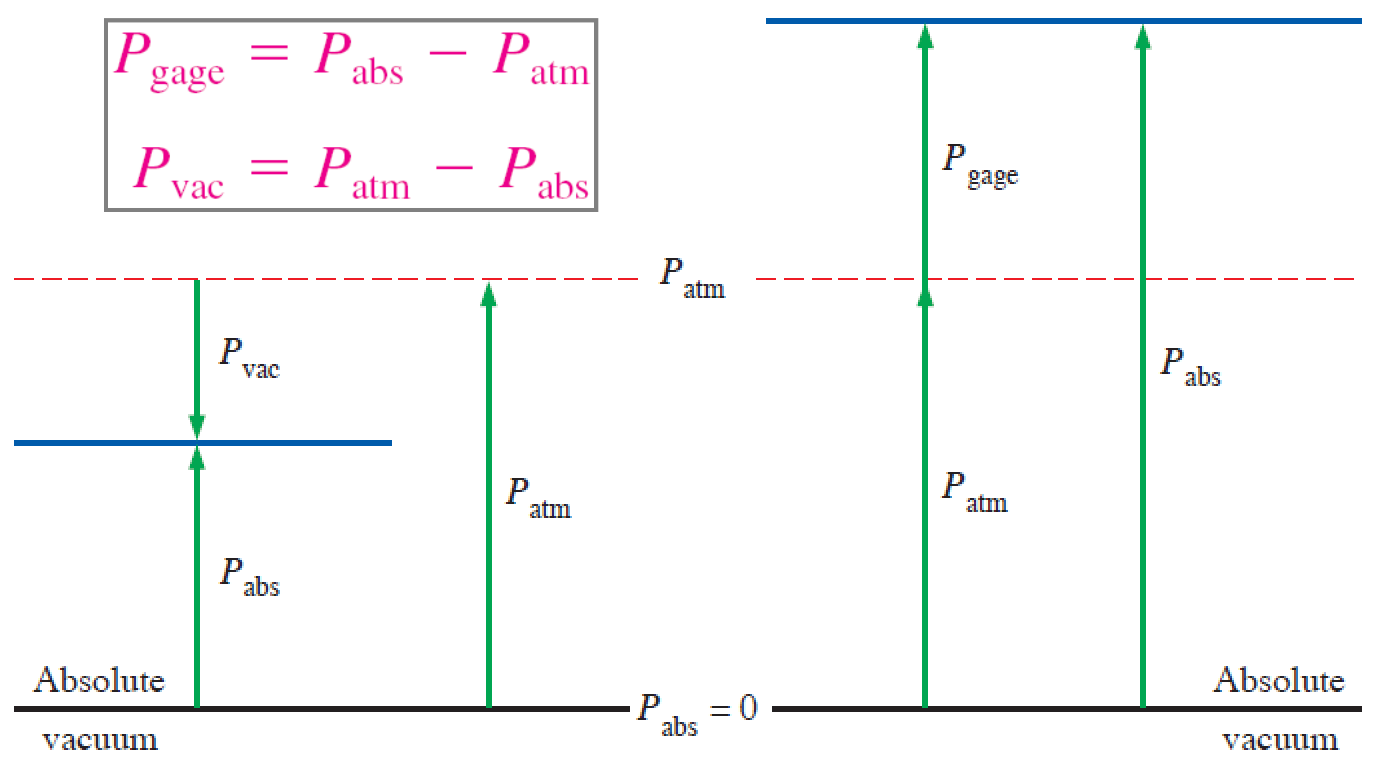
Some basic pressure gages.

Absolute pressure: The actual pressure at a given position. It is measured relative to absolute vacuum (i.e., absolute zero pressure).

Gage pressure: The difference between the absolute pressure and the local atmospheric pressure. Most pressure-measuring devices are calibrated to read zero in the atmosphere, and so they indicate gage pressure.

Vacuum pressures: Pressures below atmospheric pressure.

Throughout this text, the pressure ***P*** will denote ***absolute pressure*** unless specified otherwise.



EXAMPLE 3–1 **Absolute Pressure of a Vacuum Chamber**

A vacuum gage connected to a chamber reads 40 kPa at a location where the atmospheric pressure is 100 kPa. Determine the absolute pressure in the chamber.

SOLUTION The gage pressure of a vacuum chamber is given. The absolute pressure in the chamber is to be determined.

Analysis The absolute pressure is easily determined from Eq. 3–2 to be

$$P_{abs} = P_{atm} - P_{vac} = 100 - 40 = 60 \text{ kPa}$$

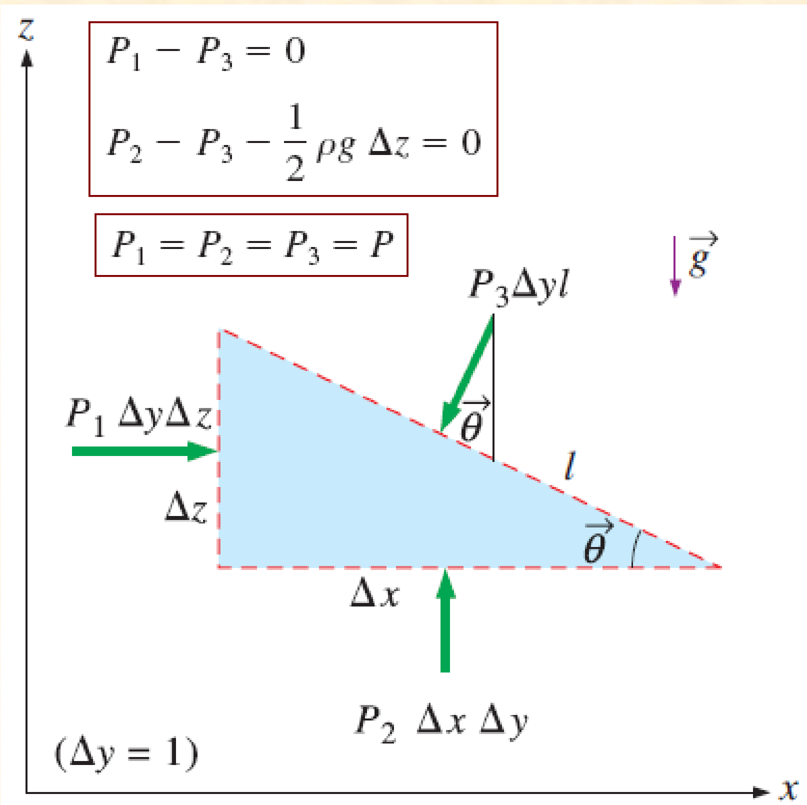
Discussion Note that the *local* value of the atmospheric pressure is used when determining the absolute pressure.

Pressure at a Point

$$\sum F_x = ma_x = 0: \qquad P_1 \Delta y \Delta z - P_3 \Delta y l \sin\theta = 0$$

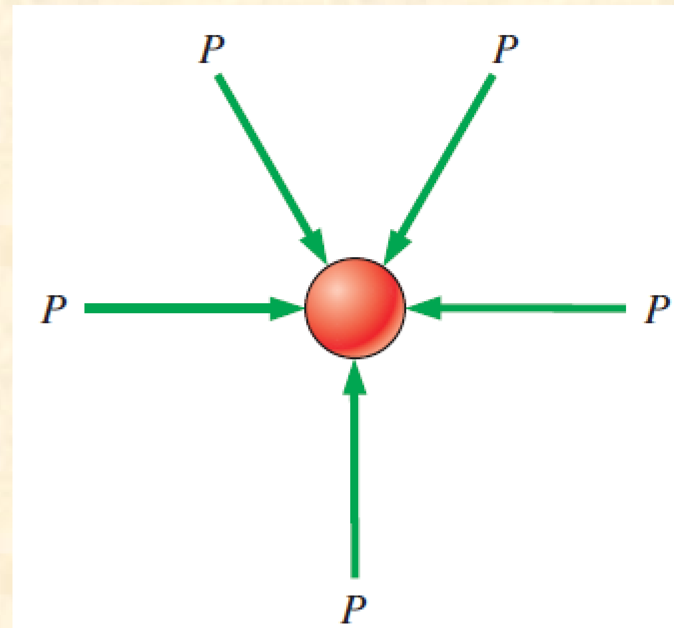
$$\sum F_z = ma_z = 0: \qquad P_2 \Delta y \Delta x - P_3 \Delta y l \cos\theta - \frac{1}{2}\rho g \, \Delta x \, \Delta y \, \Delta z = 0$$

$$W = mg = \rho g \, \Delta x \, \Delta y \, \Delta z/2 \qquad \Delta z = l \sin\theta \qquad \Delta x = l \cos\theta$$



Forces acting on a wedge-shaped fluid element in equilibrium.

Pressure is the *compressive force* per unit area but it is not a vector. *Pressure at any point in a fluid is the same in all directions.* Pressure has magnitude but not a specific direction, and thus it is a scalar quantity.



Pressure is a *scalar* quantity, not a vector; the pressure at a point in a fluid is the same in all directions.

Variation of Pressure with Depth

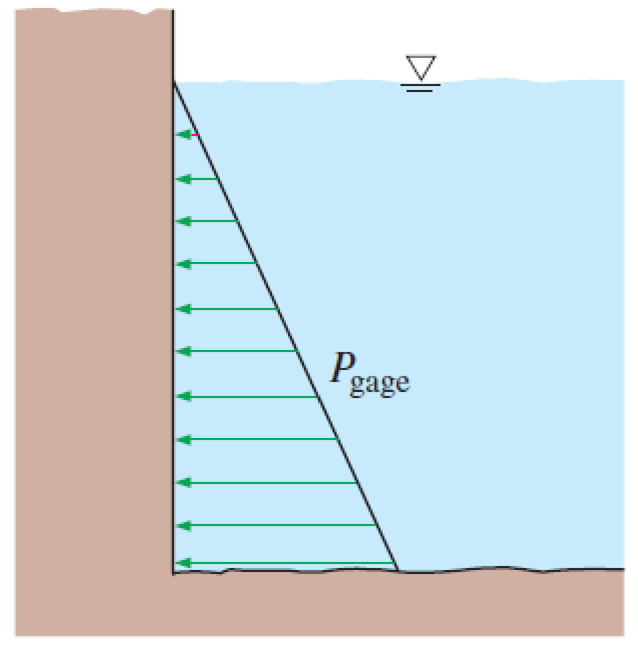
$$\Delta P = P_2 - P_1 = \rho g\, \Delta z = \gamma_s\, \Delta z$$

$$P_{\text{below}} = P_{\text{above}} + \rho g |\Delta z| = P_{\text{above}} + \gamma_s |\Delta z|$$

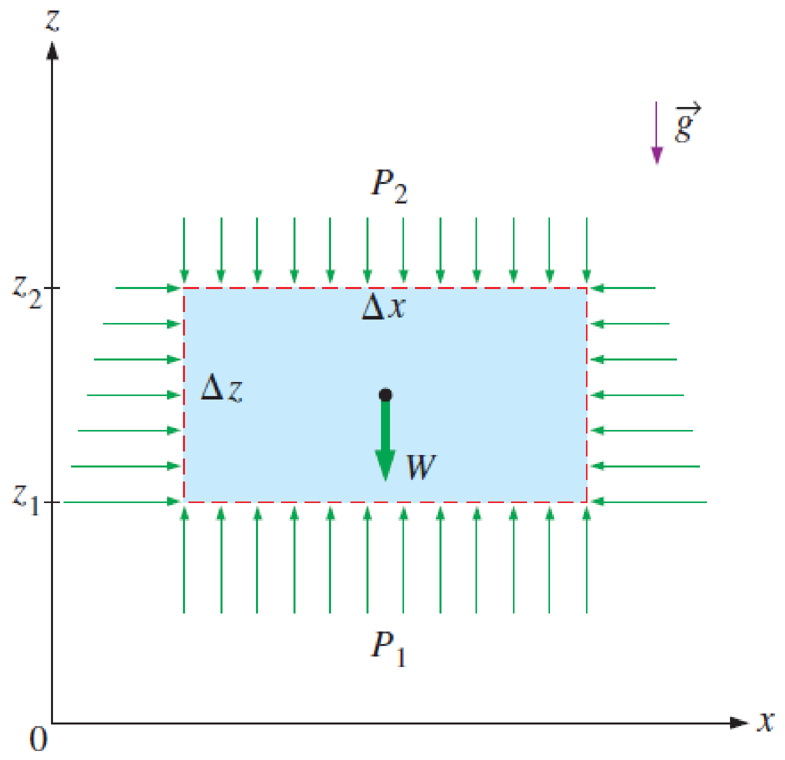
$$P = P_{\text{atm}} + \rho g h \quad \text{or} \quad P_{\text{gage}} = \rho g h$$

When the variation of density with elevation is known

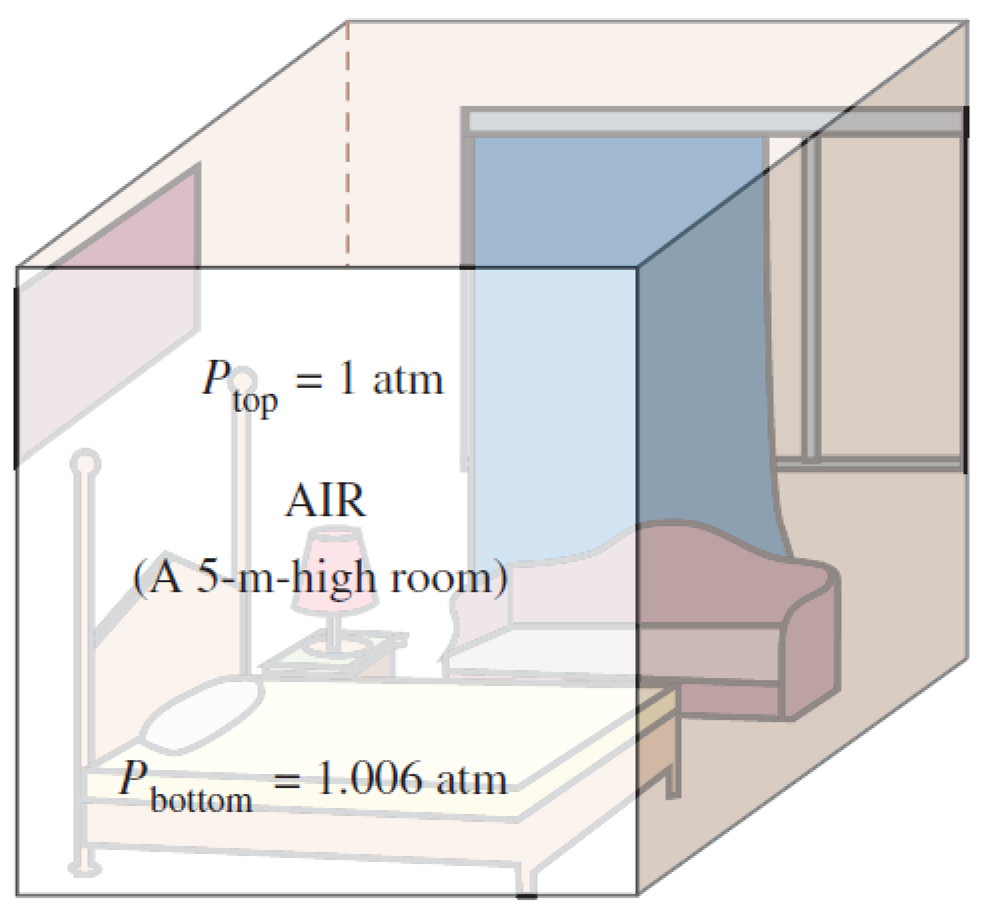
$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g\, dz$$



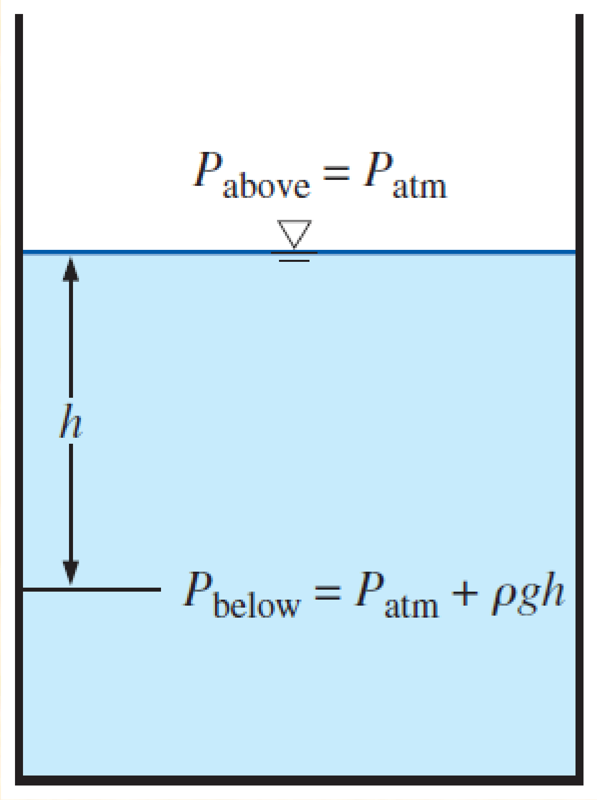
The pressure of a fluid at rest increases with depth (as a result of added weight).



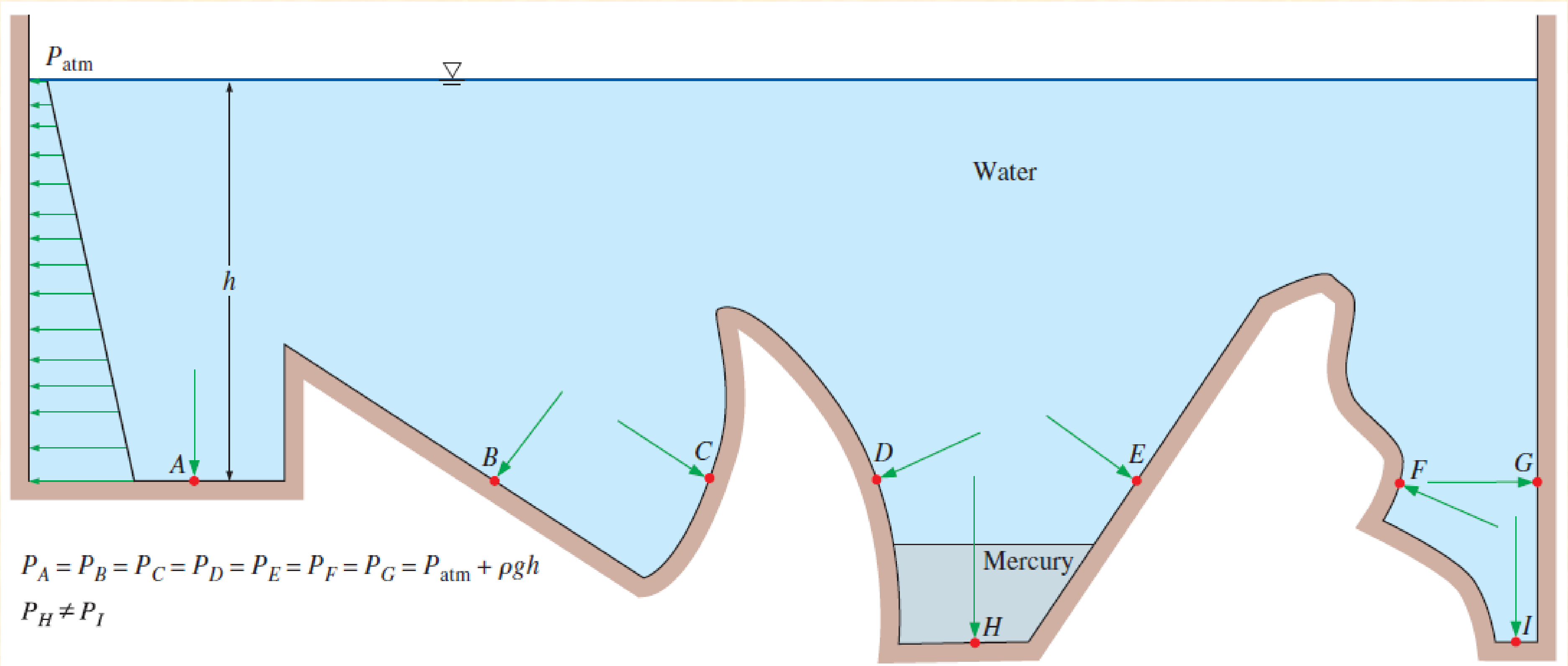
Free-body diagram of a rectangular fluid element in equilibrium.



In a room filled with a gas, the variation of pressure with height is negligible.



Pressure in a liquid at rest increases linearly with distance from the free surface.

