Chapter 4

FLUID KINEMATICS

Objectives

- Understand the role of the material derivative in transforming between Lagrangian and Eulerian descriptions
- Distinguish between various types of flow visualizations and methods of plotting the characteristics of a fluid flow
- Appreciate the many ways that fluids move and deform
- Distinguish between rotational and irrotational regions of flow based on the flow property vorticity
- Understand the usefulness of the Reynolds transport theorem

4–1 ■ LAGRANGIAN AND EULERIAN DESCRIPTIONS

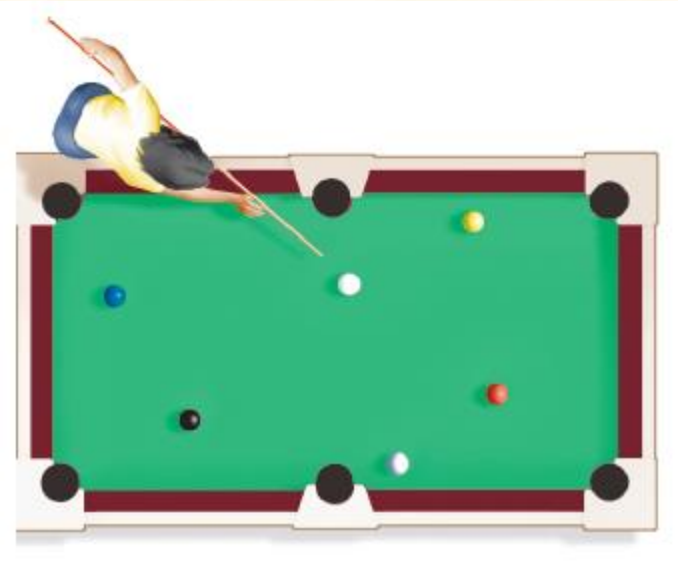
Kinematics: The study of motion.

Fluid kinematics: The study of how fluids flow and how to describe fluid motion.

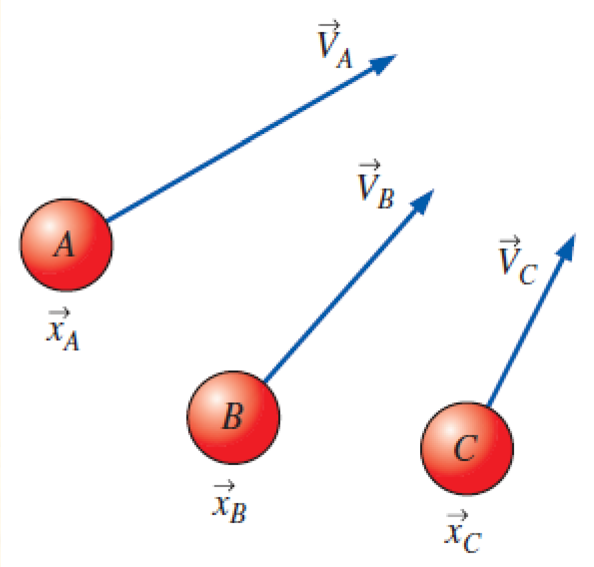
There are two distinct ways to describe motion: Lagrangian and Eulerian

Lagrangian description: To follow the path of individual objects.

This method requires us to track the position and velocity of each individual fluid parcel (**fluid particle**) and take to be a parcel of fixed identity.



With a small number of objects, such as billiard balls on a pool table, individual objects can be tracked.



In the Lagrangian description, one must keep track of the position and velocity of individual particles.

- A more common method is **Eulerian description** of fluid motion.
- In the Eulerian description of fluid flow, a finite volume called a **flow domain** or **control volume** is defined, through which fluid flows in and out.
- Instead of tracking individual fluid particles, we define **field variables**, functions of space and time, within the control volume.
- The field variable at a particular location at a particular time is the value of the variable for whichever fluid particle happens to occupy that location at that time.
- For example, the **pressure field** is a **scalar field variable**. We define the **velocity field** as a **vector field variable**.

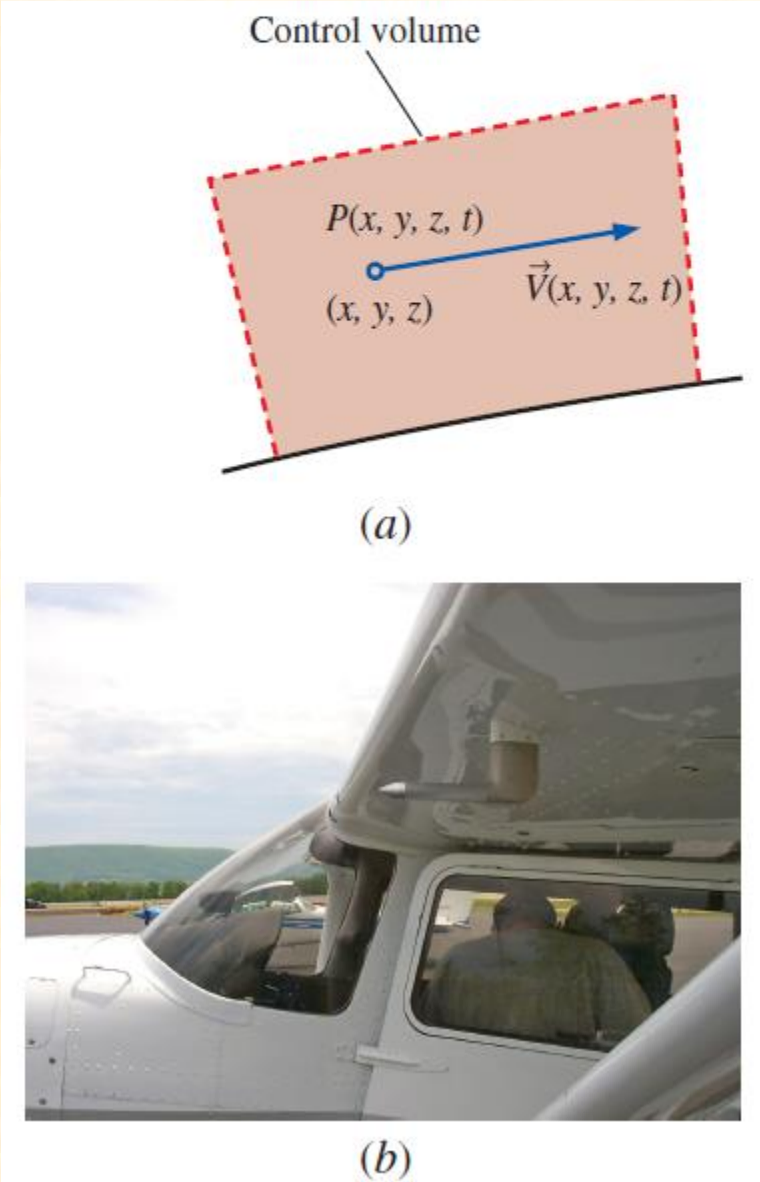
Pressure field: $$P = P(x, y, z, t)$$

Velocity field: $$\vec{V} = \vec{V}(x, y, z, t)$$

Acceleration field: $$\vec{a} = \vec{a}(x, y, z, t)$$

Collectively, these (and other) field variables define the **flow field**. The velocity field can be expanded in Cartesian coordinates as

$$\vec{V} = (u, v, w) = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$



- In the Eulerian description we don’t really care what happens to individual fluid particles; rather we are concerned with the pressure, velocity, acceleration, etc., of whichever fluid particle happens to be at the location of interest at the time of interest.
- While there are many occasions in which the Lagrangian description is useful, the Eulerian description is often more convenient for fluid mechanics applications.
- Experimental measurements are generally more suited to the Eulerian description.

(*a) In the Eulerian description, we* define field variables, such as the pressure field and the velocity field, at any location and instant in time. (*b) For example, the air speed probe* mounted under the wing of an airplane measures the air speed at that location.

EXAMPLE 4–1 A Steady Two-Dimensional Velocity Field

A steady, incompressible, two-dimensional velocity field is given by

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j} \quad (1)$$

where the *x*- and *y*-coordinates are in meters and the magnitude of velocity is in m/s. A **stagnation point** is defined as *a point in the flow field where the velocity is zero.* (*a*) Determine if there are any stagnation points in this flow field and, if so, where? (*b*) Sketch velocity vectors at several locations in the domain between $x = -2$ m to 2 m and $y = 0$ m to 5 m; qualitatively describe the flow field.