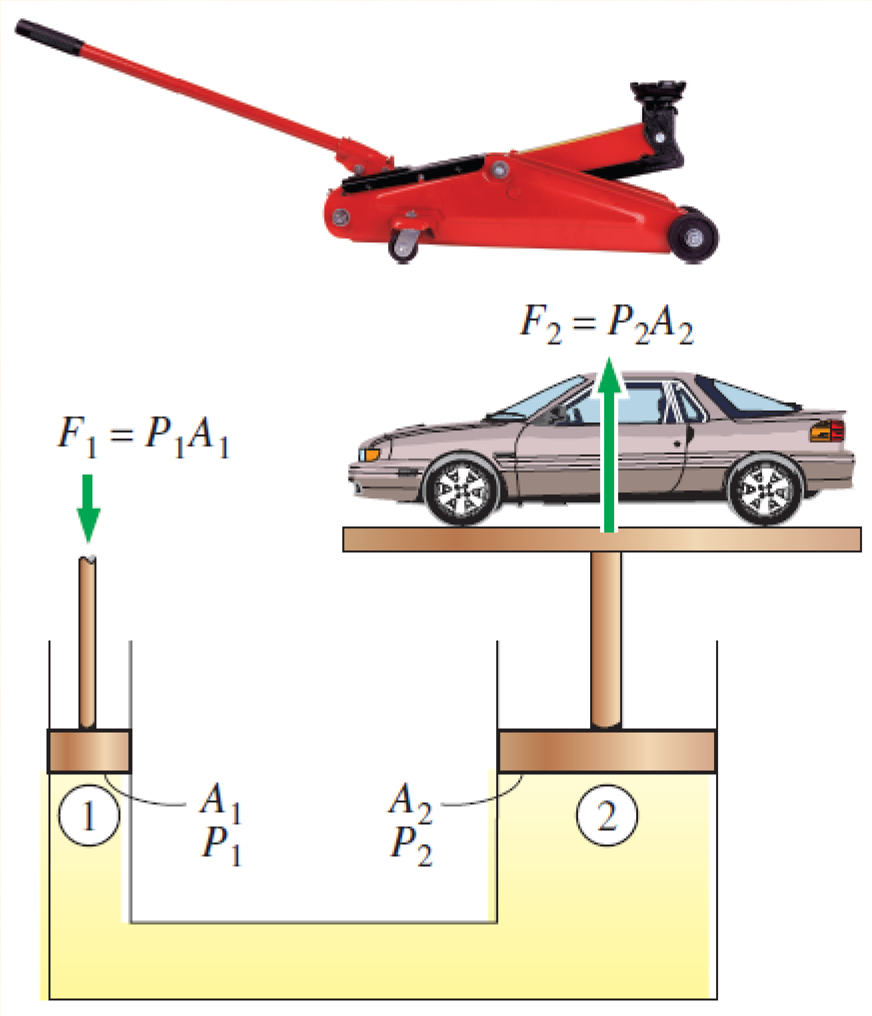


The pressure is the same at all points on a horizontal plane in a given fluid regardless of geometry, provided that the points are interconnected by the same fluid.

Pascal's law: The pressure applied to a confined fluid increases the pressure throughout by the same amount.

$$P_1 = P_2 \quad \rightarrow \quad \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \rightarrow \quad \frac{F_2}{F_1} = \frac{A_2}{A_1}$$

The area ratio A_2/A_1 is called the *ideal mechanical advantage* of the hydraulic lift.



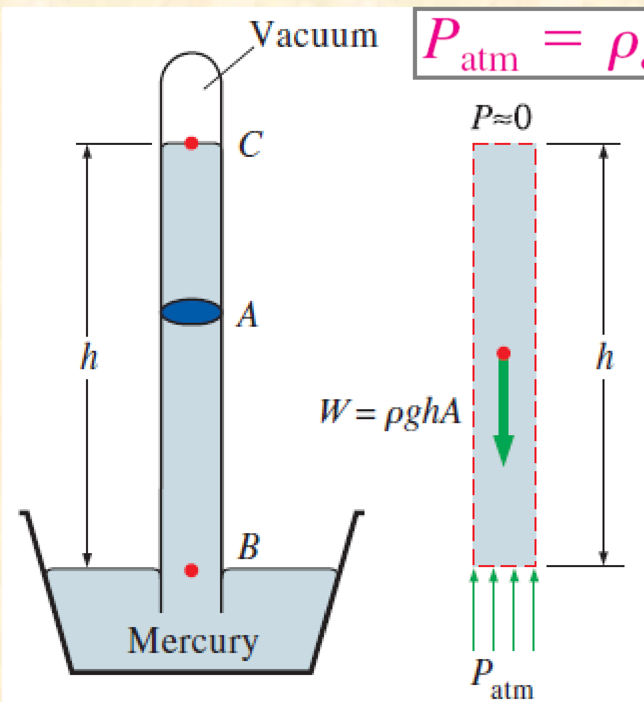
Lifting of a large weight by a small force by the application of Pascal's law.

3–2 ■ PRESSURE MEASUREMENT DEVICES

The Barometer

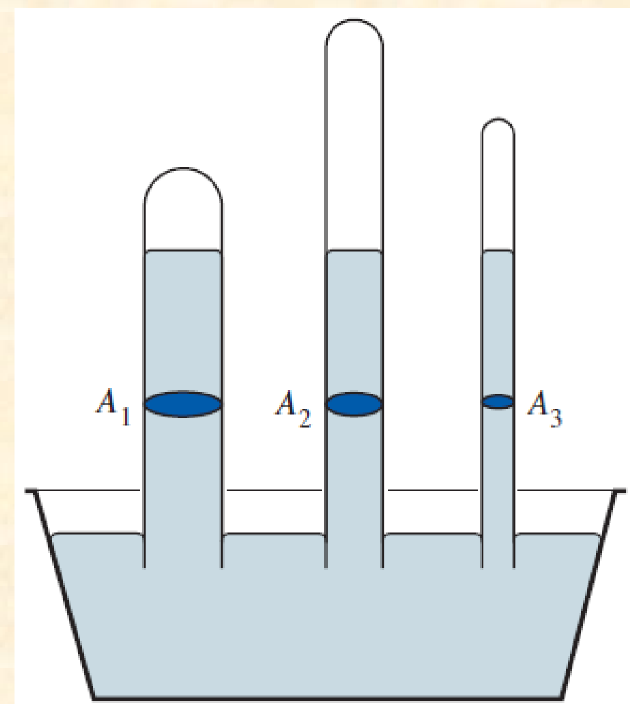
- Atmospheric pressure is measured by a device called a **barometer**; thus, the atmospheric pressure is often referred to as the ***barometric pressure***.
- A frequently used pressure unit is the *standard atmosphere*, which is defined as the pressure produced by a column of mercury 760 mm in height at 0°C (ρ_{Hg} = 13,595 kg/m^3) under standard gravitational acceleration (g = 9.807 m/s^2).

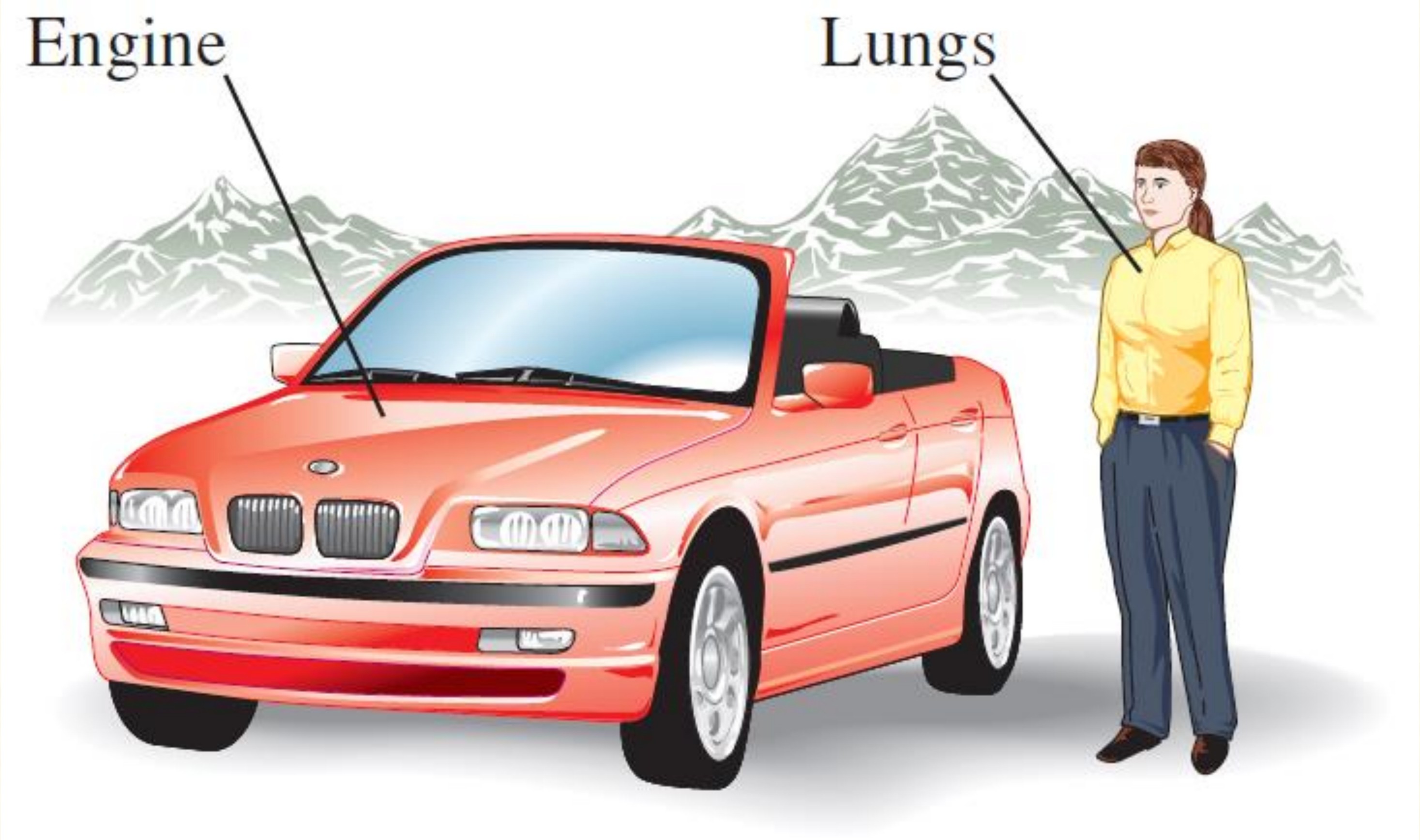
$$P_{atm} = \rho g h$$



The basic barometer.

The length or the cross-sectional area of the tube has no effect on the height of the fluid column of a barometer, provided that the tube diameter is large enough to avoid surface tension (capillary) effects.





At high altitudes, a car engine generates less power and a person gets less oxygen because of the lower density of air.

EXAMPLE 3–2 Measuring Atmospheric Pressure with a Barometer

Determine the atmospheric pressure at a location where the barometric reading is 740 mm Hg and the gravitational acceleration is $g = 9.805\ m/s^2$. Assume the temperature of mercury to be 10°C, at which its density is 13,570 kg/m^3.

SOLUTION The barometric reading at a location in height of mercury column is given. The atmospheric pressure is to be determined.
Assumptions The temperature of mercury is assumed to be 10°C.
Properties The density of mercury is given to be 13,570 kg/m^3.
Analysis From Eq. 3–12, the atmospheric pressure is determined to be

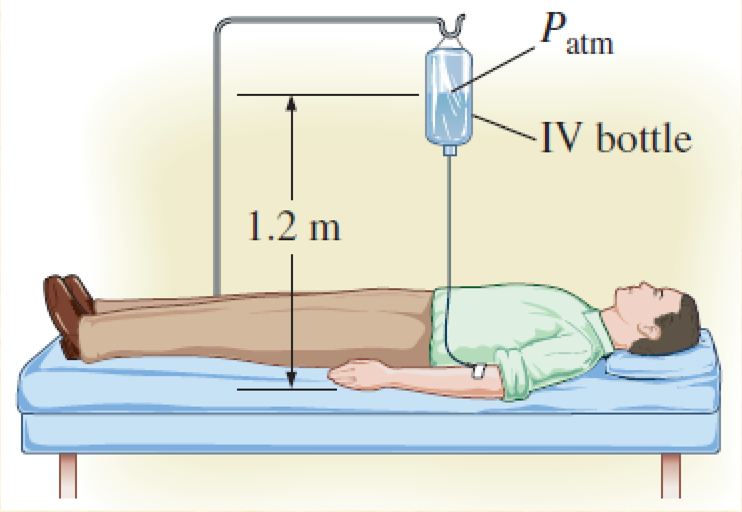
$$\begin{aligned} P_{atm} &= \rho g h \\ &= (13{,}570\ \mathrm{kg/m^3})(9.805\ \mathrm{m/s^2})(0.740\ \mathrm{m})\left(\frac{1\ \mathrm{N}}{1\ \mathrm{kg \cdot m/s^2}}\right)\left(\frac{1\ \mathrm{kPa}}{1000\ \mathrm{N/m^2}}\right) \\ &= \mathbf{98.5\ kPa} \end{aligned}$$

Discussion Note that density changes with temperature, and thus this effect should be considered in calculations.

EXAMPLE 3–3 Gravity Driven Flow from an IV Bottle

Intravenous infusions usually are driven by gravity by hanging the fluid bottle at sufficient height to counteract the blood pressure in the vein and to force the fluid into the body (Fig. 3–15). The higher the bottle is raised, the higher the flow rate of the fluid will be. (*a*) If it is observed that the fluid and the blood pressures balance each other when the bottle is 1.2 m above the arm level, determine the gage pressure of the blood. (*b*) If the gage pressure of the fluid at the arm level needs to be 20 kPa for sufficient flow rate, determine how high the bottle must be placed. Take the density of the fluid to be 1020 kg/m^3.

SOLUTION It is given that an IV fluid and the blood pressures balance each other when the bottle is at a certain height. The gage pressure of the blood and elevation of the bottle required to maintain flow at the desired rate are to be determined.



Assumptions **1** The IV fluid is incompressible. **2** The IV bottle is open to the atmosphere.

Properties The density of the IV fluid is given to be $\rho = 1020\ \text{kg/m}^3$.

Analysis (*a*) Noting that the IV fluid and the blood pressures balance each other when the bottle is 1.2 m above the arm level, the gage pressure of the blood in the arm is simply equal to the gage pressure of the IV fluid at a depth of 1.2 m,

$$\begin{aligned} P_{\text{gage, arm}} &= P_{\text{abs}} - P_{\text{atm}} = \rho g h_{\text{arm}-\text{bottle}} \\ &= (1020\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(1.20\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1\ \text{kN/m}^2}\right) \\ &= \mathbf{12.0\ kPa} \end{aligned}$$

(*b*) To provide a gage pressure of 20 kPa at the arm level, the height of the surface of the IV fluid in the bottle from the arm level is again determined from $P_{\text{gage, arm}} = \rho g h_{\text{arm}-\text{bottle}}$ to be

$$\begin{aligned} h_{\text{arm}-\text{botttle}} &= \frac{P_{\text{gage, arm}}}{\rho g} \\ &= \frac{20\ \text{kPa}}{(1020\ \text{kg/m}^3)(9.81\ \text{m/s}^2)}\left(\frac{1000\ \text{kg}\cdot\text{m/s}^2}{1\ \text{kN}}\right)\left(\frac{1\ \text{kN/m}^2}{1\ \text{kPa}}\right) \\ &= \mathbf{2.00\ m} \end{aligned}$$

Discussion Note that the height of the reservoir can be used to control flow rates in gravity-driven flows. When there is flow, the pressure drop in the tube due to frictional effects also should be considered. For a specified flow rate, this requires raising the bottle a little higher to overcome the pressure drop.

EXAMPLE 3–4 **Hydrostatic Pressure in a Solar Pond with Variable Density**

Solar ponds are small artificial lakes of a few meters deep that are used to store solar energy. The rise of heated (and thus less dense) water to the surface is prevented by adding salt at the pond bottom. In a typical salt gradient solar pond, the density of water increases in the gradient zone, as shown in Fig. 3–16, and the density can be expressed as

$$\rho = \rho_0 \sqrt{1 + \tan^2\left(\frac{\pi}{4}\frac{s}{H}\right)}$$

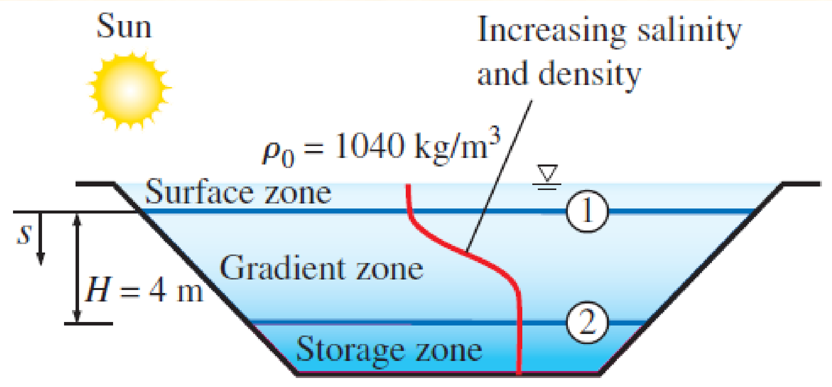
where ρ_0 is the density on the water surface, s is the vertical distance measured downward from the top of the gradient zone ($s = -z$), and H is the thickness of the gradient zone. For $H = 4$ m, $\rho_0 = 1040$ kg/m^3, and a thickness of 0.8 m for the surface zone, calculate the gage pressure at the bottom of the gradient zone.

SOLUTION The variation of density of saline water in the gradient zone of a solar pond with depth is given. The gage pressure at the bottom of the gradient zone is to be determined.

Assumptions The density in the surface zone of the pond is constant.

Properties The density of brine on the surface is given to be 1040 kg/m^3.

Analysis We label the top and the bottom of the gradient zone as 1 and 2, respectively. Noting that the density of the surface zone is constant, the



gage pressure at the bottom of the surface zone (which is the top of the gradient zone) is

$$P_1 = \rho g h_1 = (1040\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.8\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right) = 8.16\ \text{kPa}$$

since $1\ \text{kN/m}^2 = 1\ \text{kPa}$. Since $s = -z$, the differential change in hydrostatic pressure across a vertical distance of ds is given by

$$dP = \rho g\, ds$$

Integrating from the top of the gradient zone (point 1 where $s = 0$) to any location s in the gradient zone (no subscript) gives

$$P - P_1 = \int_0^s \rho g\, ds \quad \rightarrow \quad P = P_1 + \int_0^s \rho_0 \sqrt{1 + \tan^2\left(\frac{\pi}{4}\frac{s}{H}\right)}\, g\, ds$$

Performing the integration gives the variation of gage pressure in the gradient zone to be

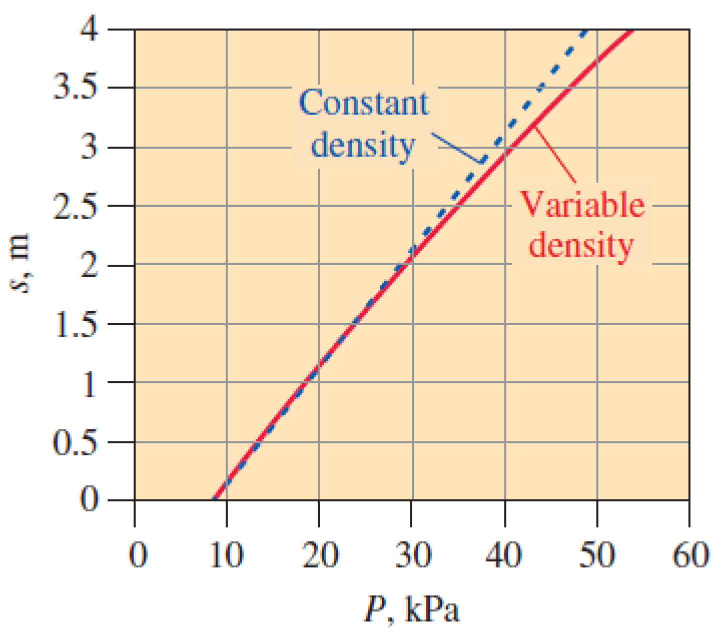
$$P = P_1 + \rho_0 g \frac{4H}{\pi} \sinh^{-1}\left(\tan \frac{\pi}{4}\frac{s}{H}\right)$$

Then the pressure at the bottom of the gradient zone ($s = H = 4$ m) becomes

$$P_2 = 8.16\ \text{kPa} + (1040\ \text{kg/m}^3)(9.81\ \text{m/s}^2)\frac{4(4\ \text{m})}{\pi} \sinh^{-1}\left(\tan \frac{\pi}{4}\frac{4}{4}\right)\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= \mathbf{54.0\ kPa\ (gage)}$$

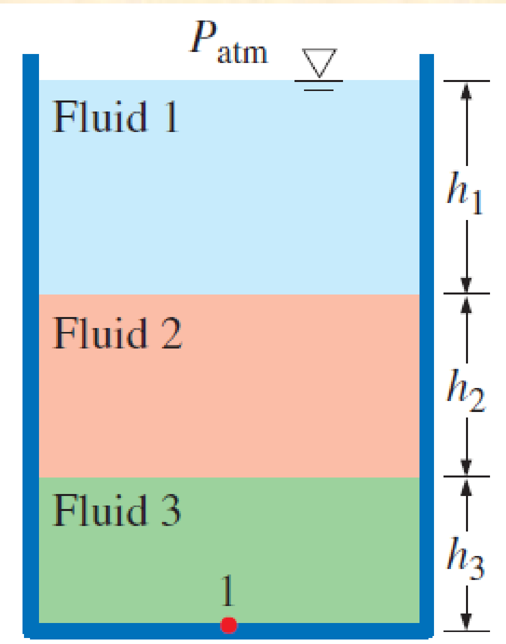
Discussion The variation of gage pressure in the gradient zone with depth is plotted in Fig. 3–17. The dashed line indicates the hydrostatic pressure for the case of constant density at 1040 kg/m³ and is given for reference. Note that the variation of pressure with depth is not linear when density varies with depth. That is why integration was required.



The variation of gage pressure with depth in the gradient zone of the solar pond.

The Manometer

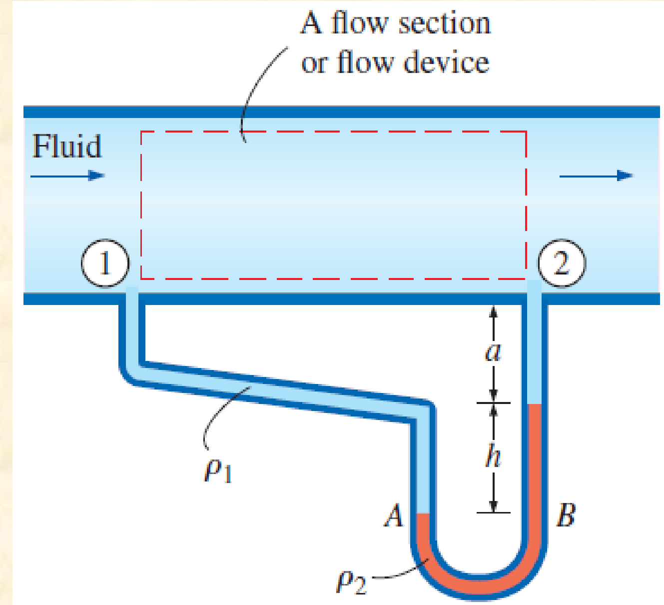
It is commonly used to measure small and moderate pressure differences. A manometer contains one or more fluids such as mercury, water, alcohol, or oil.



$$P_{atm} + \rho_1 g h_1 + \rho_2 g h_2 + \rho_3 g h_3 = P_1$$

In stacked-up fluid layers, the pressure change across a fluid layer of density ρ and height h is ρgh.

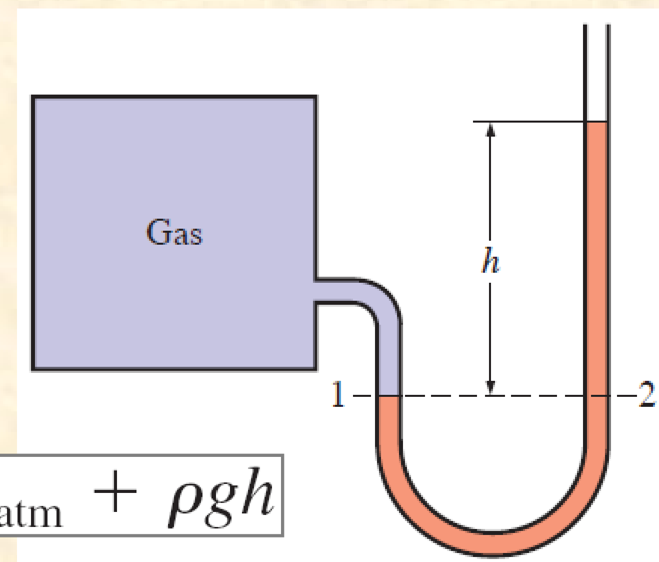
Measuring the pressure drop across a flow section or a flow device by a differential manometer.



$$P_1 + \rho_1 g(a + h) - \rho_2 gh - \rho_1 ga = P_2$$

$$P_1 - P_2 = (\rho_2 - \rho_1)gh$$

The basic manometer.



$$P_2 = P_{atm} + \rho gh$$

A simple U-tube manometer, with high pressure applied to the right side.

EXAMPLE 3–5 **Measuring Pressure with a Manometer**

A manometer is used to measure the pressure of a gas in a tank. The fluid used has a specific gravity of 0.85, and the manometer column height is 55 cm, as shown in Fig. 3–20. If the local atmospheric pressure is 96 kPa, determine the absolute pressure within the tank.

SOLUTION The reading of a manometer attached to a tank and the atmospheric pressure are given. The absolute pressure in the tank is to be determined.

Assumptions The density of the gas in the tank is much lower than the density of the manometer fluid.

Properties The specific gravity of the manometer fluid is given to be 0.85. We take the standard density of water to be 1000 kg/m^3.

Analysis The density of the fluid is obtained by multiplying its specific gravity by the density of water,

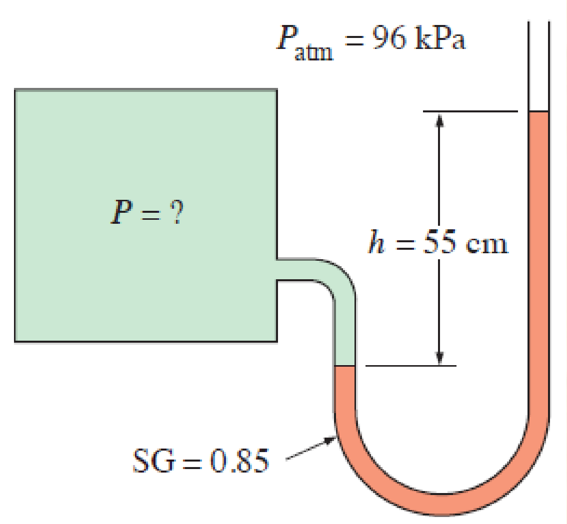
$$\rho = \text{SG}\,(\rho_{H_2O}) = (0.85)(1000\ \text{kg/m}^3) = 850\ \text{kg/m}^3$$

Then from Eq. 3–13,

$$P = P_{atm} + \rho g h$$

$$= 96\ \text{kPa} + (850\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.55\ \text{m})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1000\ \text{N/m}^2}\right)$$

$$= \mathbf{100.6\ kPa}$$



Discussion Note that the gage pressure in the tank is 4.6 kPa.

EXAMPLE 3–6 Measuring Pressure with a Multifluid Manometer

The water in a tank is pressurized by air, and the pressure is measured by a multifluid manometer as shown in Fig. 3–23. The tank is located on a mountain at an altitude of 1400 m where the atmospheric pressure is 85.6 kPa. Determine the air pressure in the tank if $h_1 = 0.1$ m, $h_2 = 0.2$ m, and $h_3 = 0.35$ m. Take the densities of water, oil, and mercury to be 1000 kg/m^3, 850 kg/m^3, and 13,600 kg/m^3, respectively.

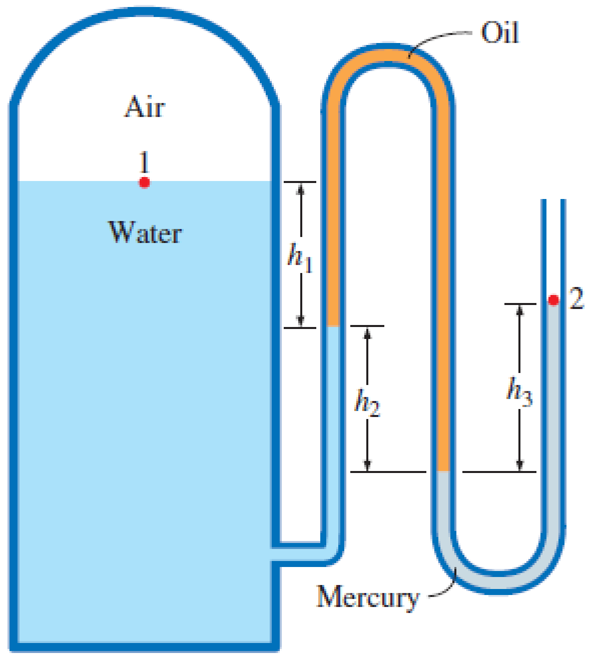
SOLUTION The pressure in a pressurized water tank is measured by a multifluid manometer. The air pressure in the tank is to be determined.

Assumption The air pressure in the tank is uniform (i.e., its variation with elevation is negligible due to its low density), and thus we can determine the pressure at the air–water interface.

Properties The densities of water, oil, and mercury are given to be 1000 kg/m^3, 850 kg/m^3, and 13,600 kg/m^3, respectively.

Analysis Starting with the pressure at point 1 at the air–water interface, moving along the tube by adding or subtracting the ρgh terms until we reach point 2, and setting the result equal to P_{atm} since the tube is open to the atmosphere gives

$$P_1 + \rho_{water} g h_1 + \rho_{oil} g h_2 - \rho_{mercury} g h_3 = P_2 = P_{atm}$$



Solving for P_1 and substituting,

$$P_1 = P_{atm} - \rho_{water} g h_1 - \rho_{oil} g h_2 + \rho_{mercury} g h_3$$

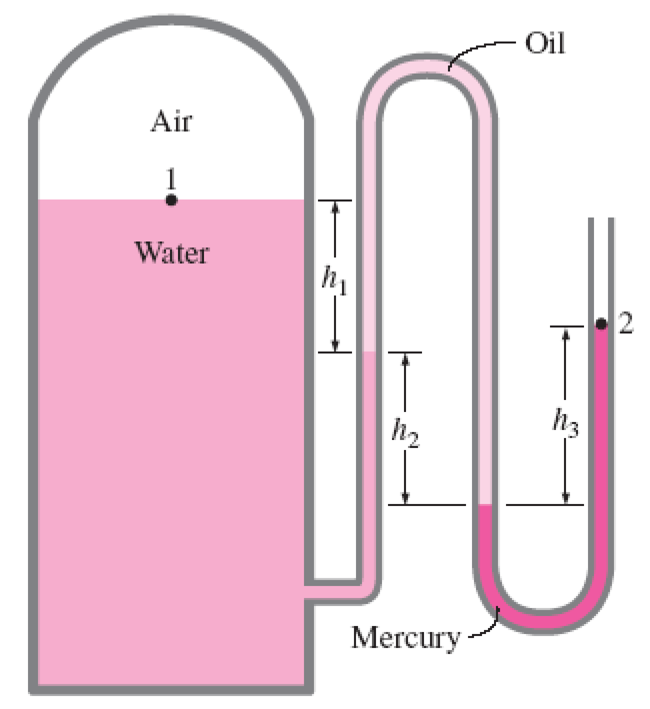
$$= P_{atm} + g(\rho_{mercury} h_3 - \rho_{water} h_1 - \rho_{oil} h_2)$$

$$= 85.6 \text{ kPa} + (9.81 \text{ m/s}^2)[(13{,}600 \text{ kg/m}^3)(0.35 \text{ m}) - (1000 \text{ kg/m}^3)(0.1 \text{ m})$$

$$- (850 \text{ kg/m}^3)(0.2 \text{ m})]\left(\frac{1 \text{ N}}{1 \text{ kg·m/s}^2}\right)\left(\frac{1 \text{ kPa}}{1000 \text{ N/m}^2}\right)$$

$$= \mathbf{130 \text{ kPa}}$$

Discussion Note that jumping horizontally from one tube to the next and realizing that pressure remains the same in the same fluid simplifies the analysis considerably. Also note that mercury is a toxic fluid, and mercury manometers and thermometers are being replaced by ones with safer fluids because of the risk of exposure to mercury vapor during an accident.



EXAMPLE 3–7 **Analyzing a Multifluid Manometer with EES**

Reconsider the multifluid manometer discussed in Example 3–6. Determine the air pressure in the tank using EES. Also determine what the differential fluid height h_3 would be for the same air pressure if the mercury in the last column were replaced by seawater with a density of 1030 kg/m^3.

SOLUTION The pressure in a water tank is measured by a multifluid manometer. The air pressure in the tank and the differential fluid height h_3 if mercury is replaced by seawater are to be determined using EES.

Analysis We start the EES program, open a new file, and type the following on the blank screen that appears (we express the atmospheric pressure in Pa for unit consistency):

```
g=9.81
Patm=85600
h1=0.1;  h2=0.2;  h3=0.35
rw=1000;  roil=850;  rm=13600
P1+rw*g*h1+roil*g*h2−rm*g*h3=Patm
```

Here P1 is the only unknown, and it is determined by EES to be

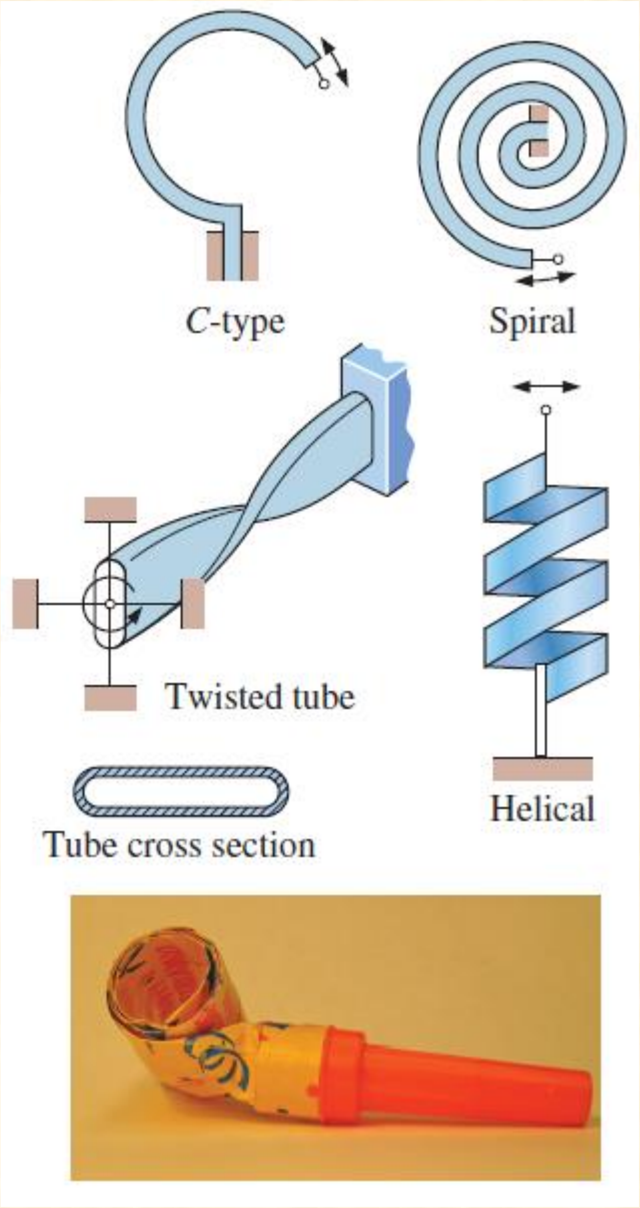
$$P_1 = 129647 \text{ Pa} \cong \mathbf{130\ kPa}$$

which is identical to the result obtained in Example 3–6. The height of the fluid column h_3 when mercury is replaced by seawater is determined easily by replacing "h3=0.35" by "P1=129647" and "rm=13600" by "rm=1030," and clicking on the calculator symbol. It gives

$$h_3 = \mathbf{4.62\ m}$$

Other Pressure Measurement Devices

- **Bourdon tube**: Consists of a hollow metal tube bent like a hook whose end is closed and connected to a dial indicator needle.
- **Pressure transducers**: Use various techniques to convert the pressure effect to an electrical effect such as a change in voltage, resistance, or capacitance.
- Pressure transducers are smaller and faster, and they can be more sensitive, reliable, and precise than their mechanical counterparts.
- **Strain-gage pressure transducers:** Work by having a diaphragm deflect between two chambers open to the pressure inputs.
- **Piezoelectric transducers**: Also called solid-state pressure transducers, work on the principle that an electric potential is generated in a crystalline substance when it is subjected to mechanical pressure.



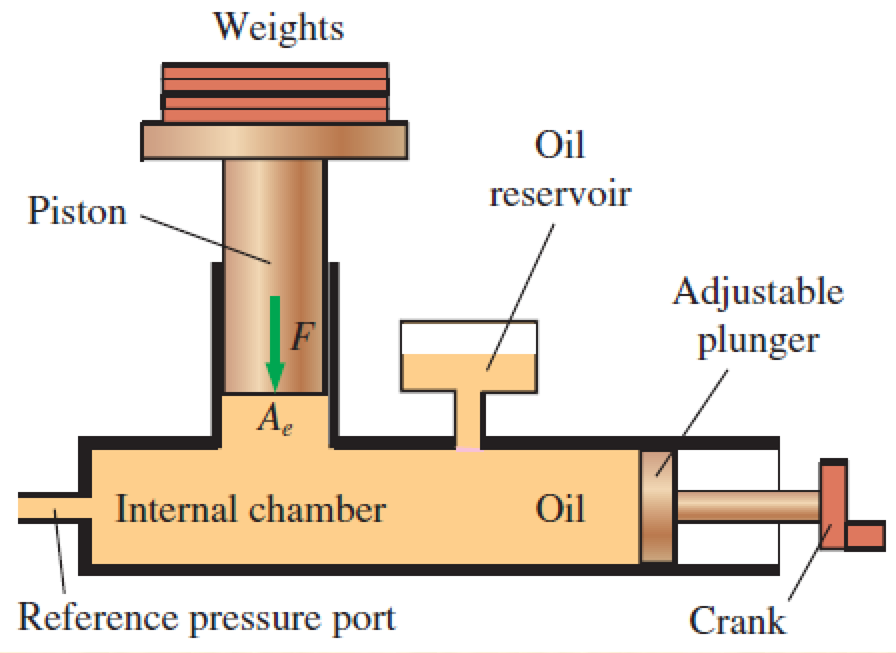
Various types of Bourdon tubes used to measure pressure.

Deadweight tester: Another type of mechanical pressure gage. It is used primarily for *calibration* and can measure extremely high pressures.

A deadweight tester measures pressure *directly* through application of a weight that provides a force per unit area—the fundamental definition of pressure.

It is constructed with an internal chamber filled with a fluid (usually oil), along with a tight-fitting piston, cylinder, and plunger.

Weights are applied to the top of the piston, which exerts a force on the oil in the chamber. The total force *F* acting on the oil at the piston–oil interface is the sum of the weight of the piston plus the applied weights.



A deadweight tester is able to measure extremely high pressures (up to 70 MPa in some applications).

3–3 ■ INTRODUCTION TO FLUID STATICS

Fluid statics: Deals with problems associated with fluids at rest.

The fluid can be either gaseous or liquid.

Hydrostatics: When thye fluid is a liquid.

Aerostatics: When the fluid is a gas.

In fluid statics, there is no relative motion between adjacent fluid layers, and thus there are no shear (tangential) stresses in the fluid trying to deform it.

The only stress we deal with in fluid statics is the *normal stress*, which is the pressure, and the variation of pressure is due only to the weight of the fluid.

The topic of fluid statics has significance only in gravity fields.

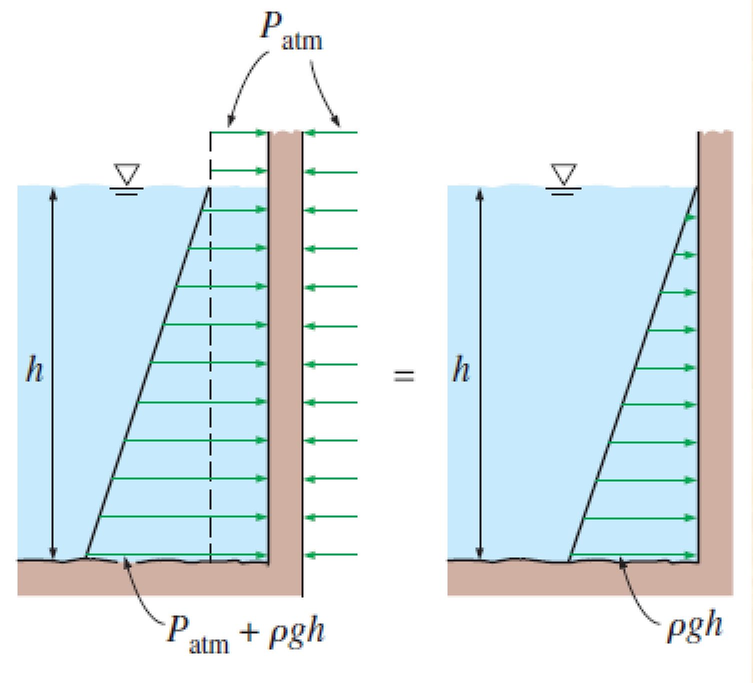
The design of many engineering systems such as water dams and liquid storage tanks requires the determination of the forces acting on the surfaces using fluid statics.

3–4 ■ HYDROSTATIC FORCES ON SUBMERGED PLANE SURFACES

Hoover Dam.

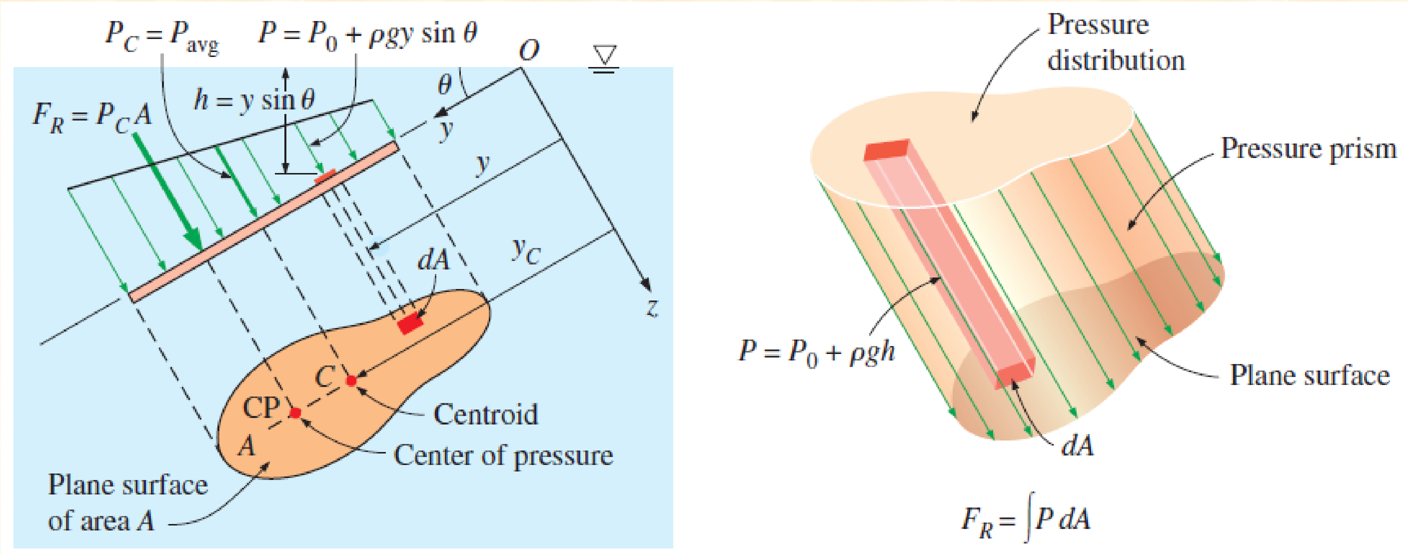
A plate, such as a gate valve in a dam, the wall of a liquid storage tank, or the hull of a ship at rest, is subjected to fluid pressure distributed over its surface when exposed to a liquid.