Analysis (*a*) Since $\vec{V}$ is a vector, *all* its components must equal zero in order for $\vec{V}$ itself to be zero. Using Eq. 4–4 and setting Eq. 1 equal to zero,

Stagnation point:

$$u = 0.5 + 0.8x = 0 \quad \rightarrow \quad x = -0.625 \text{ m}$$
$$v = 1.5 - 0.8y = 0 \quad \rightarrow \quad y = 1.875 \text{ m}$$

Yes. There is one stagnation point located at $\mathbf{x = -0.625}$ **m**, $\mathbf{y = 1.875}$ **m.**

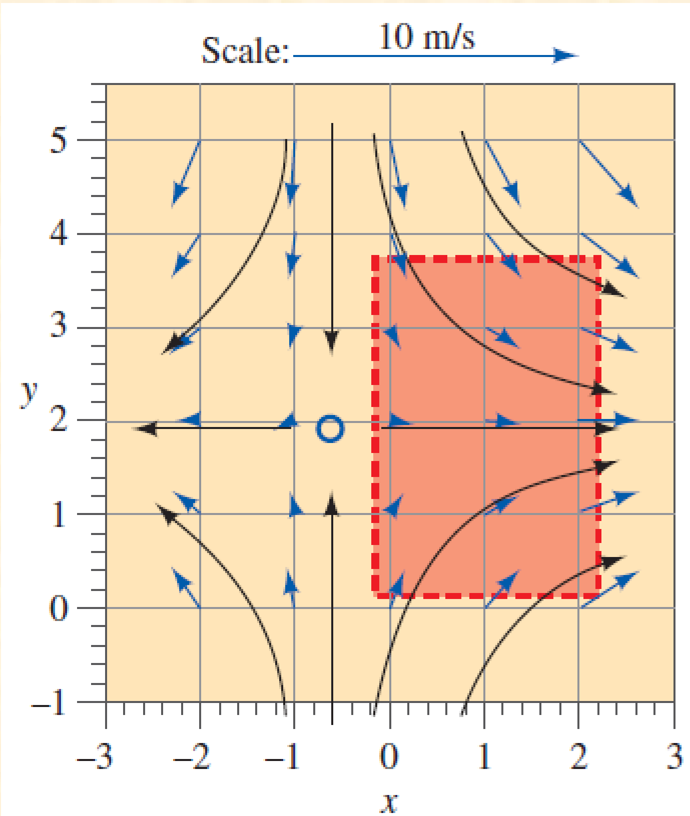
(*b*) The *x*- and *y*-components of velocity are calculated from Eq. 1 for several (*x, y*) locations in the specified range. For example, at the point ($x = 2$ m, $y = 3$ m), $u = 2.10$ m/s and $v = -0.900$ m/s. The magnitude of velocity (the *speed*) at that point is 2.28 m/s. At this and at an array of other locations, the velocity vector is constructed from its two components, the results of which are shown in Fig. 4–4. The flow can be described as stagnation point flow in which flow enters from the top and bottom and spreads out to the right and left about a horizontal line of symmetry at $y = 1.875$ m. The stagnation point of part (*a*) is indicated by the blue circle in Fig. 4–4.

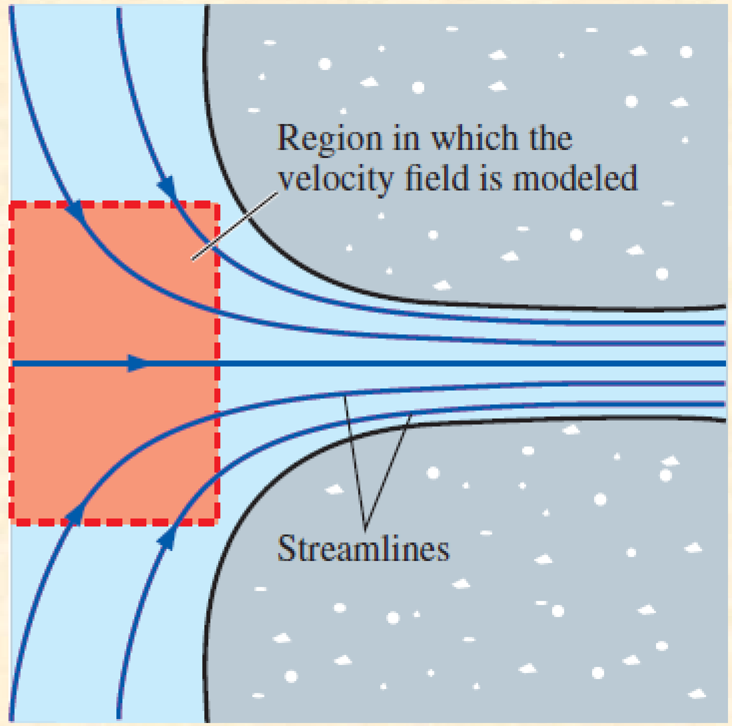
If we look only at the shaded portion of Fig. 4–4, this flow field models a converging, accelerating flow from the left to the right. Such a flow might be encountered, for example, near the submerged bell mouth inlet of a hydroelectric dam (Fig. 4–5). The useful portion of the given velocity field may be thought of as a first-order approximation of the shaded portion of the physical flow field of Fig. 4–5.

Discussion It can be verified from the material in Chap. 9 that this flow field is physically valid because it satisfies the differential equation for conservation of mass.

A Steady Two-Dimensional Velocity Field

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j}$$





Flow field near the bell mouth inlet of a hydroelectric dam; a portion of the velocity field of Example 4-1 may be used as a first-order approximation of this physical flow field.

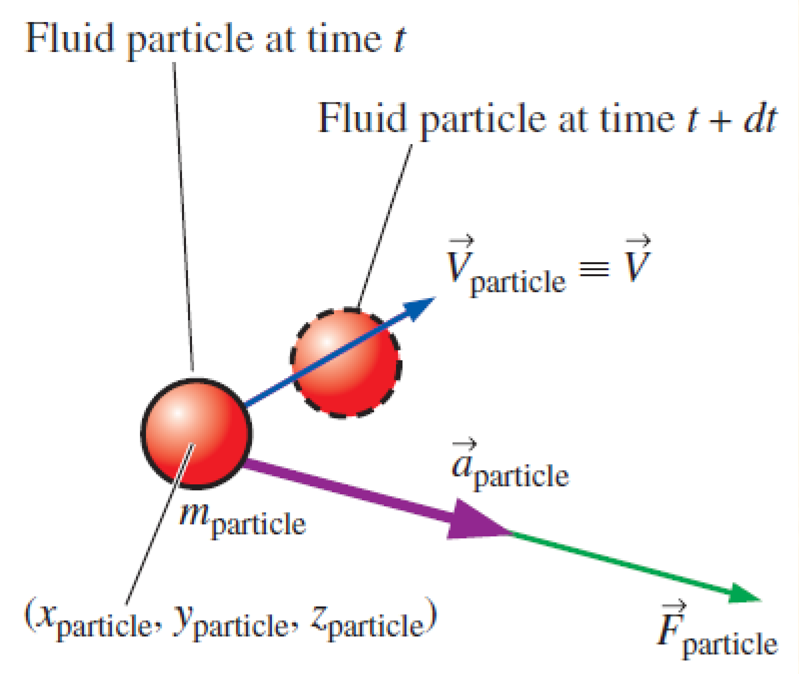
Velocity vectors for the velocity field of Example 4–1. The scale is shown by the top arrow, and the solid black curves represent the approximate shapes of some streamlines, based on the calculated velocity vectors. The stagnation point is indicated by the blue circle. The shaded region represents a portion of the flow field that can approximate flow into an inlet.

Acceleration Field

The equations of motion for fluid flow (such as Newton's second law) are written for a fluid particle, which we also call a **material particle**.

If we were to follow a particular fluid particle as it moves around in the flow, we would be employing the Lagrangian description, and the equations of motion would be directly applicable.

For example, we would define the particle's location in space in terms of a **material position vector** $(x_{particle}(t), y_{particle}(t), z_{particle}(t))$.



Newton's second law applied to a fluid particle; the acceleration vector (purple arrow) is in the same direction as the force vector (green arrow), but the velocity vector (blue arrow) may act in a different direction.