On a *plane* surface, the hydrostatic forces form a system of parallel forces, and we often need to determine the *magnitude* of the force and its *point of application*, which is called the **center of pressure**.



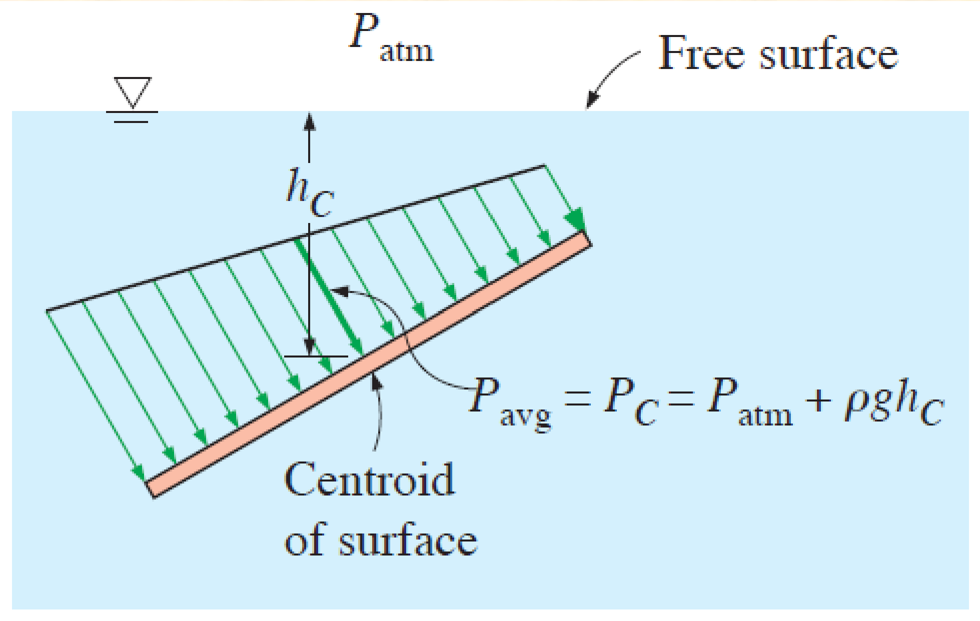
(*a*) P_{atm} considered (*b*) P_{atm} subtracted

When analyzing hydrostatic forces on submerged surfaces, the atmospheric pressure can be subtracted for simplicity when it acts on both sides of the structure.



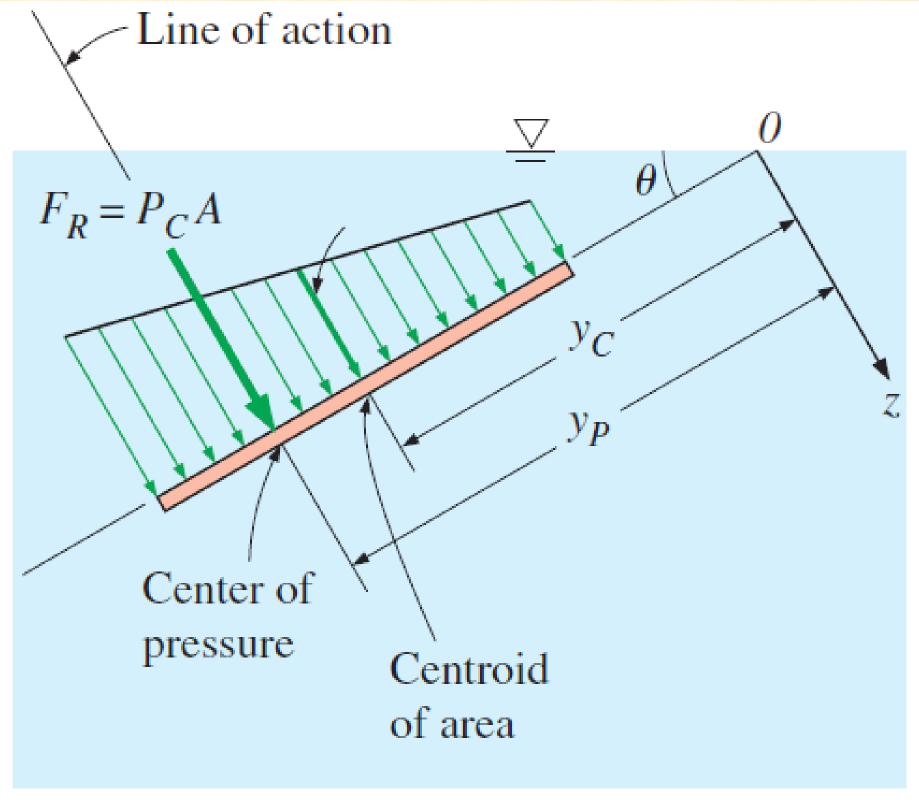
Hydrostatic force on an inclined plane surface completely submerged in a liquid.

$$F_R = (P_0 + \rho g y_C \sin\theta)A = (P_0 + \rho g h_C)A = P_C A = P_{avg} A$$



The pressure at the centroid of a surface is equivalent to the *average* pressure on the surface.

The resultant force acting on a plane surface is equal to the product of the pressure at the centroid of the surface and the surface area, and its line of action passes through the center of pressure.



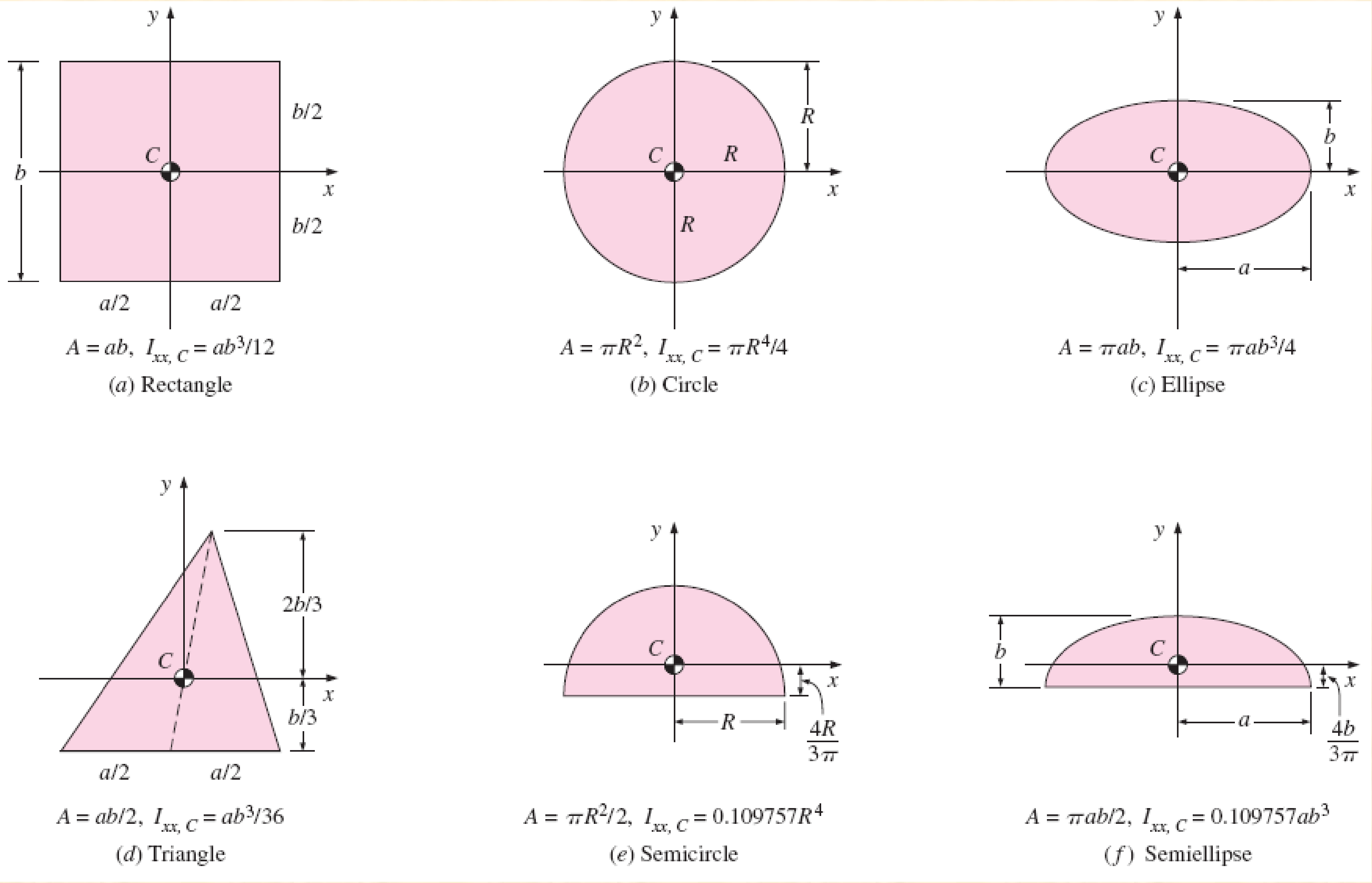
$$y_P = y_C + \frac{I_{xx,C}}{[y_C + P_0/(\rho g \sin\theta)]A}$$

$$y_P = y_C + \frac{I_{xx,C}}{y_C A}$$

$$I_{xx,O} = \int_A y^2 \, dA$$

second moment of area (*area moment of inertia*) about the *x*-axis.

$$I_{xx,O} = I_{xx,C} + y_C^2 A$$



$A = ab,\ I_{xx,C} = ab^3/12$
(*a*) Rectangle

$A = \pi R^2,\ I_{xx,C} = \pi R^4/4$
(*b*) Circle

$A = \pi ab,\ I_{xx,C} = \pi ab^3/4$
(*c*) Ellipse

$A = ab/2,\ I_{xx,C} = ab^3/36$
(*d*) Triangle

$A = \pi R^2/2,\ I_{xx,C} = 0.109757R^4$
(*e*) Semicircle

$A = \pi ab/2,\ I_{xx,C} = 0.109757ab^3$
(*f*) Semiellipse

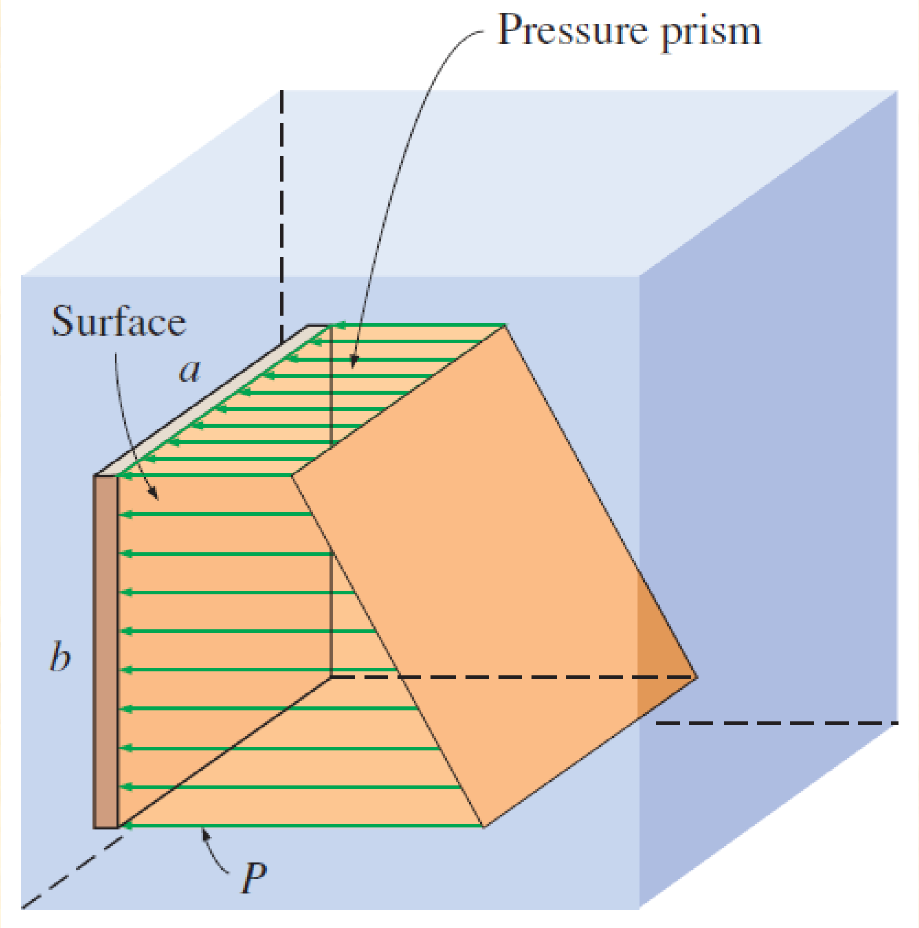
The centroid and the centroidal moments of inertia for some common geometries.

Pressure acts normal to the surface, and the hydrostatic forces acting on a flat plate of any shape form a volume whose base is the plate area and whose length is the linearly varying pressure.

This virtual **pressure prism** has an interesting physical interpretation: its *volume* is equal to the *magnitude* of the resultant hydrostatic force acting on the plate since $F_R = \int PdA$, and the line of action of this force passes through the *centroid* of this homogeneous prism.

The projection of the centroid on the plate is the *pressure center.*

Therefore, with the concept of pressure prism, the problem of describing the resultant hydrostatic force on a plane surface reduces to finding the volume and the two coordinates of the centroid of this pressure prism.



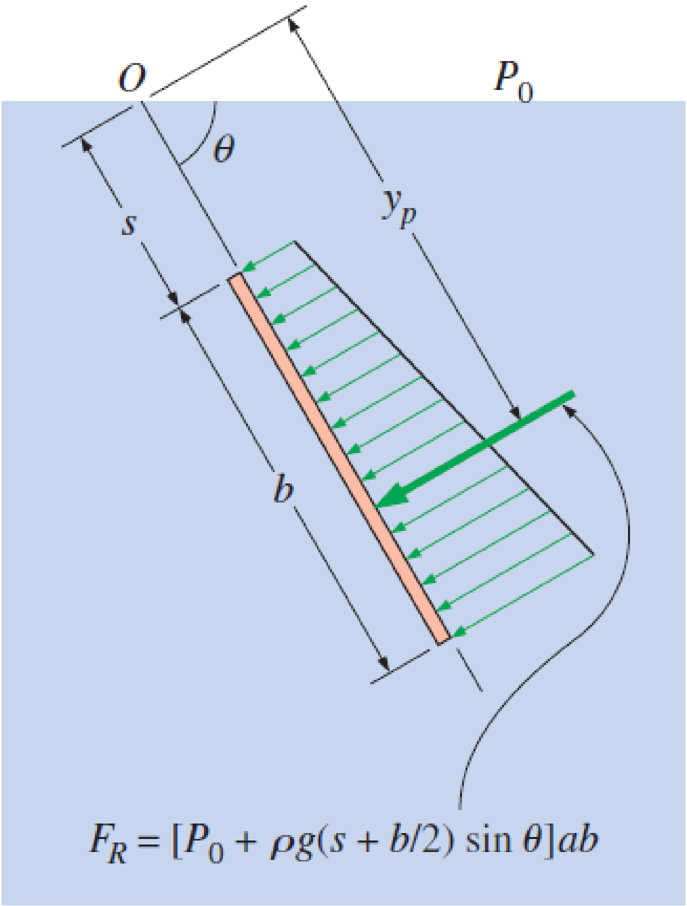
The hydrostatic forces acting on a plane surface form a pressure prism whose base (left face) is the surface and whose length is the pressure.

Special Case: Submerged Rectangular Plate

Hydrostatic force acting on the top surface of a submerged tilted rectangular plate.

$$y_P = s + \frac{b}{2} + \frac{ab^3/12}{[s + b/2 + P_0/(\rho g \sin\theta)]ab}$$

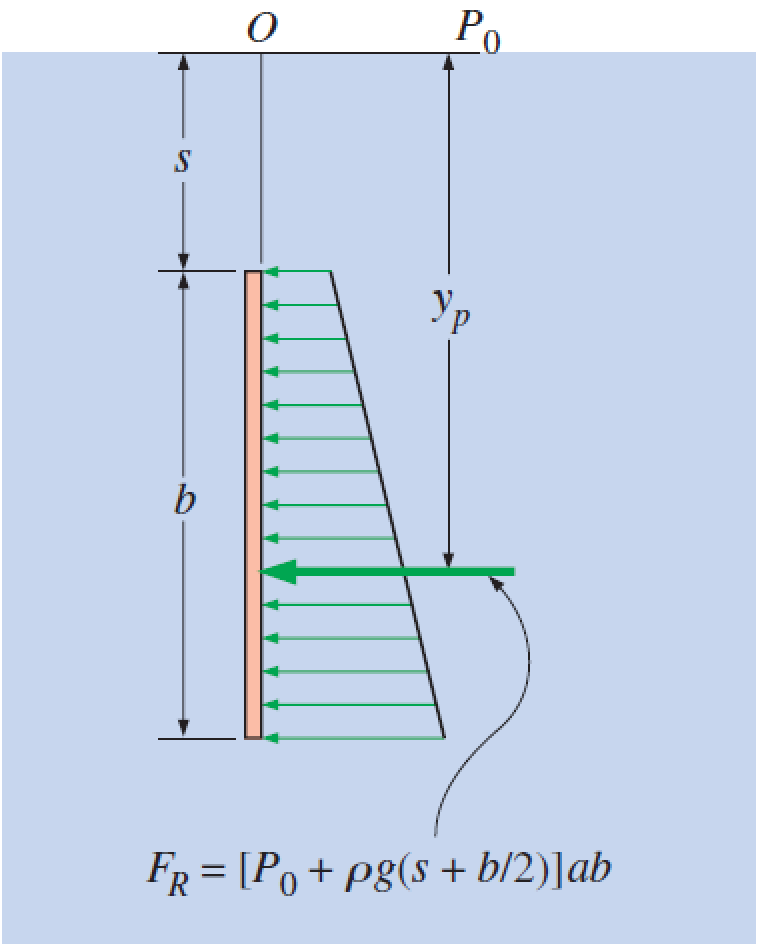
$$= s + \frac{b}{2} + \frac{b^2}{12[s + b/2 + P_0/(\rho g \sin\theta)]}$$



(a) Tilted plate

Tilted rectangular plate: $F_R = P_C A = [P_0 + \rho g(s + b/2)\sin\theta]ab$

Tilted rectangular plate ($s = 0$): $F_R = [P_0 + \rho g(b\sin\theta)/2]ab$



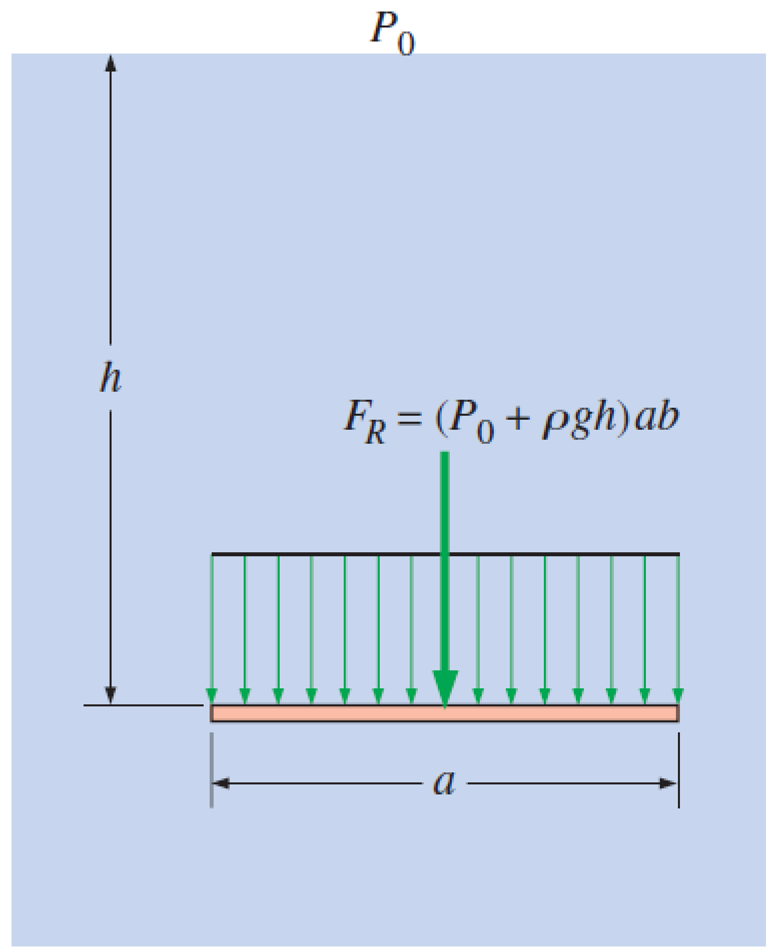
(b) Vertical plate

Hydrostatic force acting on the top surface of a submerged vertical rectangular plate.

Vertical rectangular plate: $F_R = [P_0 + \rho g(s + b/2)]ab$

Vertical rectangular plate (s = 0): $F_R = (P_0 + \rho g b/2)ab$

Hydrostatic force acting on the top surface of a submerged horizontal rectangular plate.



(c) Horizontal plate

Horizontal rectangular plate: $F_R = (P_0 + \rho g h)ab$

EXAMPLE 3–8 Hydrostatic Force Acting on the Door of a Submerged Car

A heavy car plunges into a lake during an accident and lands at the bottom of the lake on its wheels (Fig. 3–34). The door is 1.2 m high and 1 m wide, and the top edge of the door is 8 m below the free surface of the water. Determine the hydrostatic force on the door and the location of the pressure center, and discuss if the driver can open the door.

SOLUTION A car is submerged in water. The hydrostatic force on the door is to be determined, and the likelihood of the driver opening the door is to be assessed.