Newton's second law:

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

Acceleration of a fluid particle:

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

$$\vec{V}_{\text{particle}}(t) \equiv \vec{V}(x_{\text{particle}}(t),\ y_{\text{particle}}(t),\ z_{\text{particle}}(t),\ t)$$

$$\vec{a}_{\text{particle}} = \frac{d\vec{V}_{\text{particle}}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}(x_{\text{particle}}, y_{\text{particle}}, z_{\text{particle}}, t)}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x_{\text{particle}}}\frac{dx_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial y_{\text{particle}}}\frac{dy_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial z_{\text{particle}}}\frac{dz_{\text{particle}}}{dt}$$

$$\vec{a}_{\text{particle}}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Acceleration of a fluid particle expressed as a field variable:

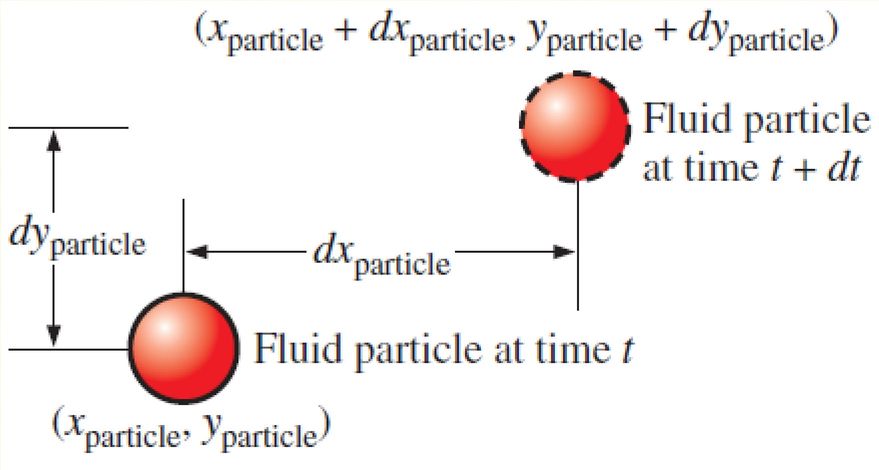
$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

$\frac{\partial \vec{V}}{\partial t}$ Local acceleration

$(\vec{V} \cdot \vec{\nabla})\vec{V}$ Advective (convective) acceleration

Gradient or del operator:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$



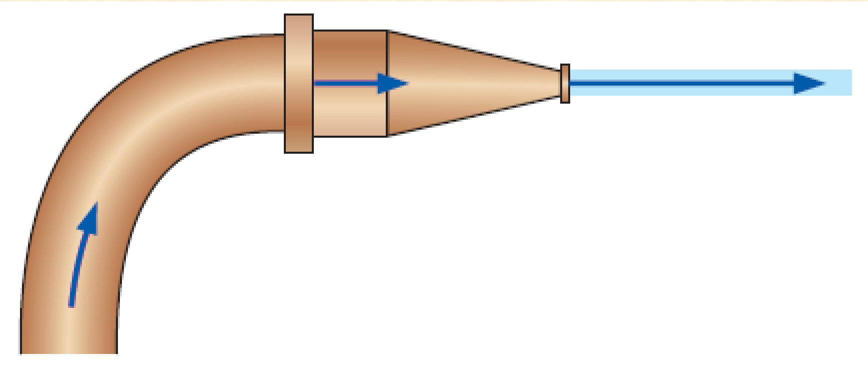
The components of the acceleration vector in cartesian coordinates:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

When following a fluid particle, the *x*-component of velocity, *u*, is defined as dx_{particle}/dt. Similarly, $v=dy_{\text{particle}}/dt$ and $w=dz_{\text{particle}}/dt$. Movement is shown here only in two dimensions for simplicity.



Flow of water through the nozzle of a garden hose illustrates that fluid particles may accelerate, even in a steady flow. In this example, the exit speed of the water is much higher than the water speed in the hose, implying that fluid particles have accelerated even though the flow is steady.

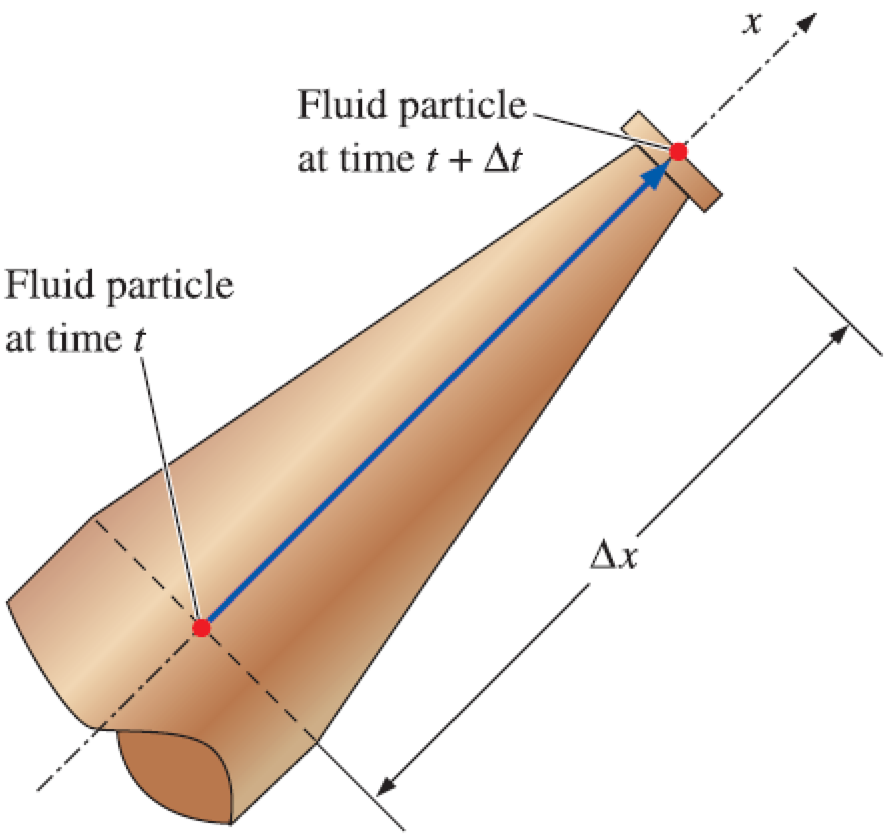


FIGURE 4–10

Residence time Δt is defined as the time it takes for a fluid particle to travel through the nozzle from inlet to outlet (distance Δx).

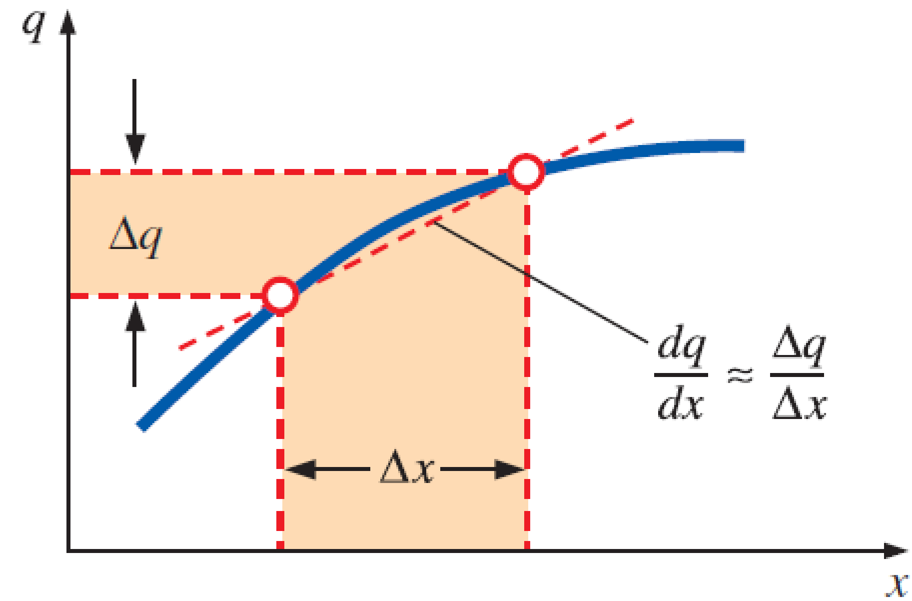


FIGURE 4–11

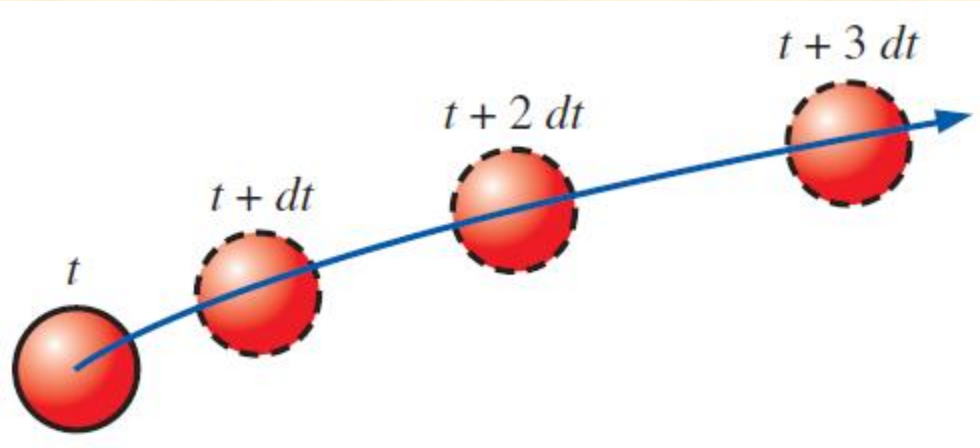
A *first-order finite difference approximation* for derivative dq/dx is simply the change in dependent variable (q) divided by the change in independent variable (x).