Assumptions **1** The bottom surface of the lake is horizontal. **2** The passenger cabin is well-sealed so that no water leaks inside. **3** The door can be approximated as a vertical rectangular plate. **4** The pressure in the passenger cabin remains at atmospheric value since there is no water leaking in, and thus no compression of the air inside. Therefore, atmospheric pressure cancels out in the calculations since it acts on both sides of the door. **5** The weight of the car is larger than the buoyant force acting on it.

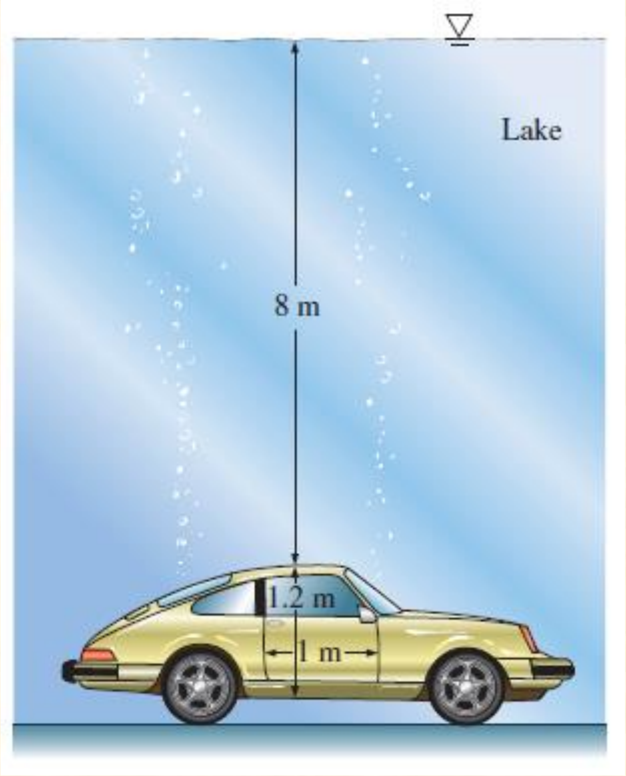
Properties We take the density of lake water to be 1000 kg/m³ throughout.

Analysis The average (gage) pressure on the door is the pressure value at the centroid (midpoint) of the door and is determined to be

$$P_{avg} = P_C = \rho g h_C = \rho g(s + b/2)$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(8 + 1.2/2\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= \mathbf{84.4\ kN/m^2}$$



Then the resultant hydrostatic force on the door becomes

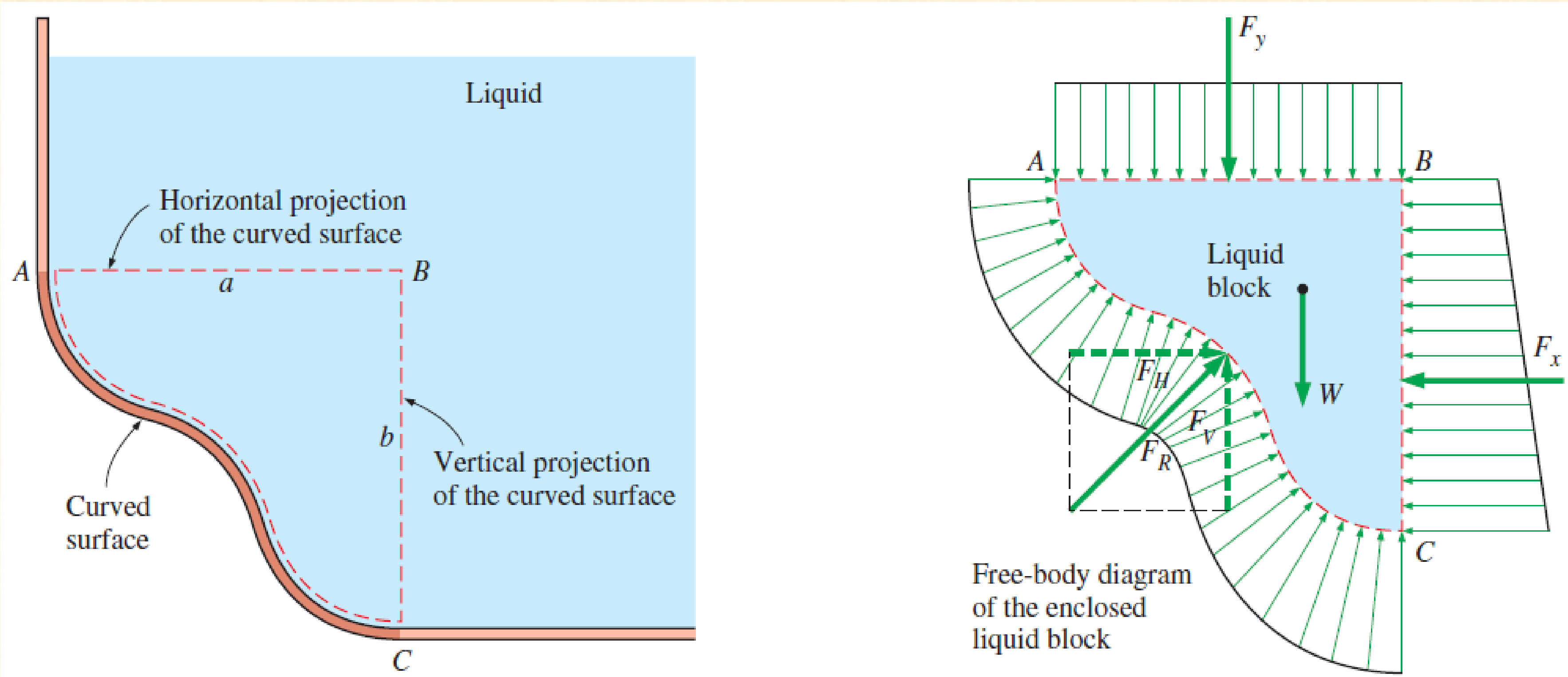
$$F_R = P_{avg}A = (84.4 \text{ kN/m}^2)(1 \text{ m} \times 1.2 \text{ m}) = \mathbf{101.3 \text{ kN}}$$

The pressure center is directly under the midpoint of the door, and its distance from the surface of the lake is determined from Eq. 3–24 by setting $P_0 = 0$, yielding

$$y_P = s + \frac{b}{2} + \frac{b^2}{12(s + b/2)} = 8 + \frac{1.2}{2} + \frac{1.2^2}{12(8 + 1.2/2)} = \mathbf{8.61 \text{ m}}$$

Discussion A strong person can lift 100 kg, which is a weight of 981 N or about 1 kN. Also, the person can apply the force at a point farthest from the hinges (1 m farther) for maximum effect and generate a moment of 1 kN·m. The resultant hydrostatic force acts under the midpoint of the door, and thus a distance of 0.5 m from the hinges. This generates a moment of 50.6 kN·m, which is about 50 times the moment the driver can possibly generate. Therefore, it is impossible for the driver to open the door of the car. The driver's best bet is to let some water in (by rolling the window down a little, for example) and to keep his or her head close to the ceiling. The driver should be able to open the door shortly before the car is filled with water since at that point the pressures on both sides of the door are nearly the same and opening the door in water is almost as easy as opening it in air.

3–5 ■ HYDROSTATIC FORCES ON SUBMERGED CURVED SURFACES

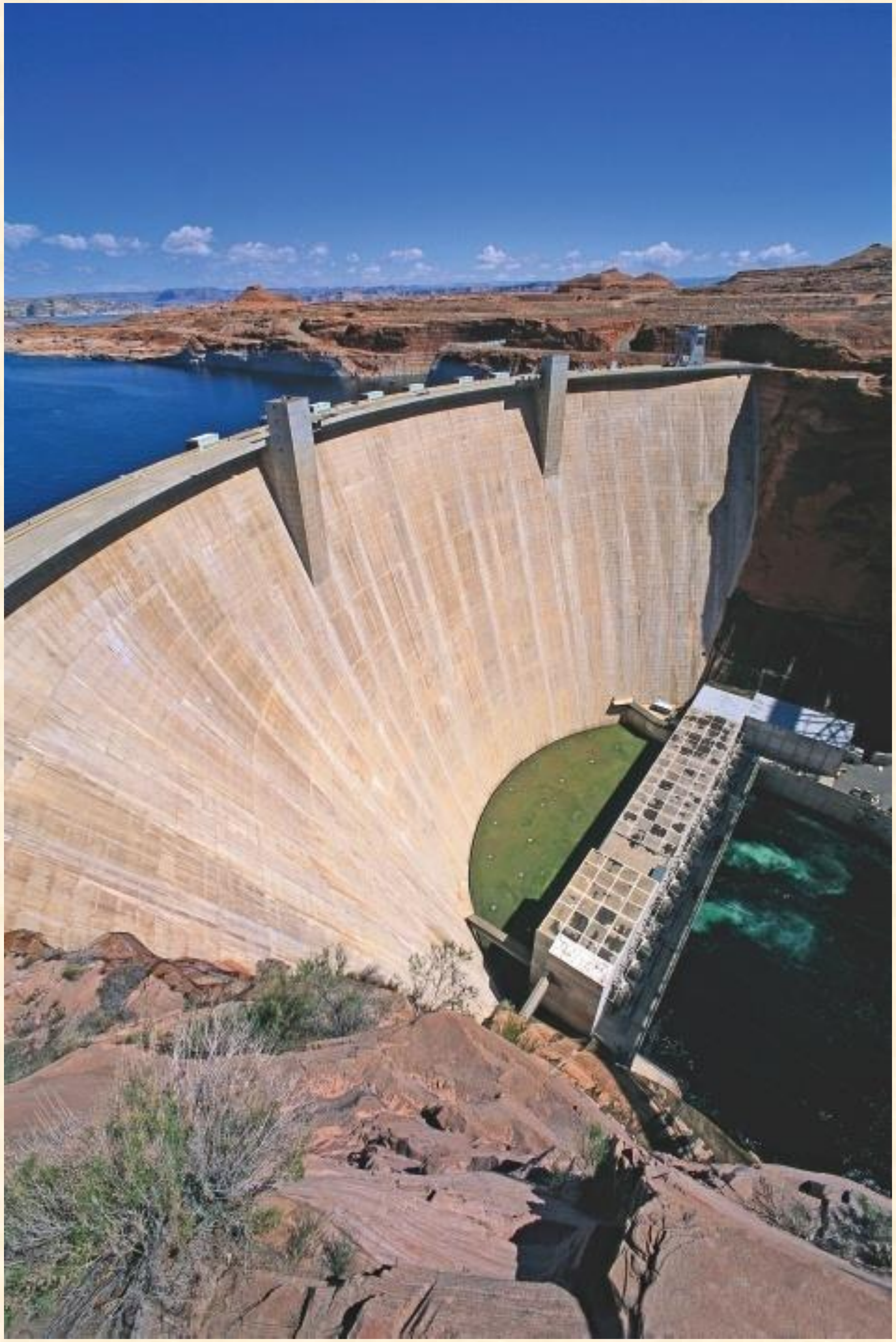


$$F_R = \sqrt{F_H^2 + F_V^2} \qquad \alpha = F_V/F_H$$

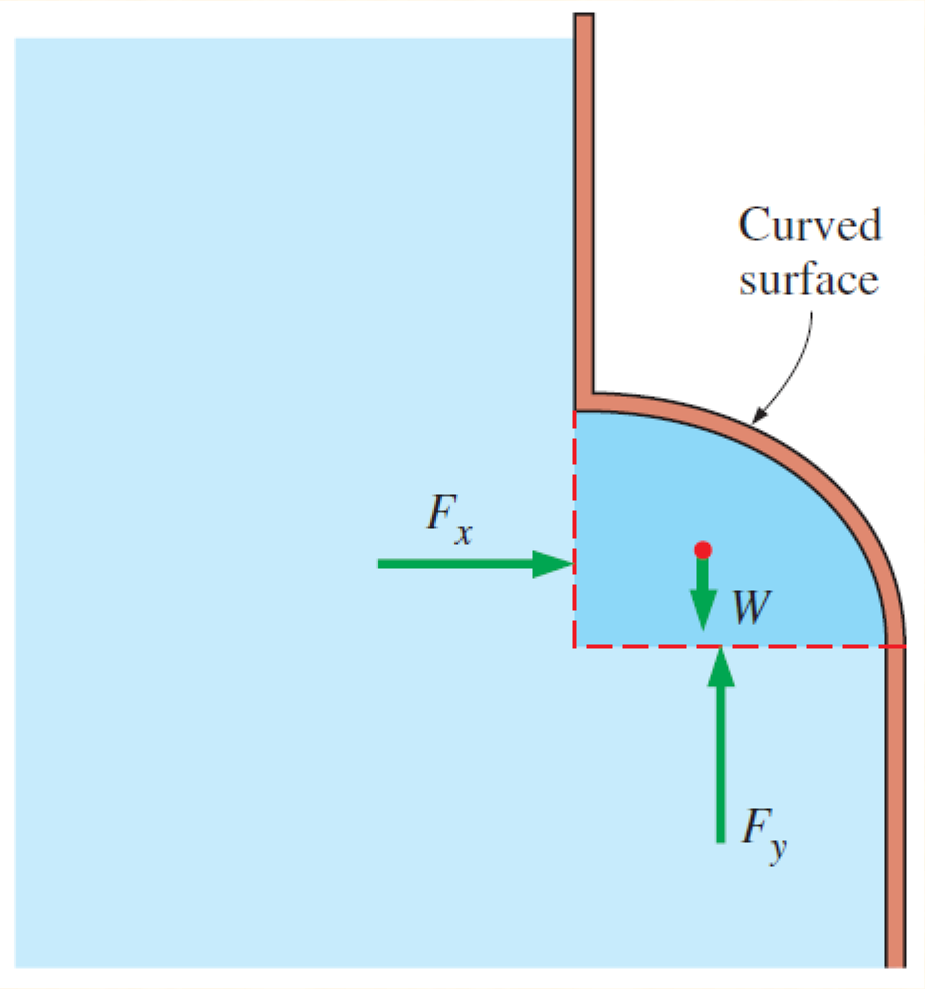
Determination of the hydrostatic force acting on a submerged curved surface.

Horizontal force component on curved surface: $F_H = F_x$

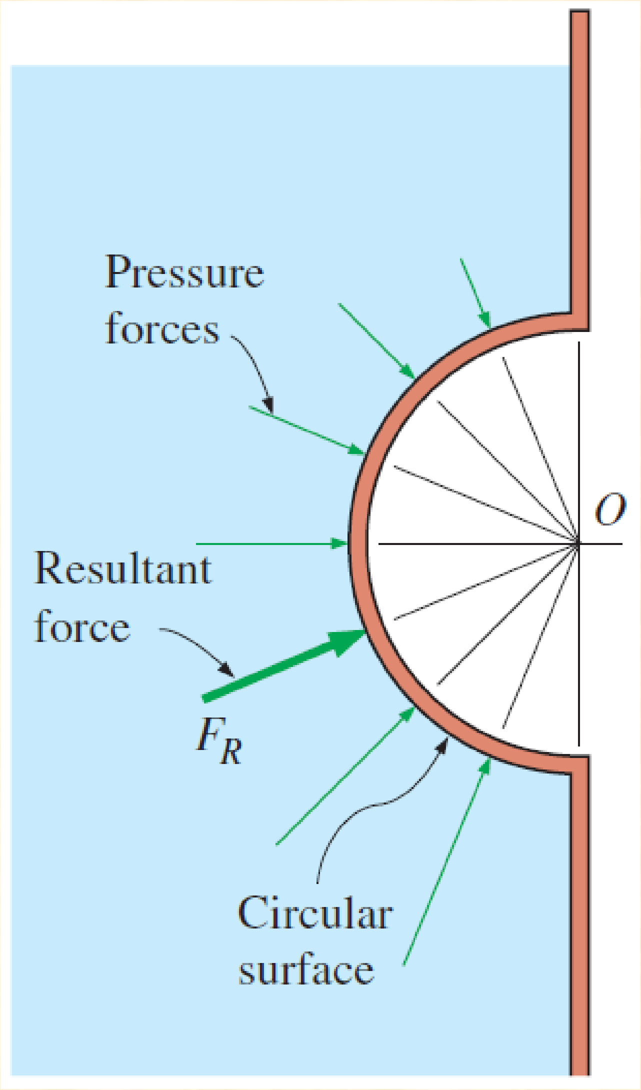
Vertical force component on curved surface: $F_V = F_y + W$



In many structures of practical application, the submerged surfaces are not flat, but curved as here at Glen Canyon Dam in Utah and Arizona.



When a curved surface is above the liquid, the weight of the liquid and the vertical component of the hydrostatic force act in the opposite directions.

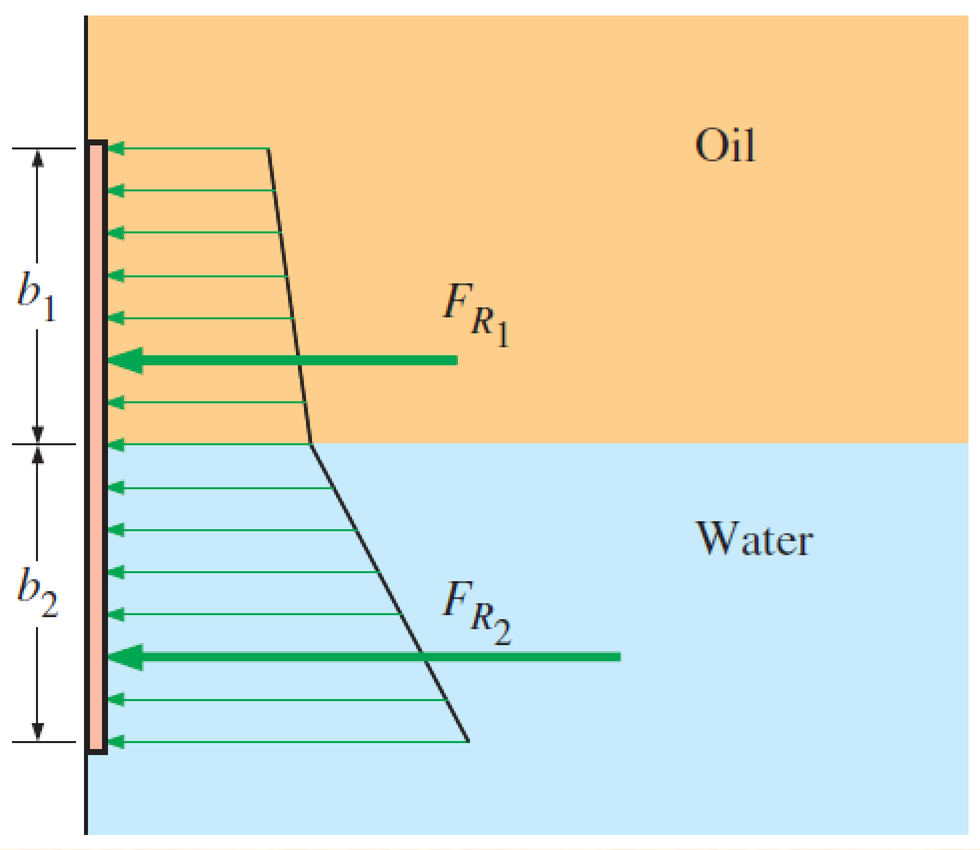


The hydrostatic force acting on a circular surface always passes through the center of the circle since the pressure forces are normal to the surface and they all pass through the center.

in a **multilayered fluid** of different densities can be determined by considering different parts of surfaces in different fluids as different surfaces, finding the force on each part, and then adding them using vector addition. For a plane surface, it can be expressed as

Plane surface in a multilayered fluid: $$F_R = \sum F_{R,i} = \sum P_{C,i} A_i$$

$$P_{C,i} = P_0 + \rho_i g h_{C,i}$$



The hydrostatic force on a surface submerged in a multilayered fluid can be determined by considering parts of the surface in different fluids as different surfaces.

EXAMPLE 3–9 A Gravity-Controlled Cylindrical Gate

A long solid cylinder of radius 0.8 m hinged at point *A* is used as an automatic gate, as shown in Fig. 3–40. When the water level reaches 5 m, the gate opens by turning about the hinge at point *A*. Determine (*a*) the hydrostatic force acting on the cylinder and its line of action when the gate opens and (*b*) the weight of the cylinder per m length of the cylinder.

SOLUTION The height of a water reservoir is controlled by a cylindrical gate hinged to the reservoir. The hydrostatic force on the cylinder and the weight of the cylinder per m length are to be determined.

Assumptions **1** Friction at the hinge is negligible. **2** Atmospheric pressure acts on both sides of the gate, and thus it cancels out.

Properties We take the density of water to be 1000 kg/m^3 throughout.