Material Derivative

$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

The total derivative operator *d*/*dt* in this equation is given a special name, the **material derivative**; it is assigned a special notation, *D*/*Dt*, in order to emphasize that it is formed by *following a fluid particle as it moves through the flow field*.

Other names for the material derivative include **total**, **particle**, **Lagrangian**, **Eulerian**, and **substantial derivative**.

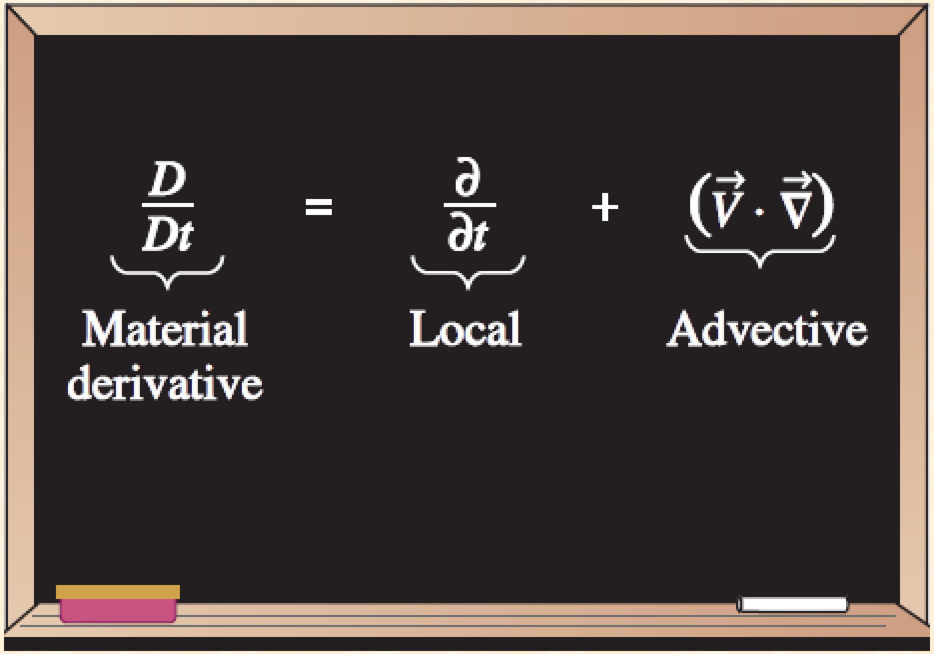


The material derivative *D/Dt* is defined by following a fluid particle as it moves throughout the flow field. In this illustration, the fluid particle is accelerating to the right as it moves up and to the right.

Material derivative: $$\frac{D}{Dt} = \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \vec{\nabla})$$

Material acceleration: $$\vec{a}(x, y, z, t) = \frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

Material derivative of pressure: $$\frac{DP}{Dt} = \frac{dP}{dt} = \frac{\partial P}{\partial t} + (\vec{V} \cdot \vec{\nabla})P$$



The material derivative *D/Dt* is composed of a *local* or *unsteady* part and a *convective* or *advective* part.

EXAMPLE 4–3 Material Acceleration of a Steady Velocity Field

Consider the steady, incompressible, two-dimensional velocity field of Example 4–1. (*a*) Calculate the material acceleration at the point ($x = 2$ m, $y = 3$ m). (*b*) Sketch the material acceleration vectors at the same array of *x*- and *y*-values as in Example 4–1.

Analysis (*a*) Using the velocity field of Eq. 1 of Example 4–1 and the equation for material acceleration components in Cartesian coordinates (Eq. 4–11), we write expressions for the two nonzero components of the acceleration vector:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= 0 + \overbrace{(0.5 + 0.8x)(0.8)} + \overbrace{(1.5 - 0.8y)(0)} + 0 = (0.4 + 0.64x)\ \text{m/s}^2$$

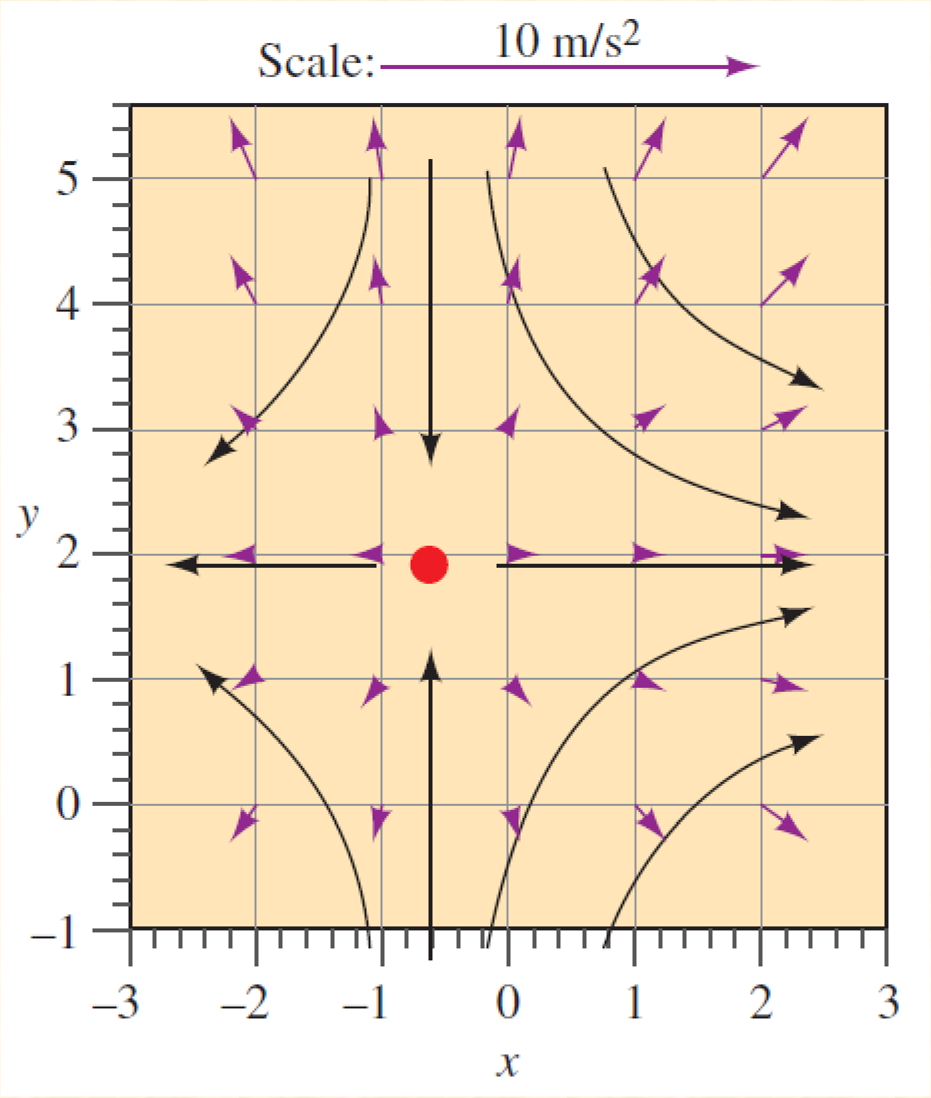
and

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= 0 + \overbrace{(0.5 + 0.8x)(0)} + \overbrace{(1.5 - 0.8y)(-0.8)} + 0 = (-1.2 + 0.64y)\ \text{m/s}^2$$

At the point ($x = 2$ m, $y = 3$ m), $a_x = 1.68\ \text{m/s}^2$ and $a_y = 0.720\ \text{m/s}^2$.
(*b*) The equations in part (*a*) are applied to an array of *x*- and *y*-values in the flow domain within the given limits, and the acceleration vectors are plotted in Fig. 4–14.

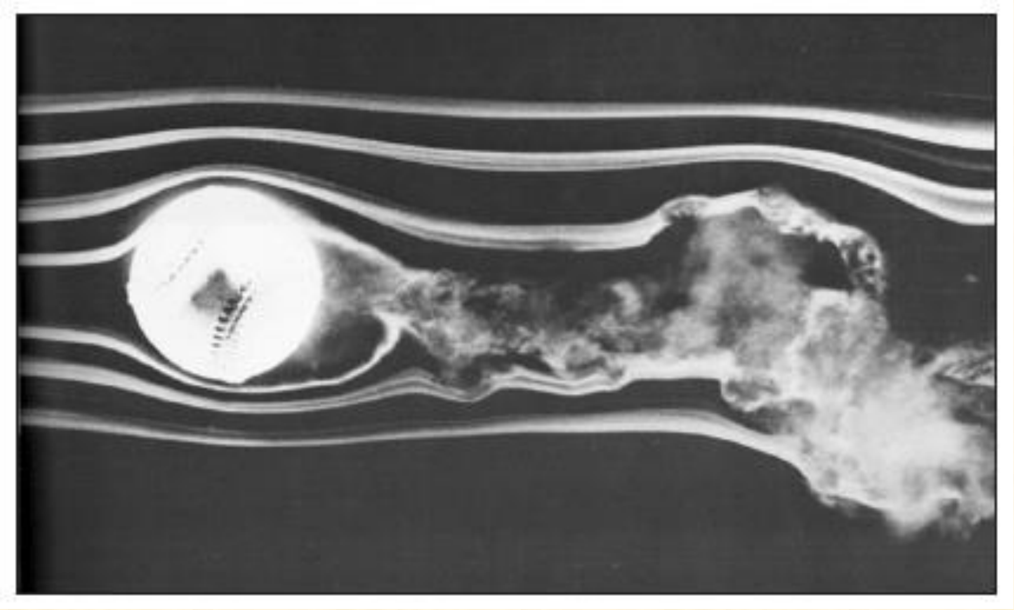
Material Acceleration of a Steady Velocity Field



Acceleration vectors for the velocity field of Examples 4–1 and 4–3. The scale is shown by the top arrow, and the solid black curves represent the approximate shapes of some streamlines, based on the calculated velocity vectors. The stagnation point is indicated by the red circle.

4–2 ■ FLOW PATTERNS AND FLOW VISUALIZATION

- **Flow visualization:** The visual examination of flow field features.
- While quantitative study of fluid dynamics requires advanced mathematics, much can be learned from flow visualization.
- Flow visualization is useful not only in physical experiments but in *numerical* solutions as well [**computational fluid dynamics (CFD)**].
- In fact, the very first thing an engineer using CFD does after obtaining a numerical solution is simulate some form of flow visualization.



Spinning baseball. The late F. N. M. Brown devoted many years to developing and using smoke visualization in wind tunnels at the University of Notre Dame. Here the flow speed is about 23 m/s and the ball is rotated at 630 rpm.

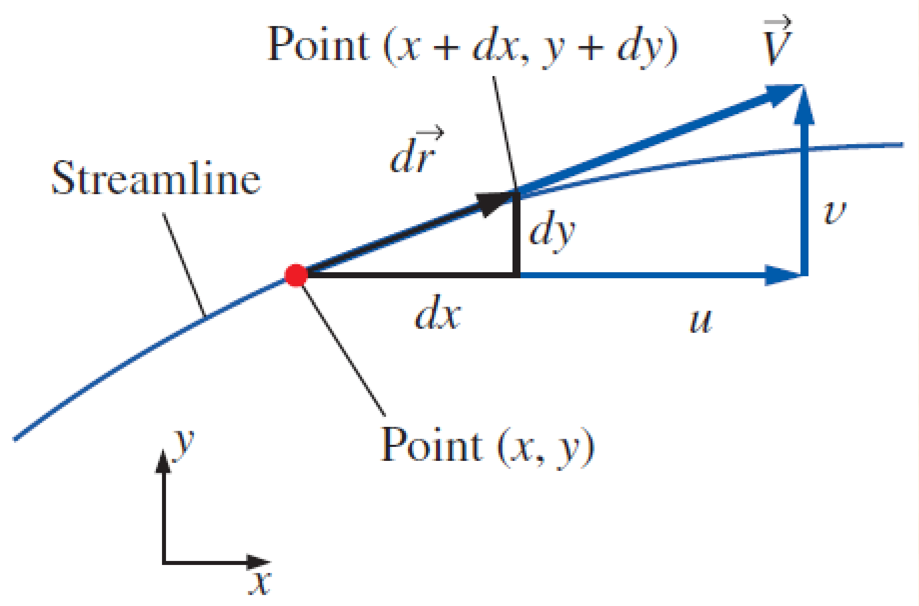
Streamlines and Streamtubes

Streamline: A curve that is everywhere tangent to the instantaneous local velocity vector.

Streamlines are useful as indicators of the instantaneous direction of fluid motion throughout the flow field.

For example, regions of recirculating flow and separation of a fluid off of a solid wall are easily identified by the streamline pattern.

Streamlines cannot be directly observed experimentally except in steady flow fields.



For two-dimensional flow in the xy-plane, arc length $d\vec{r} = (dx, dy)$ along a *streamline* is everywhere tangent to the local instantaneous velocity vector $\vec{V} = (u, v)$.

Consider an infinitesimal arc length $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ along a streamline; $d\vec{r}$ must be parallel to the local velocity vector $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$ by definition of the streamline. By simple geometric arguments using similar triangles, we know that the components of $d\vec{r}$ must be proportional to those of $\vec{V}$ (Fig. 4–16). Hence,

Equation for a streamline:

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{4–15}$$

where dr is the magnitude of $d\vec{r}$ and V is the speed, the magnitude of velocity vector $\vec{V}$. Equation 4–15 is illustrated in two dimensions for simplicity in Fig. 4–16. For a known velocity field, we integrate Eq. 4–15 to obtain equations for the streamlines. In two dimensions, (x, y), (u, v), the following differential equation is obtained:

Streamline in the xy-plane:

$$\left(\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u} \tag{4–16}$$

In some simple cases, Eq. 4–16 may be solvable analytically; in the general case, it must be solved numerically. In either case, an arbitrary constant of integration appears. Each chosen value of the constant represents a different streamline. The *family* of curves that satisfy Eq. 4–16 therefore represents streamlines of the flow field.

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j}$$

EXAMPLE 4–4 Streamlines in the *xy*-Plane—An Analytical Solution

For the steady, incompressible, two-dimensional velocity field of Example 4–1, plot several streamlines in the right half of the flow ($x > 0$) and compare to the velocity vectors plotted in Fig. 4–4.

SOLUTION An analytical expression for streamlines is to be generated and plotted in the upper-right quadrant.

Assumptions **1** The flow is steady and incompressible. **2** The flow is two-dimensional, implying no z-component of velocity and no variation of u or v with z.

Analysis Equation 4–16 is applicable here; thus, along a streamline,

$$\frac{dy}{dx} = \frac{v}{u} = \frac{1.5 - 0.8y}{0.5 + 0.8x}$$

We solve this differential equation by separation of variables:

$$\frac{dy}{1.5 - 0.8y} = \frac{dx}{0.5 + 0.8x} \quad \rightarrow \quad \int \frac{dy}{1.5 - 0.8y} = \int \frac{dx}{0.5 + 0.8x}$$

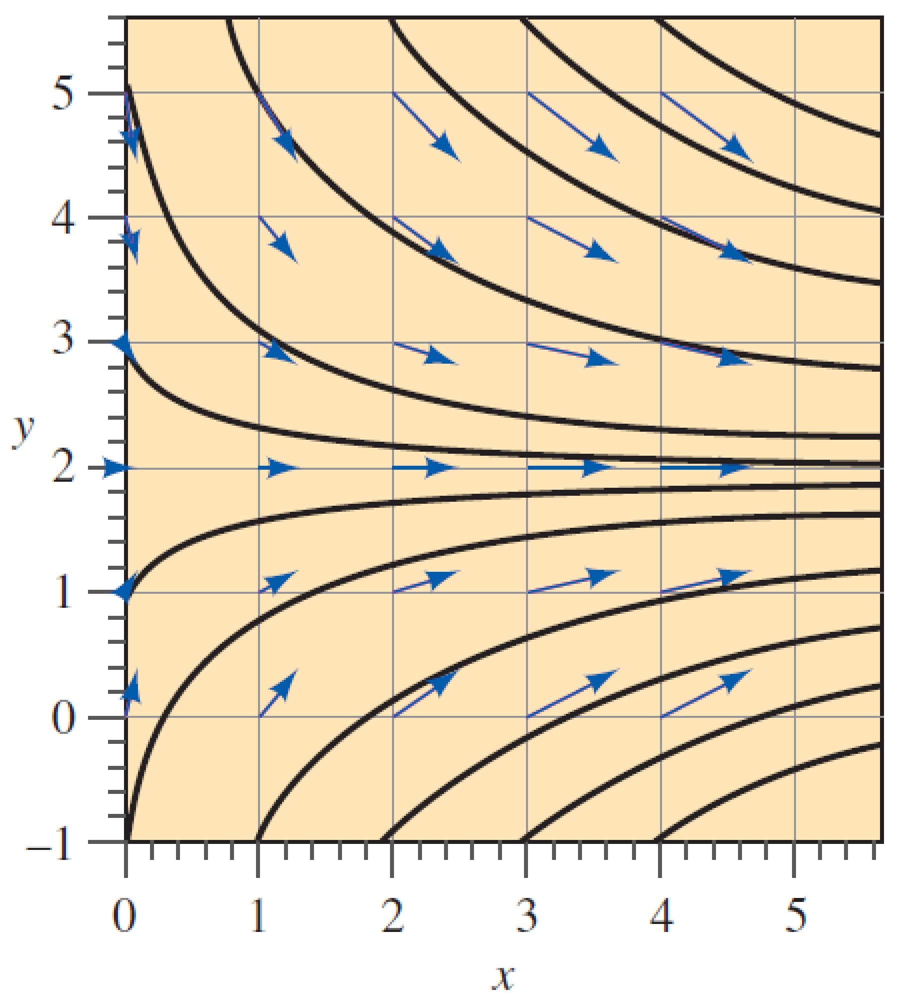
After some algebra, we solve for y as a function of x along a streamline,

$$y = \frac{C}{0.8(0.5 + 0.8x)} + 1.875$$

where C is a constant of integration that can be set to various values in order to plot the streamlines. Several streamlines of the given flow field are shown in Fig. 4–17.