Analysis (*a*) We consider the free-body diagram of the liquid block enclosed by the circular surface of the cylinder and its vertical and horizontal projections. The hydrostatic forces acting on the vertical and horizontal plane surfaces as well as the weight of the liquid block are determined as

Horizontal force on vertical surface:

$$F_H = F_x = P_{avg}A = \rho g h_C A = \rho g(s + R/2)A$$

$$= (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(4.2 + 0.8/2 \text{ m})(0.8 \text{ m} \times 1 \text{ m})\left(\frac{1 \text{ kN}}{1000 \text{ kg·m/s}^2}\right)$$

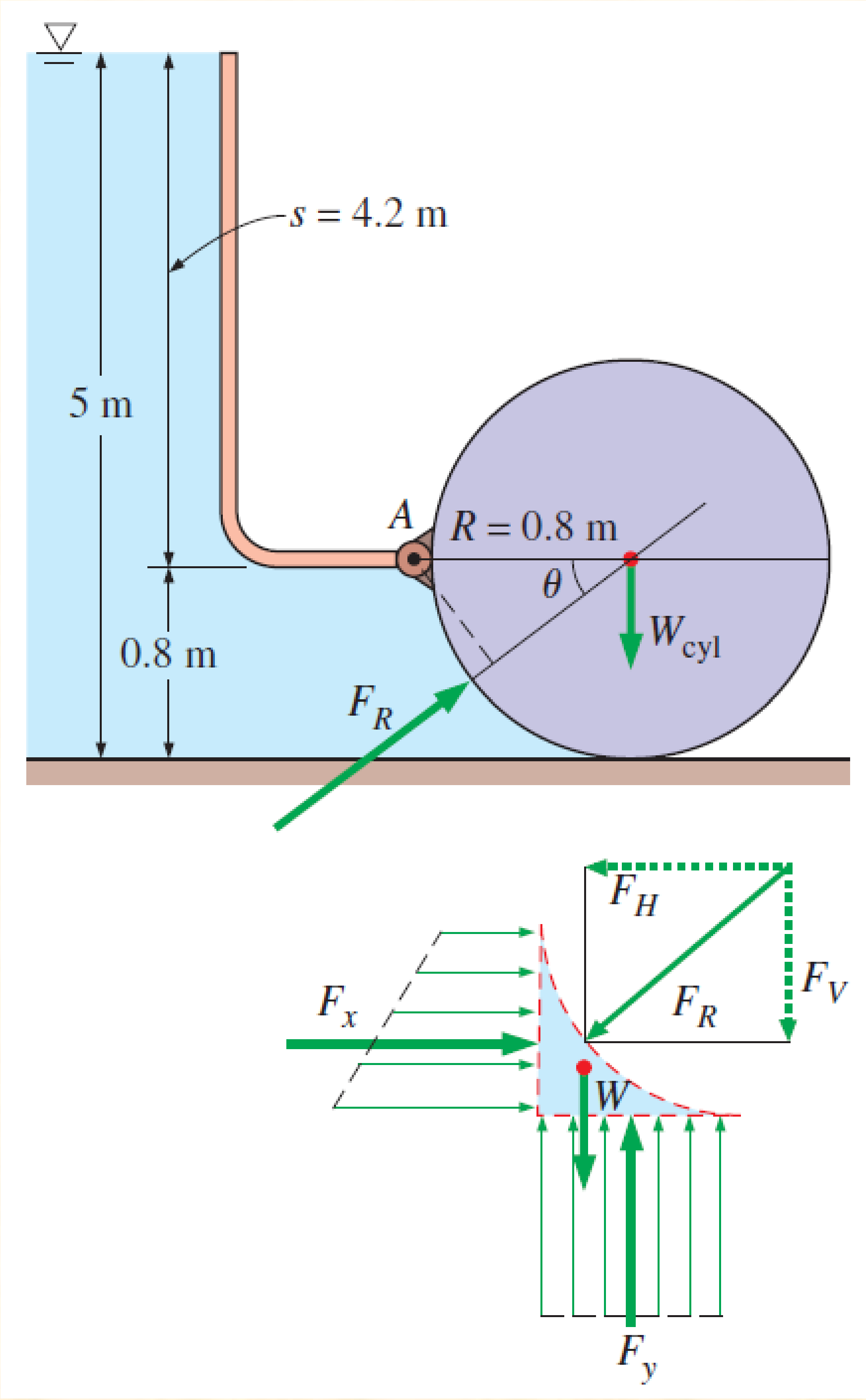
$$= 36.1 \text{ kN}$$

Vertical force on horizontal surface (upward):

$$F_y = P_{avg}A = \rho g h_C A = \rho g h_{bottom} A$$

$$= (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(5 \text{ m})(0.8 \text{ m} \times 1 \text{ m})\left(\frac{1 \text{ kN}}{1000 \text{ kg·m/s}^2}\right)$$

$$= 39.2 \text{ kN}$$



Schematic for Example 3–9 and the free-body diagram of the liquid underneath the cylinder.

Weight (downward) of fluid block for one m width into the page:

$$W = mg = \rho g V = \rho g(R^2 - \pi R^2/4)(1\ \text{m})$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.8\ \text{m})^2(1 - \pi/4)(1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 1.3\ \text{kN}$$

Therefore, the net upward vertical force is

$$F_V = F_y - W = 39.2 - 1.3 = 37.9\ \text{kN}$$

Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{36.1^2 + 37.9^2} = \mathbf{52.3\ kN}$$

$$\tan\theta = F_V/F_H = 37.9/36.1 = 1.05 \rightarrow \theta = 46.4°$$

Therefore, the magnitude of the hydrostatic force acting on the cylinder is 52.3 kN per m length of the cylinder, and its line of action passes through the center of the cylinder making an angle 46.4° with the horizontal.

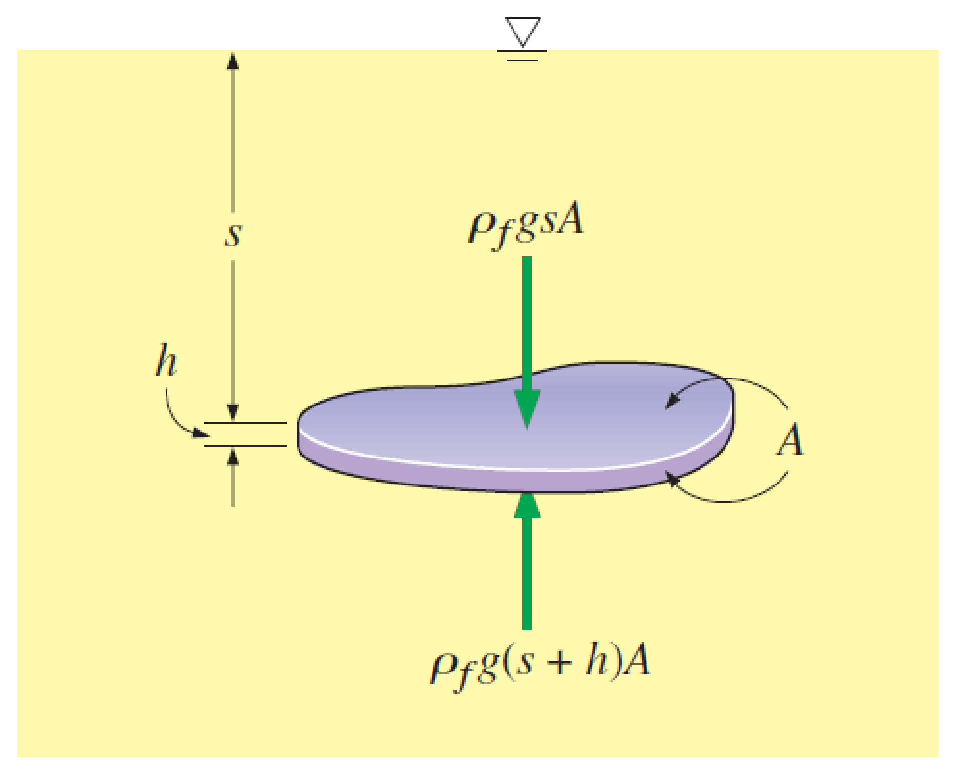
(*b*) When the water level is 5 m high, the gate is about to open and thus the reaction force at the bottom of the cylinder is zero. Then the forces other than those at the hinge acting on the cylinder are its weight, acting through the center, and the hydrostatic force exerted by water. Taking a moment about point *A* at the location of the hinge and equating it to zero gives

$$F_R R \sin\theta - W_{\text{cyl}} R = 0 \rightarrow W_{\text{cyl}} = F_R \sin\theta = (52.3\ \text{kN}) \sin 46.4° = \mathbf{37.9\ kN}$$

Discussion The weight of the cylinder per m length is determined to be 37.9 kN. It can be shown that this corresponds to a mass of 3863 kg per m length and to a density of 1921 kg/m^3 for the material of the cylinder.

3–6 ■ BUOYANCY AND STABILITY

Buoyant force: The upward force a fluid exerts on a body immersed in it. The buoyant force is caused by the increase of pressure with depth in a fluid.



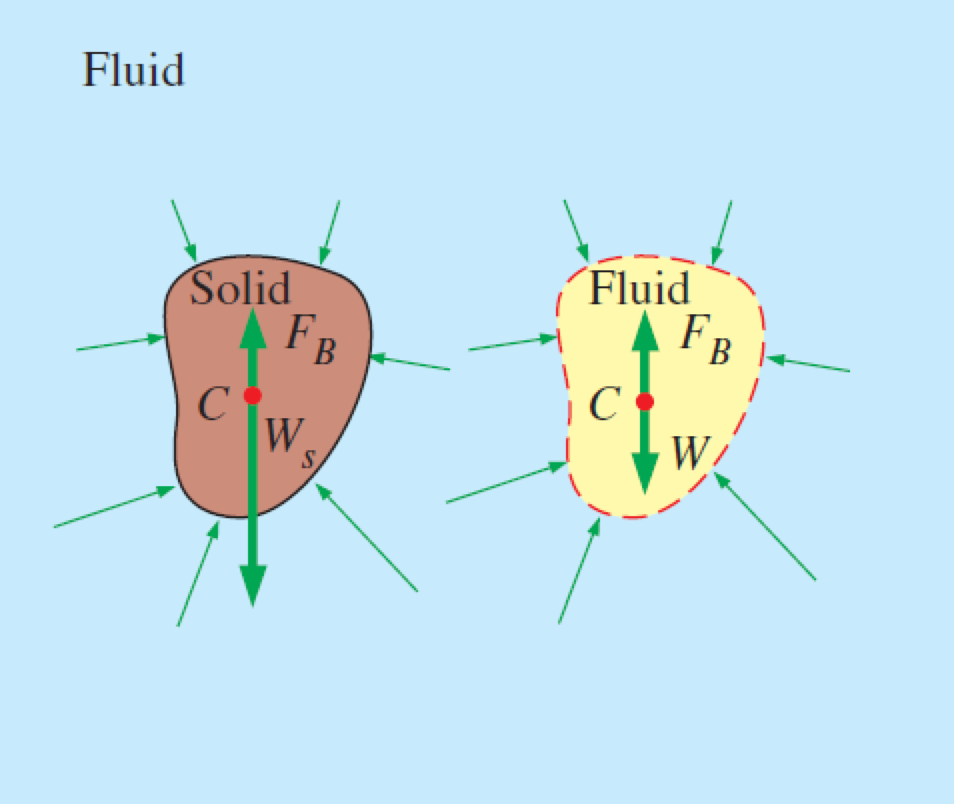
A flat plate of uniform thickness *h* submerged in a liquid parallel to the free surface.

The buoyant force acting on the plate is equal to the weight of the liquid displaced by the plate.

For a fluid with constant density, the buoyant force is independent of the distance of the body from the free surface.

It is also independent of the density of the solid body.

$$F_B = F_{\text{bottom}} - F_{\text{top}} = \rho_f g(s+h)A - \rho_f g s A = \rho_f g h A = \rho_f g V$$

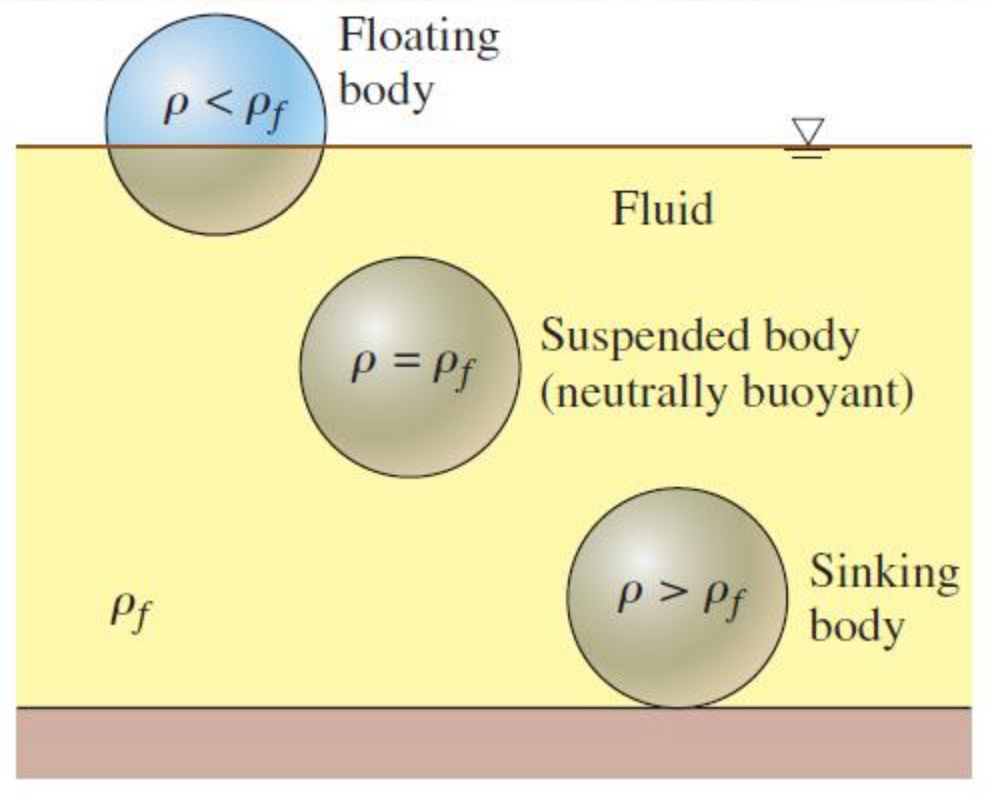


The buoyant forces acting on a solid body submerged in a fluid and on a fluid body of the same shape at the same depth are identical. The buoyant force F_B acts upward through the centroid *C* of the displaced volume and is equal in magnitude to the weight *W* of the displaced fluid, but is opposite in direction. For a solid of uniform density, its weight W_s also acts through the centroid, but its magnitude is not necessarily equal to that of the fluid it displaces. (Here $W_s > W$ and thus $W_s > F_B$; this solid body would sink.)

Archimedes' principle: The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume.

For *floating* bodies, the weight of the entire body must be equal to the buoyant force, which is the weight of the fluid whose volume is equal to the volume of the submerged portion of the floating body:

$$F_B = W \rightarrow \rho_f g V_{\text{sub}} = \rho_{\text{avg, body}} g V_{\text{total}} \rightarrow \frac{V_{\text{sub}}}{V_{\text{total}}} = \frac{\rho_{\text{avg, body}}}{\rho_f}$$

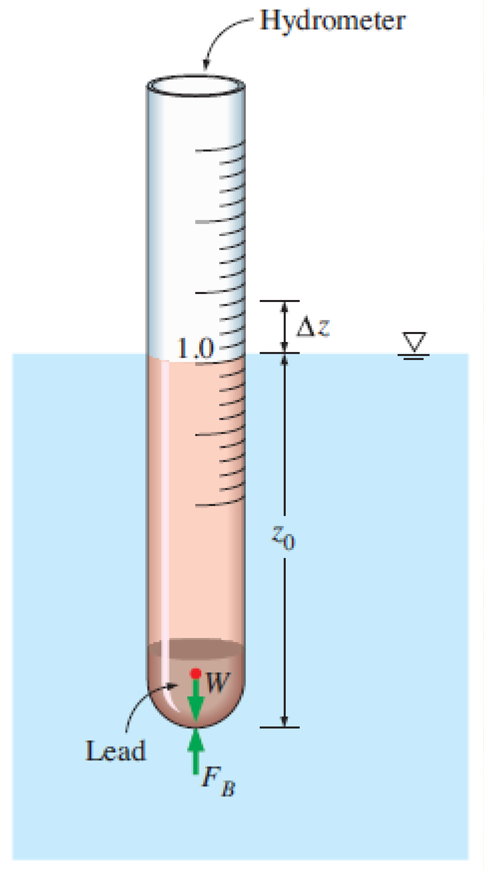


A solid body dropped into a fluid will sink, float, or remain at rest at any point in the fluid, depending on its average density relative to the density of the fluid.

The altitude of a hot air balloon is controlled by the temperature difference between the air inside and outside the balloon, since warm air is less dense than cold air. When the balloon is neither rising nor falling, the upward buoyant force exactly balances the downward weight.

EXAMPLE 3–10 **Measuring Specific Gravity by a Hydrometer**

If you have a seawater aquarium, you have probably used a small cylindrical glass tube with a lead-weight at its bottom to measure the salinity of the water by simply watching how deep the tube sinks. Such a device that floats in a vertical position and is used to measure the specific gravity of a liquid is called a *hydrometer* (Fig. 3–45). The top part of the hydrometer extends above the liquid surface, and the divisions on it allow one to read the specific gravity directly. The hydrometer is calibrated such that in pure water it reads exactly 1.0 at the air–water interface. (*a*) Obtain a relation for the specific gravity of a liquid as a function of distance Δz from the mark corresponding to pure water and (*b*) determine the mass of lead that must be poured into a 1-cm-diameter, 20-cm-long hydrometer if it is to float halfway (the 10-cm mark) in pure water.



SOLUTION The specific gravity of a liquid is to be measured by a hydrometer. A relation between specific gravity and the vertical distance from the reference level is to be obtained, and the amount of lead that needs to be added into the tube for a certain hydrometer is to be determined.

Assumptions **1** The weight of the glass tube is negligible relative to the weight of the lead added. **2** The curvature of the tube bottom is disregarded.

Properties We take the density of pure water to be 1000 kg/m^3.

Analysis (*a*) Noting that the hydrometer is in static equilibrium, the buoyant force F_B exerted by the liquid must always be equal to the weight W of the hydrometer. In pure water (subscript w), we let the vertical distance between the bottom of the hydrometer and the free surface of water be z_0. Setting $F_{B,w} = W$ in this case gives

$$W_{\text{hydro}} = F_{B,w} = \rho_w g V_{\text{sub}} = \rho_w g A z_0 \qquad (1)$$

where A is the cross-sectional area of the tube, and ρ_w is the density of pure water.

In a fluid lighter than water ($\rho_f < \rho_w$), the hydrometer will sink deeper, and the liquid level will be a distance of Δz above z_0. Again setting $F_B = W$ gives

$$W_{\text{hydro}} = F_{B,f} = \rho_f g V_{\text{sub}} = \rho_f g A(z_0 + \Delta z) \quad \textbf{(2)}$$

This relation is also valid for fluids heavier than water by taking Δz to be a negative quantity. Setting Eqs. (1) and (2) here equal to each other since the weight of the hydrometer is constant and rearranging gives

$$\rho_w g A z_0 = \rho_f g A(z_0 + \Delta z) \quad \rightarrow \quad \text{SG}_f = \frac{\rho_f}{\rho_w} = \frac{z_0}{z_0 + \Delta z}$$

which is the relation between the specific gravity of the fluid and Δz. Note that z_0 is constant for a given hydrometer and Δz is negative for fluids heavier than pure water.

(*b*) Disregarding the weight of the glass tube, the amount of lead that needs to be added to the tube is determined from the requirement that the weight of the lead be equal to the buoyant force. When the hydrometer is floating with half of it submerged in water, the buoyant force acting on it is

$$F_B = \rho_w g V_{\text{sub}}$$

Equating F_B to the weight of lead gives

$$W = mg = \rho_w g V_{\text{sub}}$$

Solving for m and substituting, the mass of lead is determined to be

$$m = \rho_w V_{\text{sub}} = \rho_w(\pi R^2 h_{\text{sub}}) = (1000 \text{ kg/m}^3)[\pi(0.005 \text{ m})^2(0.1 \text{ m})] = \mathbf{0.00785 \text{ kg}}$$

Discussion Note that if the hydrometer were required to sink only 5 cm in water, the required mass of lead would be one-half of this amount. Also, the assumption that the weight of the glass tube is negligible is questionable since the mass of lead is only 7.85 g.

EXAMPLE 3–11 **Weight Loss of an Object in Seawater**

A crane is used to lower weights into the sea (density = 1025 kg/m^3) for an underwater construction project (Fig. 3–46). Determine the tension in the rope of the crane due to a rectangular 0.4-m × 0.4-m × 3-m concrete block (density = 2300 kg/m^3) when it is (*a*) suspended in the air and (*b*) completely immersed in water.

SOLUTION A concrete block is lowered into the sea. The tension in the rope is to be determined before and after the block is in water.

Assumptions **1** The buoyant force in air is negligible. **2** The weight of the ropes is negligible.

Properties The densities are given to be 1025 kg/m^3 for seawater and 2300 kg/m^3 for concrete.