Streamlines for a steady, incompressible, two-dimensional velocity field

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j}$$



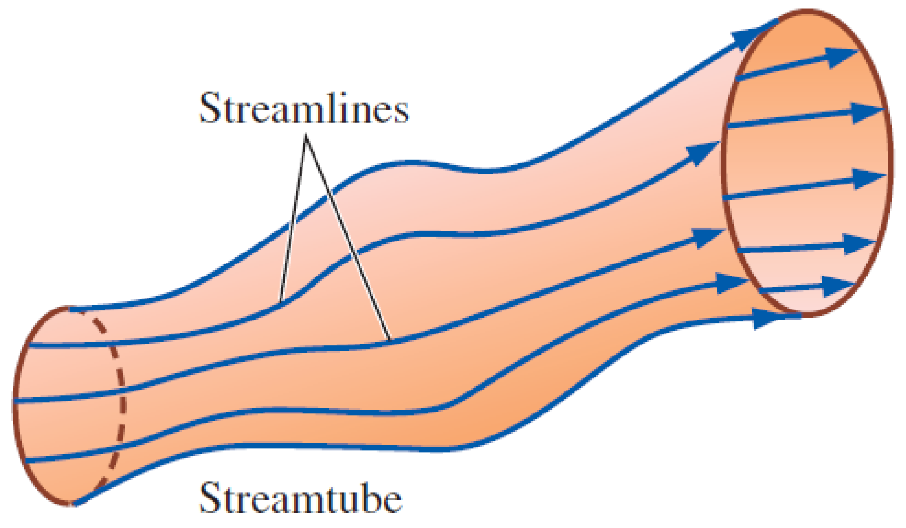
Streamlines (solid black curves) for the velocity field of Example 4–4; velocity vectors (blue arrows) are superimposed for comparison.

The agreement is excellent in the sense that the velocity vectors point everywhere tangent to the streamlines. Note that speed cannot be determined directly from the streamlines alone.

A **streamtube** consists of a bundle of streamlines much like a communications cable consists of a bundle of fiber-optic cables.

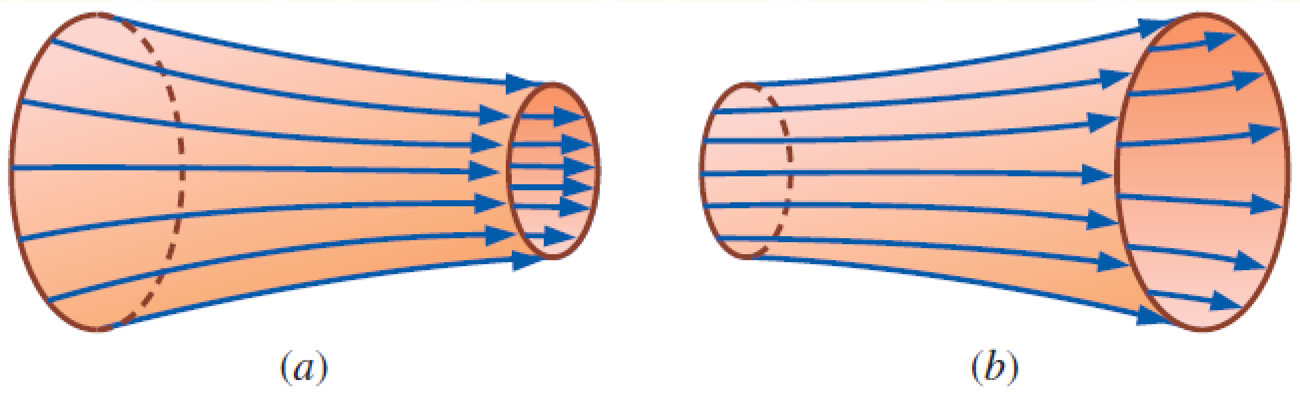
Since streamlines are everywhere parallel to the local velocity, fluid cannot cross a streamline by definition.

Fluid within a streamtube must remain there and cannot cross the boundary of the streamtube.



A *streamtube* consists of a bundle of individual streamlines.

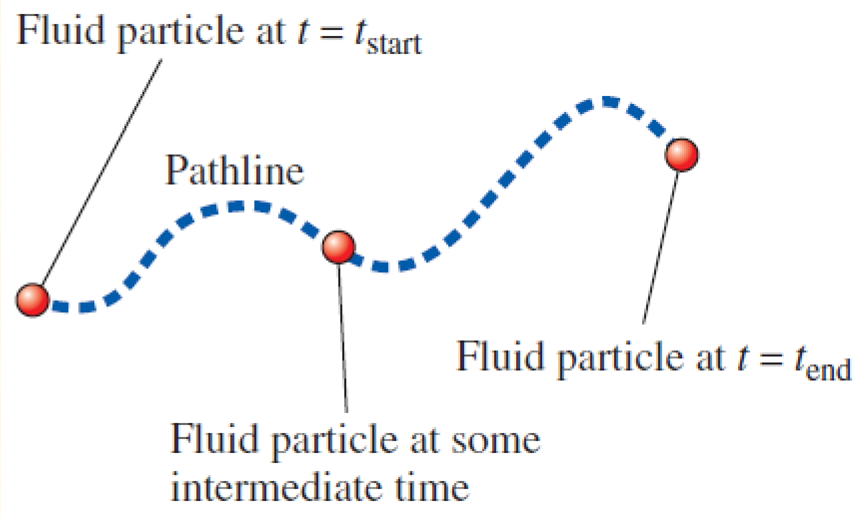
Both streamlines and streamtubes are instantaneous quantities, defined at a particular instant in time according to the velocity field at that instant.



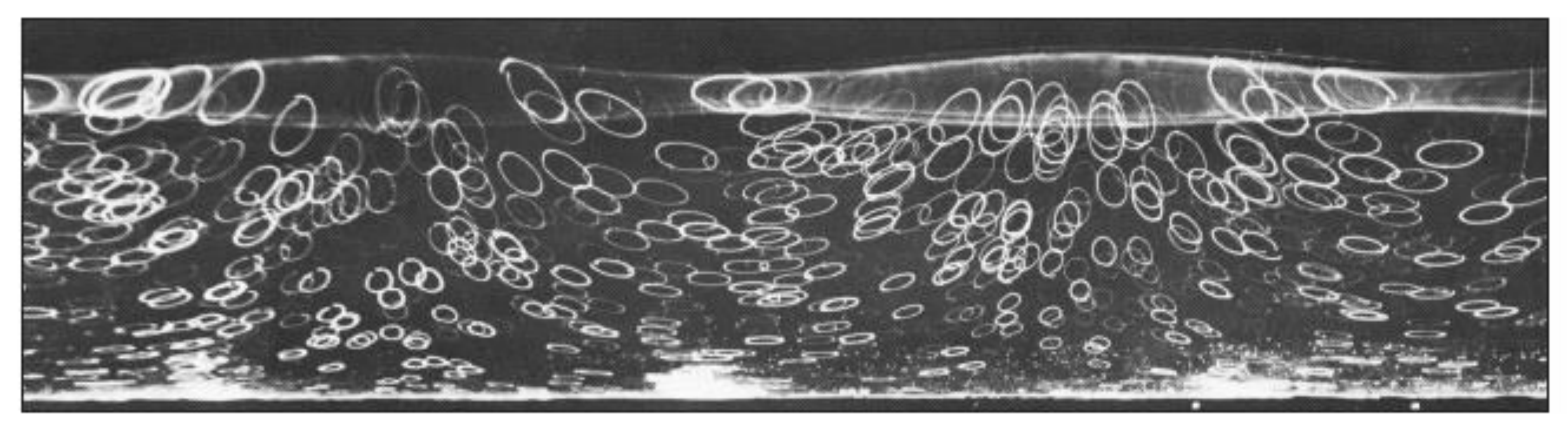
In an incompressible flow field, a streamtube (*a*) decreases in diameter as the flow accelerates or converges and (*b*) increases in diameter as the flow decelerates or diverges.

Pathlines

- **Pathline:** The actual path traveled by an individual fluid particle over some time period.
- A pathline is a Lagrangian concept in that we simply follow the path of an individual fluid particle as it moves around in the flow field.
- Thus, a pathline is the same as the fluid particle's material position vector ($x_{particle}(t)$, $y_{particle}(t)$, $z_{particle}(t)$) traced out over some finite time interval.



A *pathline* is formed by following the actual path of a fluid particle.



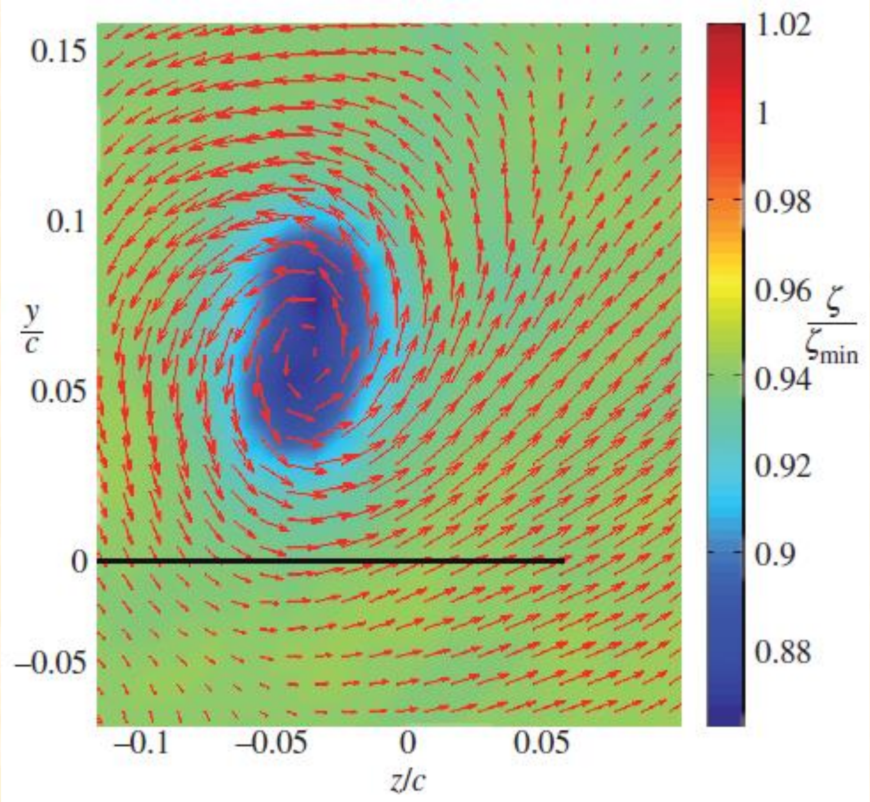
Pathlines produced by white tracer particles suspended in water and captured by time-exposure photography; as waves pass horizontally, each particle moves in an elliptical path during one wave period.

Particle image velocimetry (PIV): A modern experimental technique that utilizes short segments of particle pathlines to measure the velocity field over an entire plane in a flow.

Recent advances also extend the technique to three dimensions.

In PIV, tiny tracer particles are suspended in the fluid. However, the flow is illuminated by two flashes of light (usually a light sheet from a laser) to produce two bright spots (recorded by a camera) for each moving particle.

Then, both the magnitude and direction of the velocity vector at each particle location can be inferred, assuming that the tracer particles are small enough that they move with the fluid.



Stereo PIV measurements of the wing tip vortex in the wake of a NACA-66 airfoil at angle of attack. Color contours denote the local vorticity, normalized by the minimum value, as indicated in the color map. Vectors denote fluid motion in the plane of measurement. The black line denotes the location of the upstream wing trailling edge. Coordinates are normalized by the airfoil chord, and the origin is the wing root.

Pathlines can also be calculated numerically for a known velocity field. Specifically, the location of the tracer particle is integrated over time from some starting location $\vec{x}_{start}$ and starting time t_{start} to some later time t.

Tracer particle location at time t:

$$\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V}\, dt \tag{11–17}$$

When Eq. 11–17 is calculated for t between t_{start} and t_{end}, a plot of $\vec{x}(t)$ is the pathline of the fluid particle during that time interval, as illustrated in Fig. 11–15. For some simple flow fields, Eq. 11–17 can be integrated analytically. For more complex flows, we must perform a numerical integration.

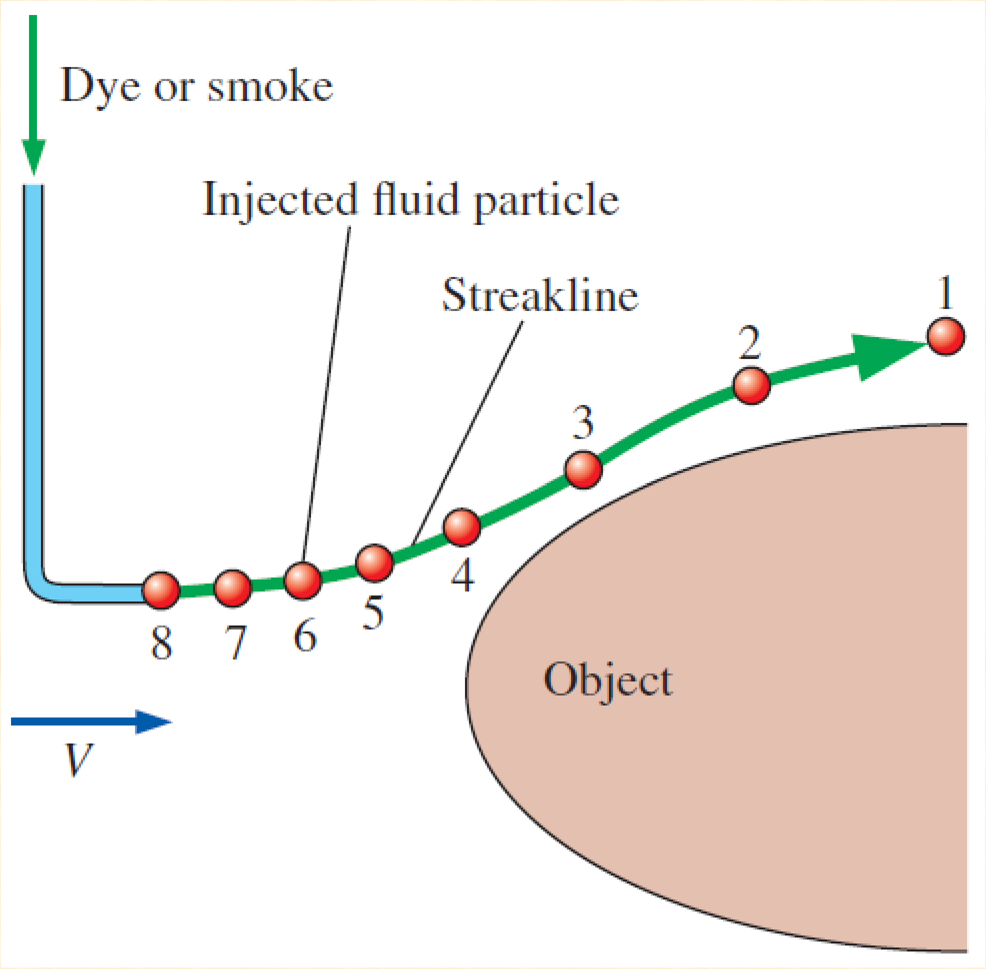
If the velocity field is steady, individual fluid particles will follow streamlines. Thus, *for steady flow, pathlines are identical to streamlines*.

Streaklines

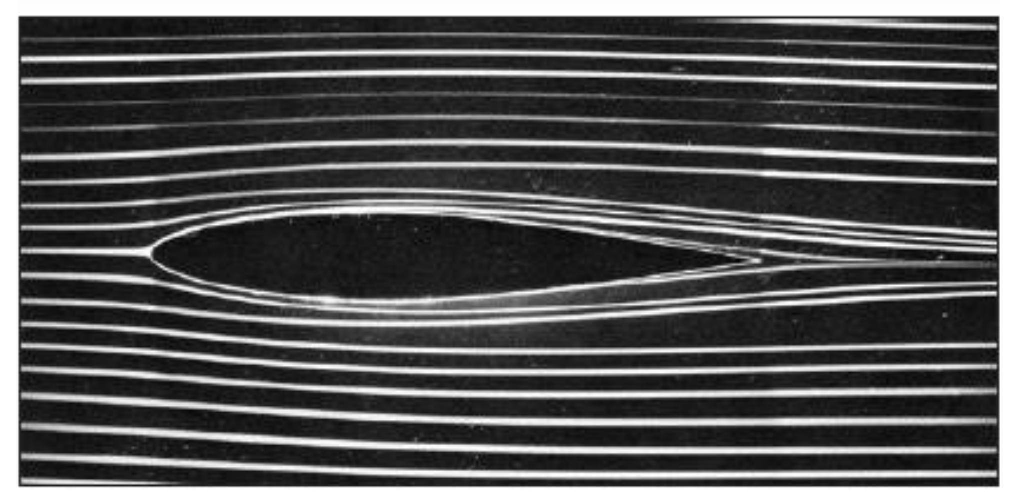
Streakline: The locus of fluid particles that have passed sequentially through a prescribed point in the flow.