Analysis (*a*) Consider a free-body diagram of the concrete block. The forces acting on the concrete block in air are its weight and the upward pull action (tension) by the rope. These two forces must balance each other, and thus the tension in the rope must be equal to the weight of the block:

$$V = (0.4\ \text{m})(0.4\ \text{m})(3\ \text{m}) = 0.48\ \text{m}^3$$

$$F_{T,\,\text{air}} = W = \rho_{\text{concrete}}\, gV$$

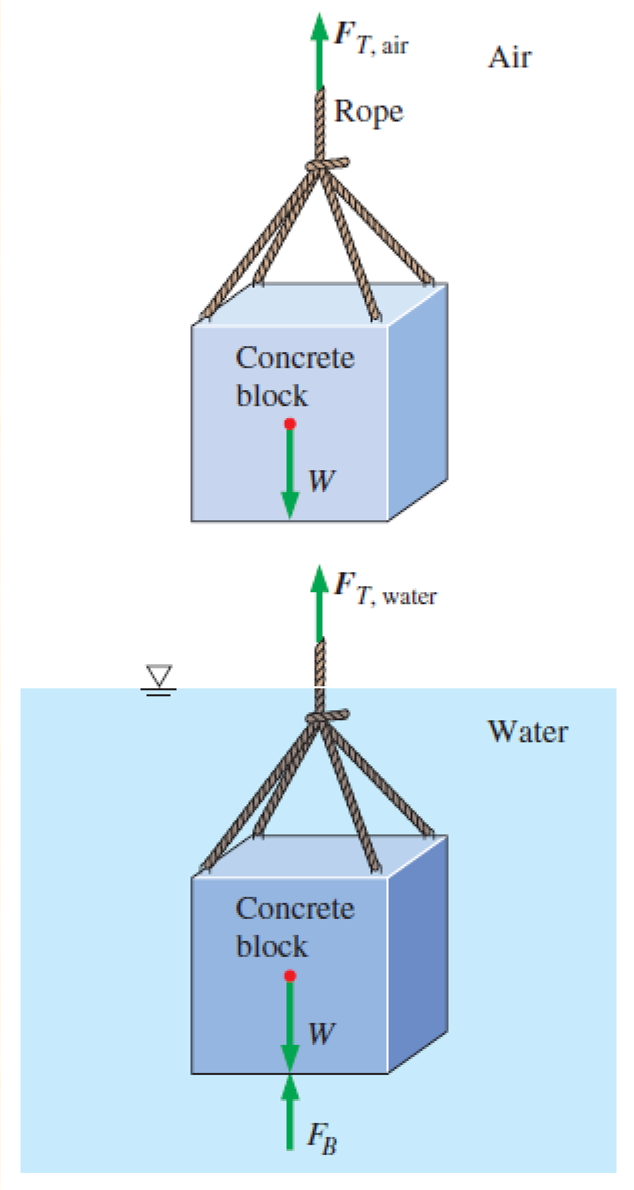
$$= (2300\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.48\ \text{m}^3)\left(\frac{1\ \text{kN}}{1000\ \text{kg·m/s}^2}\right) = \mathbf{10.8\ kN}$$

(*b*) When the block is immersed in water, there is the additional force of buoyancy acting upward. The force balance in this case gives

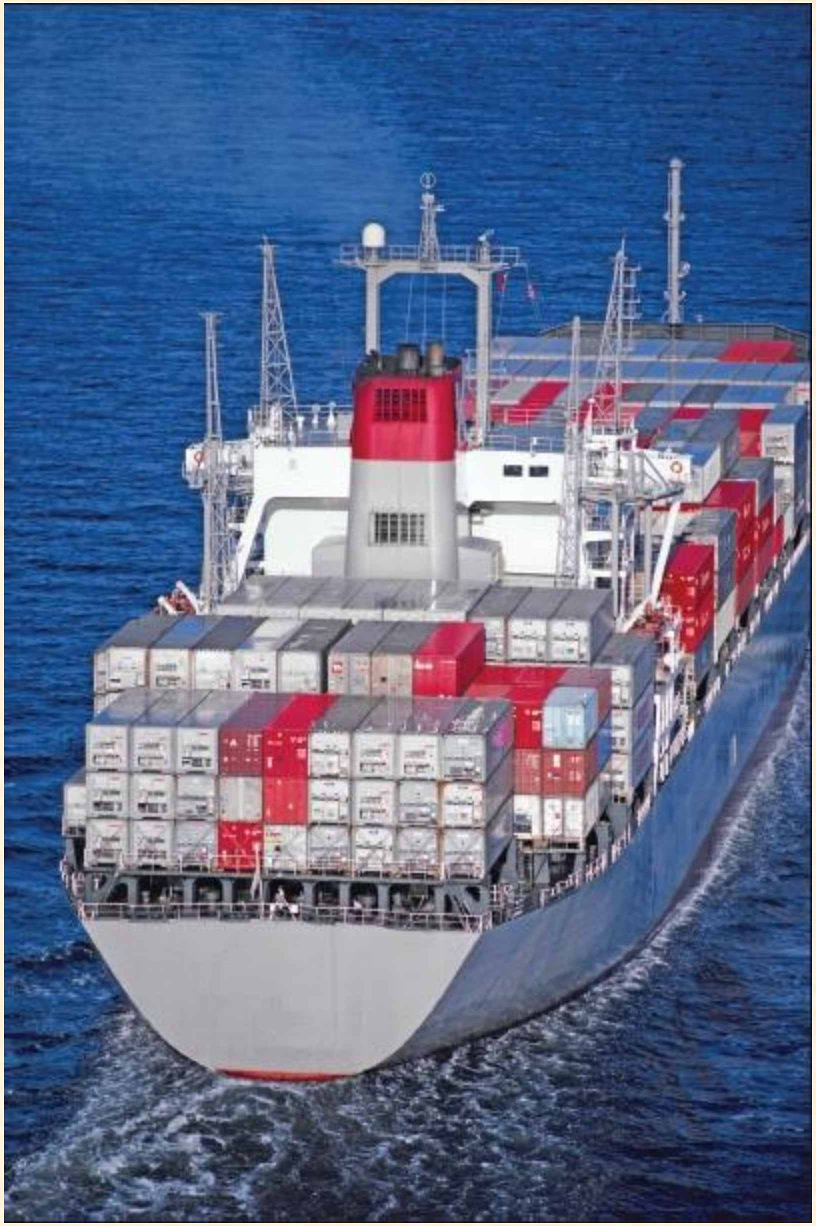
$$F_B = \rho_f\, gV = (1025\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.48\ \text{m}^3)\left(\frac{1\ \text{kN}}{1000\ \text{kg·m/s}^2}\right) = 4.8\ \text{kN}$$

$$F_{T,\,\text{water}} = W - F_B = 10.8 - 4.8 = \mathbf{6.0\ kN}$$

Discussion Note that the weight of the concrete block, and thus the tension of the rope, decreases by (10.8 − 6.0)/10.8 = 55 percent in water.

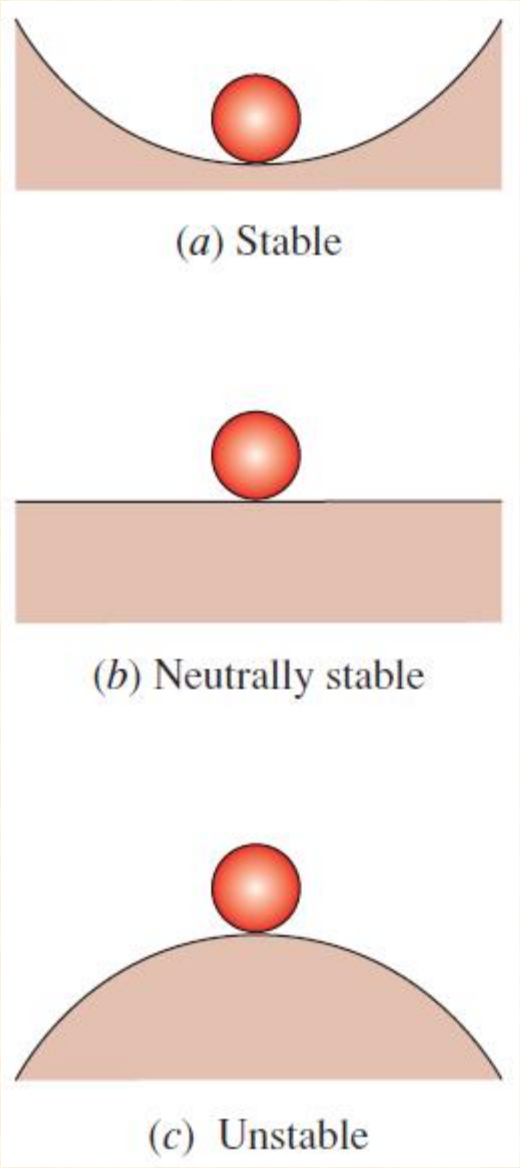


Stability of Immersed and Floating Bodies

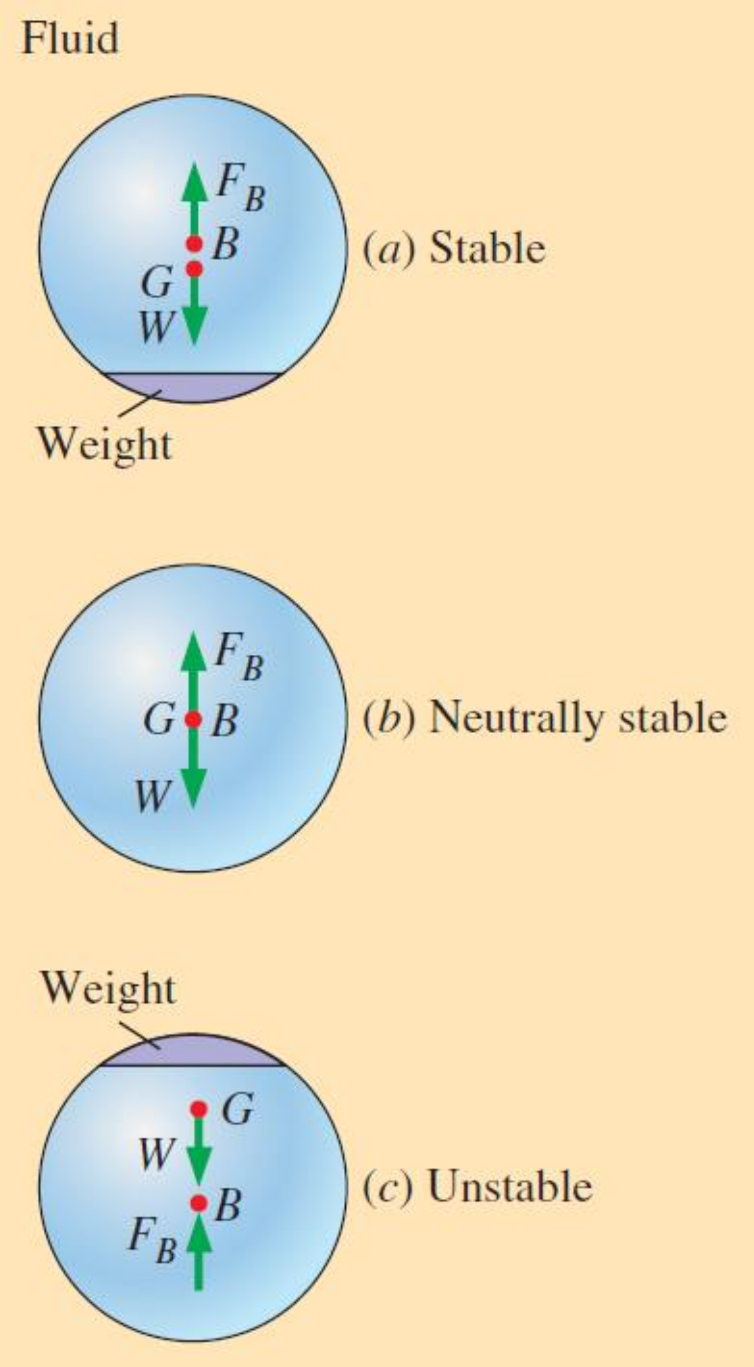


For floating bodies such as ships, stability is an important consideration for safety.

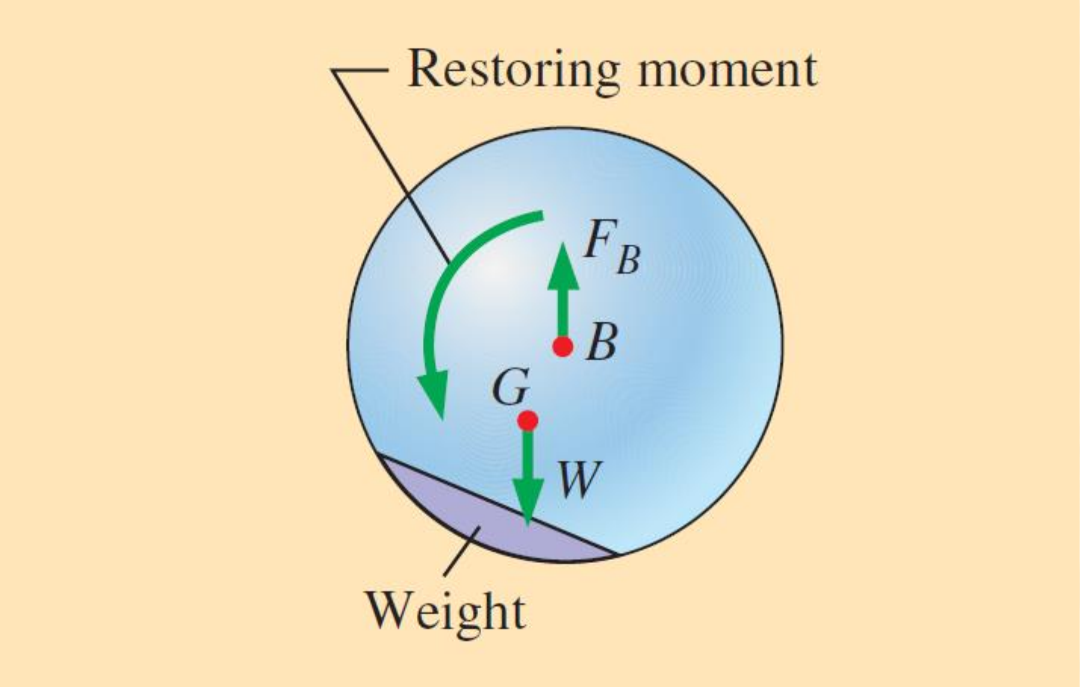
Stability is easily understood by analyzing a ball on the floor.



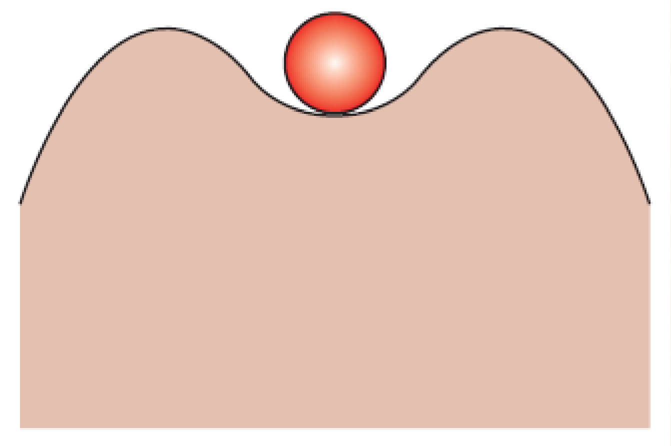
A floating body possesses vertical stability, while an immersed neutrally buoyant body is neutrally stable since it does not return to its original position after a disturbance.



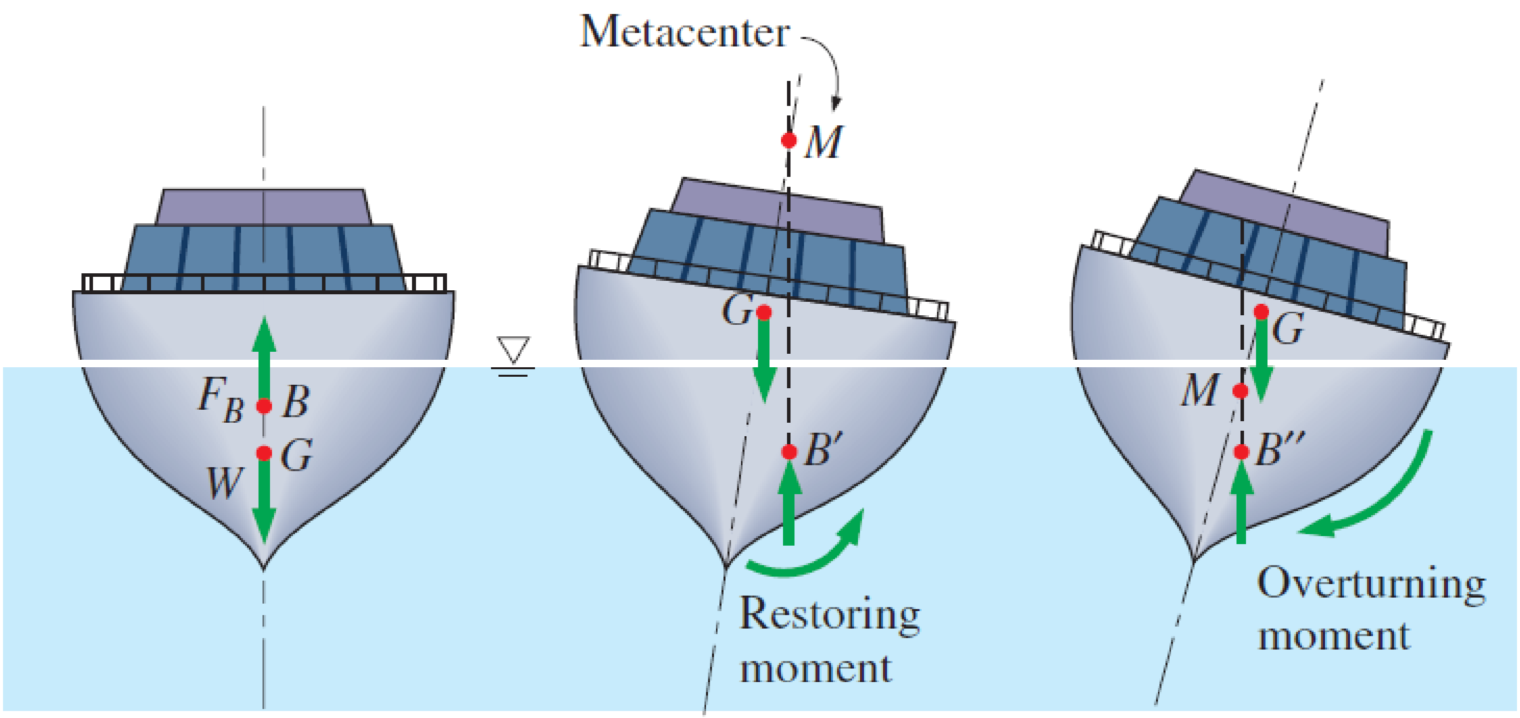
An immersed neutrally buoyant body is (*a*) stable if the center of gravity *G* is directly below the center of buoyancy *B* of the body, (*b*) neutrally stable if *G* and *B* are coincident, and (*c*) unstable if *G* is directly above *B*.



When the center of gravity *G* of an immersed neutrally buoyant body is not vertically aligned with the center of buoyancy *B* of the body, it is not in an equilibrium state and would rotate to its stable state, even without any disturbance.



A ball in a trough between two hills is stable for small disturbances, but unstable for large disturbances.



(*a*) Stable (*b*) Stable (*c*) Unstable

A floating body is *stable* if the body is bottom-heavy and thus the center of gravity *G* is below the centroid *B* of the body, or if the metacenter *M* is above point *G*. However, the body is *unstable* if point *M* is below point *G*.

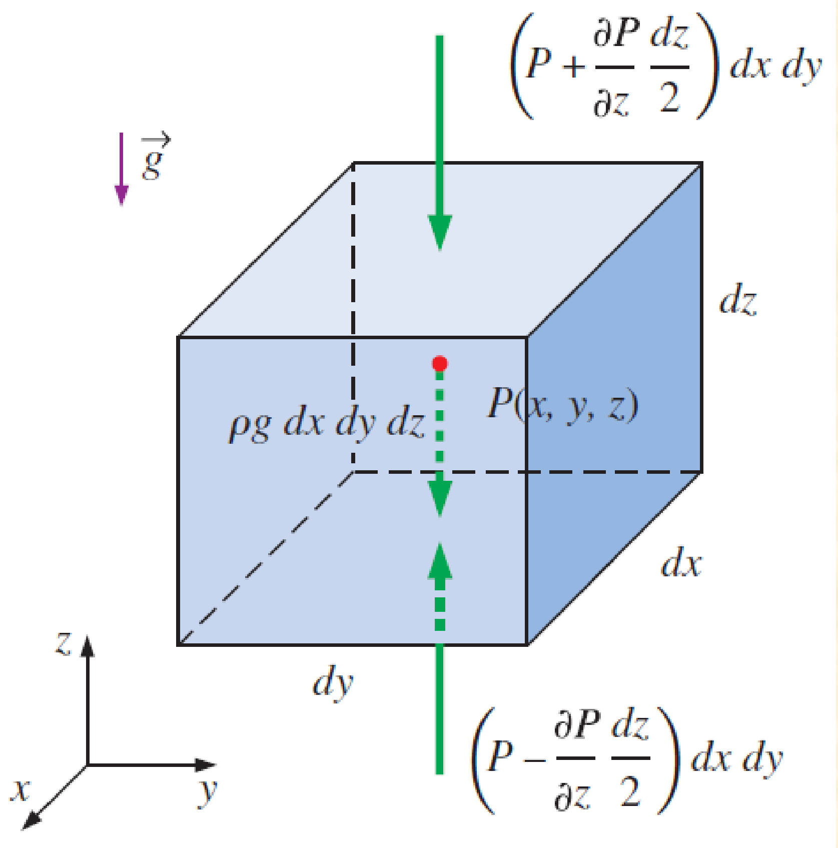
Metacentric height *GM*: The distance between the center of gravity *G* and the metacenter *M*—the intersection point of the lines of action of the buoyant force through the body before and after rotation.

The length of the metacentric height *GM* above *G* is a measure of the stability: the larger it is, the more stable is the floating body.

3–7 ■ FLUIDS IN RIGID-BODY MOTION

Pressure at a given point has the same magnitude in all directions, and thus it is a *scalar* function.

In this section we obtain relations for the variation of pressure in fluids moving like a solid body with or without acceleration in the absence of any shear stresses (i.e., no motion between fluid layers relative to each other).



$$\delta \vec{F} = \delta m \cdot \vec{a}$$

$$\delta m = \rho\, dV = \rho\, dx\, dy\, dz$$

$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z}\, dx\, dy\, dz$$

$$\delta F_{S,x} = -\frac{\partial P}{\partial x}\, dx\, dy\, dz \qquad \text{and} \qquad \delta F_{S,y} = -\frac{\partial P}{\partial y}\, dx\, dy\, dz$$

$$\delta\vec{F}_S = \delta F_{S,x}\vec{i} + \delta F_{S,y}\vec{j} + \delta F_{S,z}\vec{k}$$
$$= -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right) dx\,dy\,dz = -\vec{\nabla}P\,dx\,dy\,dz$$

$$\vec{\nabla}P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$$

$$\delta\vec{F}_{B,z} = -g\delta m\vec{k} = -\rho g\,dx\,dy\,dz\vec{k}$$

$$\delta\vec{F} = \delta\vec{F}_S + \delta\vec{F}_B = -(\vec{\nabla}P + \rho g\vec{k})\,dx\,dy\,dz$$

Rigid-body motion of fluids: $\vec{\nabla}P + \rho g\vec{k} = -\rho\vec{a}$

$$\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} + \rho g\vec{k} = -\rho(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

Accelerating fluids: $\frac{\partial P}{\partial x} = -\rho a_x,\quad \frac{\partial P}{\partial y} = -\rho a_y,\quad$ and $\quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$

Special Case 1: Fluids at Rest

For fluids at rest or moving on a straight path at constant velocity, all components of acceleration are zero, and the relations reduce to

Fluids at rest: $$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{dP}{dz} = -\rho g$$

The pressure remains constant in any horizontal direction (*P* is independent of *x* and *y*) and varies only in the vertical direction as a result of gravity [and thus $P = P(z)$]. These relations are applicable for both compressible and incompressible fluids.

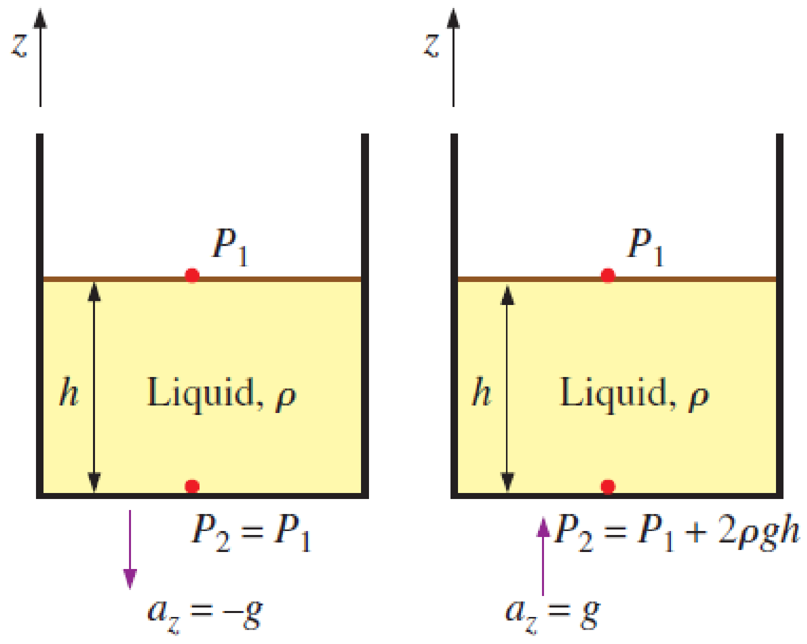
A glass of water at rest is a special case of a fluid in rigid-body motion. If the glass of water were moving at constant velocity in any direction, the hydrostatic equations would still apply.

Special Case 2: Free Fall of a Fluid Body

A freely falling body accelerates under the influence of gravity. When the air resistance is negligible, the acceleration of the body equals the gravitational acceleration, and acceleration in any horizontal direction is zero. Therefore, $a_x = a_y = 0$ and $a_z = -g$.

$$\textit{Free-falling fluids:} \qquad \frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0 \quad \rightarrow \quad P = \text{constant}$$

In a frame of reference moving with the fluid, it behaves like it is in an environment with zero gravity. Also, the gage pressure in a drop of liquid in free fall is zero throughout.



(*a*) Free fall of a liquid

(*b*) Upward acceleration of a liquid with $a_z = +g$

The effect of acceleration on the pressure of a liquid during free fall and upward acceleration.

Acceleration on a Straight Path

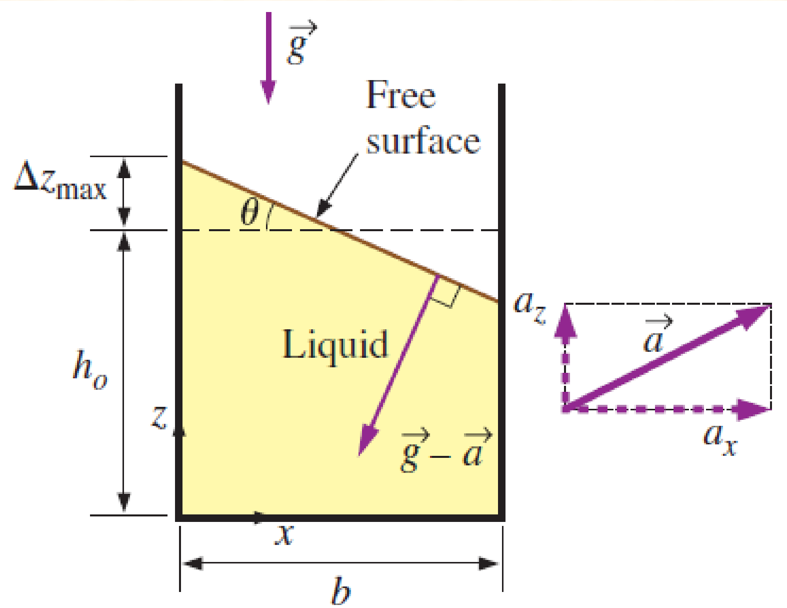
$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

$$P = P(x, z) \qquad dP = (\partial P/\partial x)\,dx + (\partial P/\partial z)\,dz$$

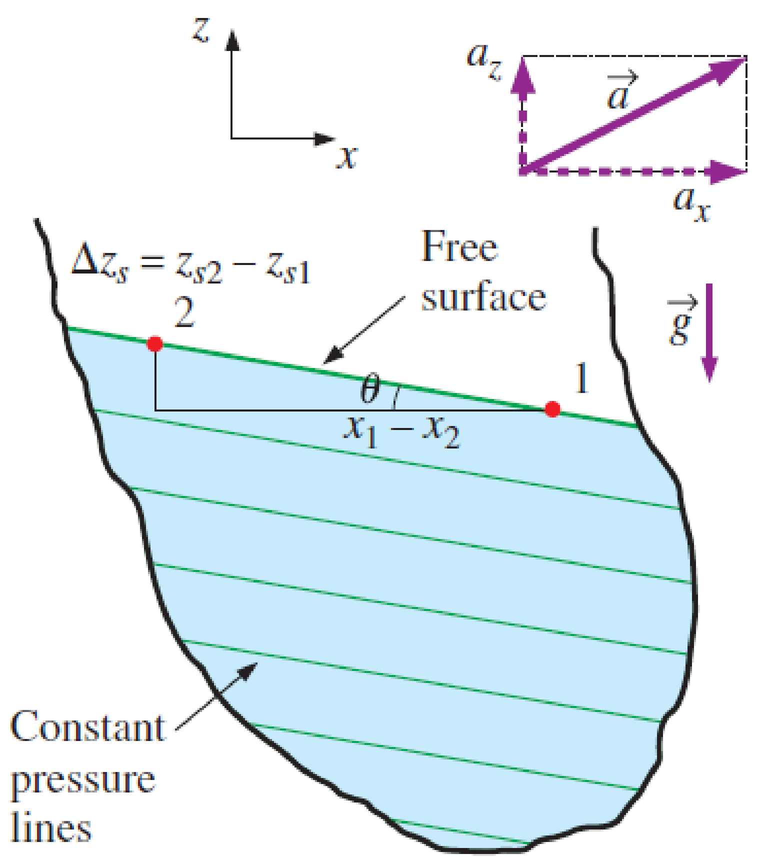
$$dP = -\rho a_x\,dx - \rho(g + a_z)\,dz$$

$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$

Pressure variation: $P = P_0 - \rho a_x x - \rho(g + a_z)z$



Rigid-body motion of a liquid in a linearly accelerating tank.



Lines of constant pressure (which are the projections of the surfaces of constant pressure on the *xz*-plane) in a linearly accelerating liquid. Also shown is the vertical rise.

Vertical rise of surface:

$$\Delta z_s = z_{s2} - z_{s1} = -\frac{a_x}{g + a_z}(x_2 - x_1)$$

Surfaces of constant pressure:

$$\frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = \text{constant}$$

Slope of isobars:

$$\text{Slope} = \frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = -\tan\theta$$

EXAMPLE 3–12 Overflow from a Water Tank During Acceleration

An 80-cm-high fish tank of cross section 2 m × 0.6 m that is partially filled with water is to be transported on the back of a truck (Fig. 3–58). The truck accelerates from 0 to 90 km/h in 10 s. If it is desired that no water spills during acceleration, determine the allowable initial water height in the tank. Would you recommend the tank to be aligned with the long or short side parallel to the direction of motion?

SOLUTION A fish tank is to be transported on a truck. The allowable water height to avoid spill of water during acceleration and the proper orientation are to be determined.

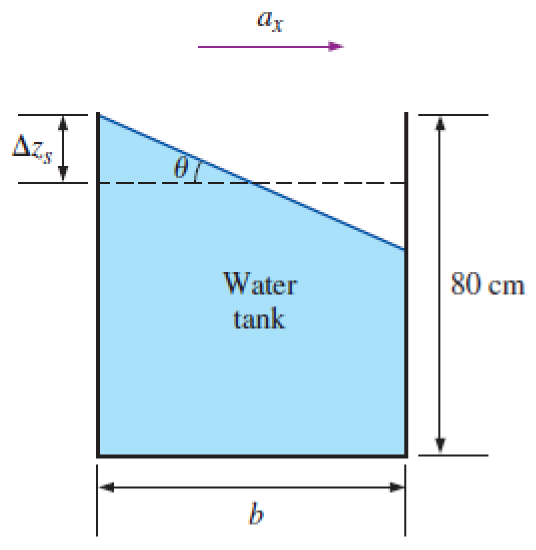
Assumptions **1** The road is horizontal during acceleration so that acceleration has no vertical component ($a_z = 0$). **2** Effects of splashing, braking, shifting gears, driving over bumps, climbing hills, etc., are assumed to be secondary and are not considered. **3** The acceleration remains constant.

Analysis We take the *x*-axis to be the direction of motion, the *z*-axis to be the upward vertical direction, and the origin to be the lower left corner of the tank. Noting that the truck goes from 0 to 90 km/h in 10 s, the acceleration of the truck is

$$a_x = \frac{\Delta V}{\Delta t} = \frac{(90 - 0)\ \text{km/h}}{10\ \text{s}}\left(\frac{1\ \text{m/s}}{3.6\ \text{km/h}}\right) = 2.5\ \text{m/s}^2$$

The tangent of the angle the free surface makes with the horizontal is

$$\tan\theta = \frac{a_x}{g + a_z} = \frac{2.5}{9.81 + 0} = 0.255 \qquad (\text{and thus } \theta = 14.3°)$$



The maximum vertical rise of the free surface occurs at the back of the tank, and the vertical midplane experiences no rise or drop during acceleration since it is a plane of symmetry. Then the vertical rise at the back of the tank relative to the midplane for the two possible orientations becomes

Case 1: The long side is parallel to the direction of motion:

$$\Delta z_{s1} = (b_1/2)\tan\theta = [(2\text{ m})/2] \times 0.255 = 0.255\text{ m} = \mathbf{25.5\text{ cm}}$$

Case 2: The short side is parallel to the direction of motion:

$$\Delta z_{s2} = (b_2/2)\tan\theta = [(0.6\text{ m})/2] \times 0.255 = 0.076\text{ m} = \mathbf{7.6\text{ cm}}$$

Therefore, assuming tipping is not a problem, **the tank should definitely be oriented such that its short side is parallel to the direction of motion.** Emptying the tank such that its free surface level drops just 7.6 cm in this case will be adequate to avoid spilling during acceleration.

Discussion Note that the orientation of the tank is important in controlling the vertical rise. Also, the analysis is valid for any fluid with constant density, not just water, since we used no information that pertains to water in the solution.

Rotation in a Cylindrical Container

Consider a vertical cylindrical container partially filled with a liquid. The container is now rotated about its axis at a constant angular velocity of ω. After initial transients, the liquid will move as a rigid body together with the container. There is no deformation, and thus there can be no shear stress, and every fluid particle in the container moves with the same angular velocity.

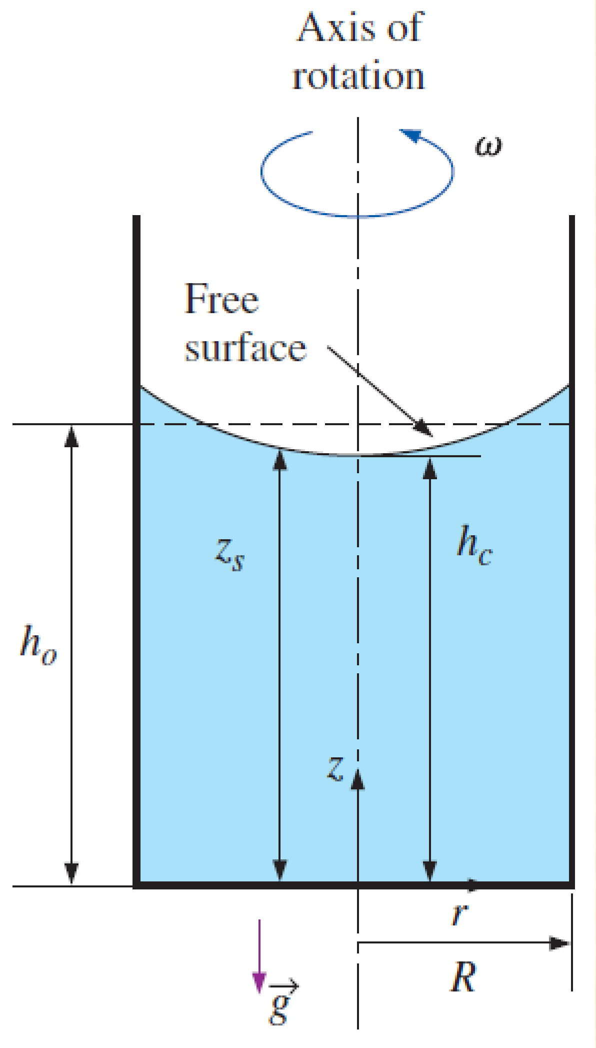
$$\frac{\partial P}{\partial r} = \rho r \omega^2, \qquad \frac{\partial P}{\partial \theta} = 0, \qquad \text{and} \qquad \frac{\partial P}{\partial z} = -\rho g$$

$$P = P(r, z) \qquad dP = (\partial P/\partial r)dr + (\partial P/\partial z)dz$$

$$dP = \rho r \omega^2 \, dr - \rho g \, dz$$

$$\frac{dz_{\text{isobar}}}{dr} = \frac{r\omega^2}{g}$$

Surfaces of constant pressure: $z_{\text{isobar}} = \frac{\omega^2}{2g} r^2 + C_1$



Rigid-body motion of a liquid in a rotating vertical cylindrical container.

$$z_s = \frac{\omega^2}{2g} r^2 + h_c$$

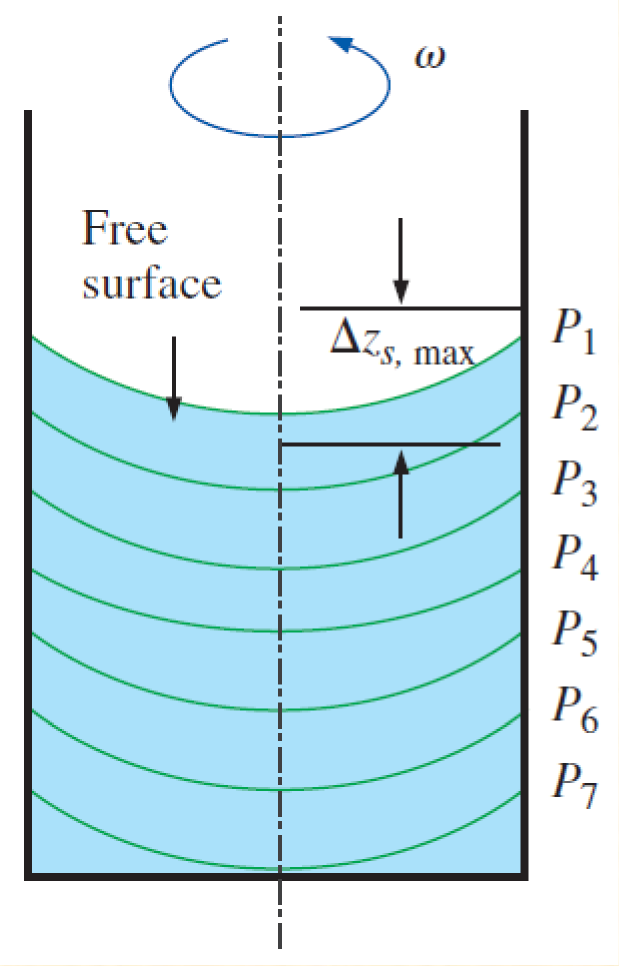
$$V = \int_{r=0}^{R} 2\pi z_s r \, dr = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^2 + h_c \right) r \, dr = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c \right)$$

$$V = \pi R^2 h_0$$

$$h_c = h_0 - \frac{\omega^2 R^2}{4g}$$

Free surface: $z_s = h_0 - \frac{\omega^2}{4g} (R^2 - 2r^2)$

The 6-meter spinning liquid-mercury mirror of the Large Zenith Telescope located near Vancouver, British Columbia.



Surfaces of constant pressure in a rotating liquid.

Maximum height difference: $\Delta z_{s,\max} = z_s(R) - z_s(0) = \dfrac{\omega^2}{2g} R^2$

$dP = \rho r \omega^2\, dr - \rho g\, dz$ $\quad P_2 - P_1 = \dfrac{\rho \omega^2}{2}(r_2^2 - r_1^2) - \rho g(z_2 - z_1)$

Pressure variation: $P = P_0 + \dfrac{\rho \omega^2}{2} r^2 - \rho g z$

Note that at a fixed radius, the pressure varies hydrostatically in the vertical direction, as in a fluid at rest.

For a fixed vertical distance *z*, the pressure varies with the square of the radial distance *r*, increasing from the centerline toward the outer edge.

In any horizontal plane, the pressure difference between the center and edge of the container of radius *R* is

$$\Delta P = \rho \omega^2 R^2 / 2$$

EXAMPLE 3–13 **Rising of a Liquid During Rotation**

A 20-cm-diameter, 60-cm-high vertical cylindrical container, shown in Fig. 3–62, is partially filled with 50-cm-high liquid whose density is 850 kg/m^3. Now the cylinder is rotated at a constant speed. Determine the rotational speed at which the liquid will start spilling from the edges of the container.

SOLUTION A vertical cylindrical container partially filled with a liquid is rotated. The angular speed at which the liquid will start spilling is to be determined.

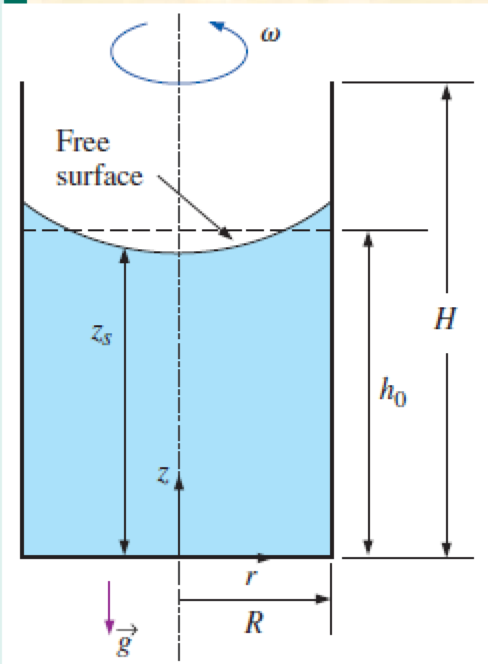
Assumptions **1** The increase in the rotational speed is very slow so that the liquid in the container always acts as a rigid body. **2** The bottom surface of the container remains covered with liquid during rotation (no dry spots).

Analysis Taking the center of the bottom surface of the rotating vertical cylinder as the origin ($r = 0$, $z = 0$), the equation for the free surface of the liquid is given as

$$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

Then the vertical height of the liquid at the edge of the container where $r = R$ becomes

$$z_s(R) = h_0 + \frac{\omega^2 R^2}{4g}$$



where $h_0 = 0.5$ m is the original height of the liquid before rotation. Just before the liquid starts spilling, the height of the liquid at the edge of the container equals the height of the container, and thus $z_s(R) = H = 0.6$ m. Solving the last equation for ω and substituting, the maximum rotational speed of the container is determined to be

$$\omega = \sqrt{\frac{4g(H - h_0)}{R^2}} = \sqrt{\frac{4(9.81 \text{ m/s}^2)[(0.6 - 0.5) \text{ m}]}{(0.1 \text{ m})^2}} = \mathbf{19.8 \text{ rad/s}}$$

Noting that one complete revolution corresponds to 2π rad, the rotational speed of the container can also be expressed in terms of revolutions per minute (rpm) as

$$\dot{n} = \frac{\omega}{2\pi} = \frac{19.8 \text{ rad/s}}{2\pi \text{ rad/rev}}\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = \mathbf{189 \text{ rpm}}$$

Therefore, the rotational speed of this container should be limited to 189 rpm to avoid any spill of liquid as a result of the centrifugal effect.

Discussion Note that the analysis is valid for any liquid since the result is independent of density or any other fluid property. We should also verify that our assumption of no dry spots is valid. The liquid height at the center is

$$z_s(0) = h_0 - \frac{\omega^2 R^2}{4g} = 0.4 \text{ m}$$

Since $z_s(0)$ is positive, our assumption is validated.

Summary

- Pressure
- Pressure Measurement Devices
- Introduction to Fluid Statics
- Hydrostatic Forces on Submerged Plane Surfaces
- Hydrostatic Forces on Submerged Curved Surfaces
- Buoyancy and Stability
- Fluids in Rigid-Body Motion