Streaklines are the most common flow pattern generated in a physical experiment.

If you insert a small tube into a flow and introduce a continuous stream of tracer fluid (dye in a water flow or smoke in an air flow), the observed pattern is a streakline.



A *streakline* is formed by continuous introduction of dye or smoke from a point in the flow. Labeled tracer particles (1 through 8) were introduced sequentially.



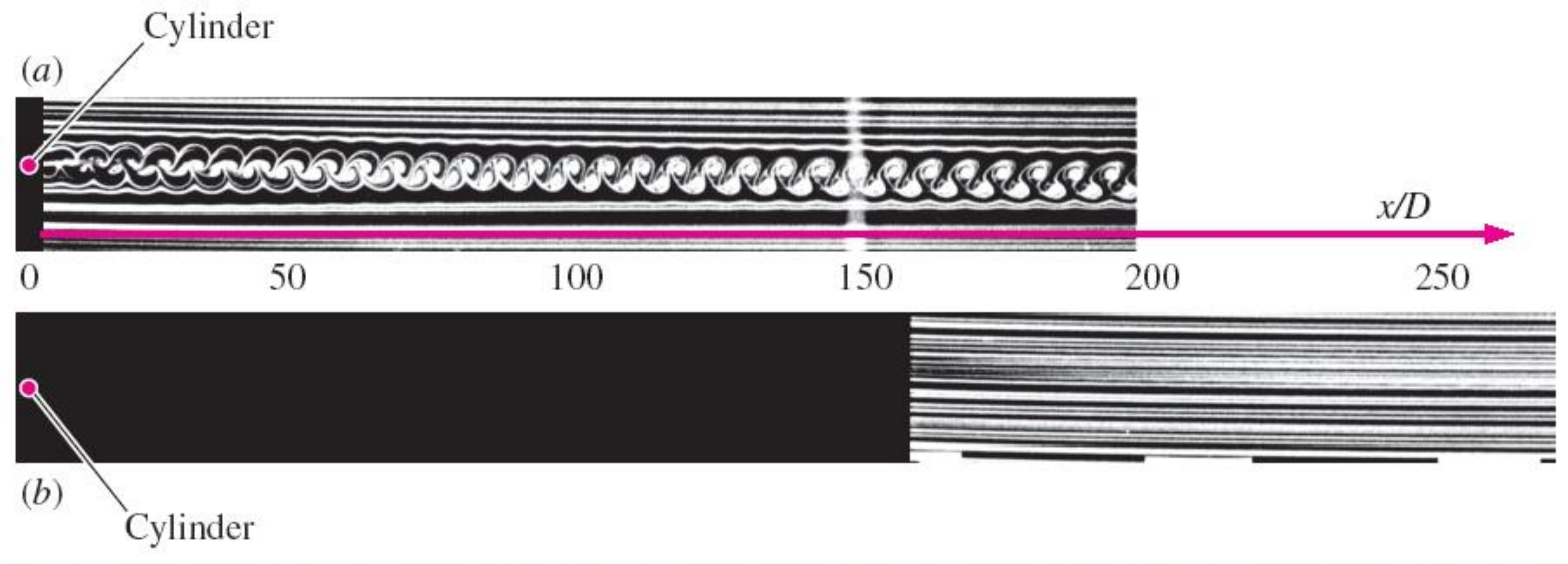
Streaklines produced by colored fluid introduced upstream; since the flow is steady, these streaklines are the same as streamlines and pathlines.

- Streaklines, streamlines, and pathlines are identical in steady flow but they can be quite different in unsteady flow.
- The main difference is that a streamline represents an *instantaneous* flow pattern at a given instant in time, while a streakline and a pathline are flow patterns that have some *age* and thus a *time history* associated with them.
- A streakline is an instantaneous snapshot of a *time-integrated* flow pattern.
- A pathline, on the other hand, is the *time-exposed* flow path of an individual particle over some time period.

In the figure, streaklines are introduced from a smoke wire located just downstream of a circular cylinder of diameter *D* aligned normal to the plane of view.

When multiple streaklines are introduced along a line, as in the figure, we refer to this as a **rake** of streaklines.

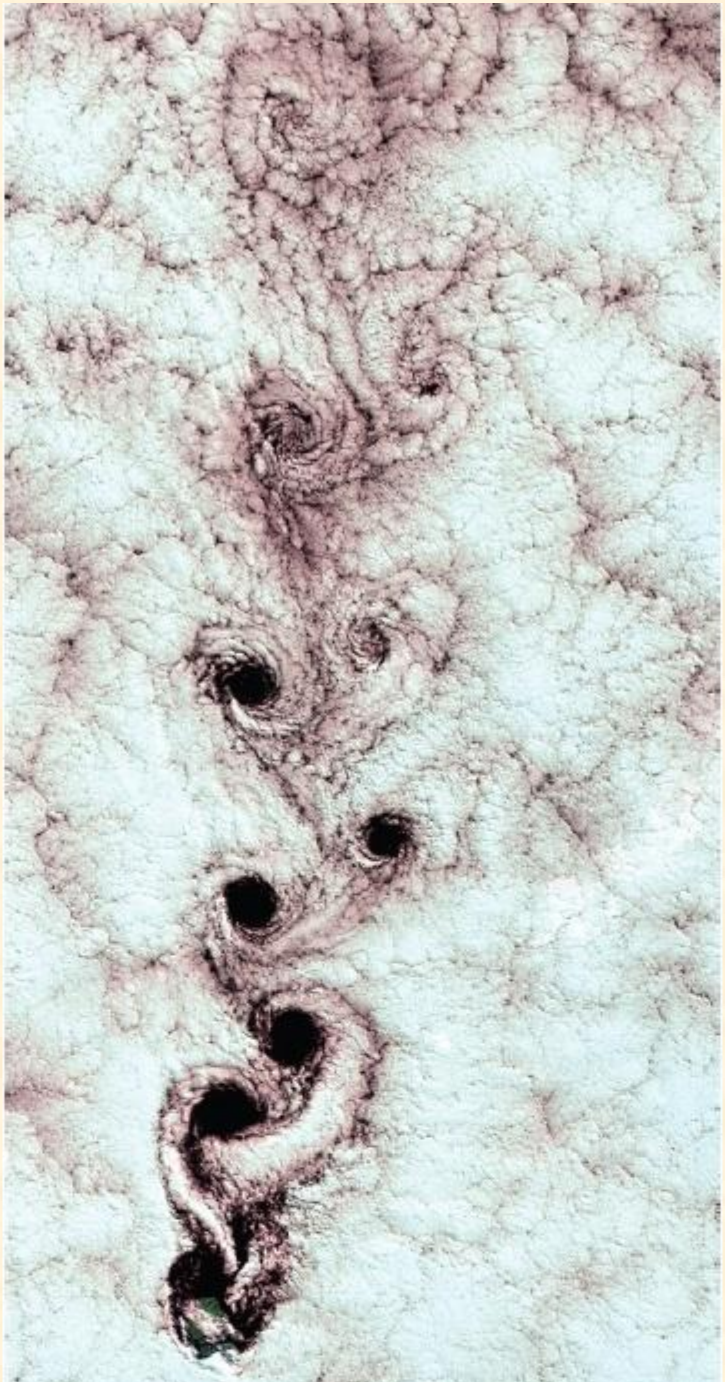
The Reynolds number of the flow is Re = 93.



Smoke streaklines introduced by a smoke wire at two different locations in the wake of a circular cylinder: (*a*) smoke wire just downstream of the cylinder and (*b*) smoke wire located at $x/D = 150$. The time-integrative nature of streaklines is clearly seen by comparing the two photographs.

Because of unsteady **vortices** shed in an alternating pattern from the cylinder, the smoke collects into a clearly defined periodic pattern called a **Kármán vortex street**.

A similar pattern can be seen at much larger scale in the air flow in the wake of an island.



Kármán vortices visible in the clouds in the wake of Alexander Selkirk Island in the southern Pacific Ocean.

For a known velocity field, a streakline can be generated numerically. We need to follow the paths of a continuous stream of tracer particles from the time of their injection into the flow until the present time, using Eq. 4–17. Mathematically, the location of a tracer particle is integrated over time from the time of its injection t_{inject} to the present time t_{present}. Equation 4–17 becomes

Integrated tracer particle location:
$$\vec{x} = \vec{x}_{\text{injection}} + \int_{t_{\text{inject}}}^{t_{\text{present}}} \vec{V}\,dt \qquad \textbf{(4–18)}$$

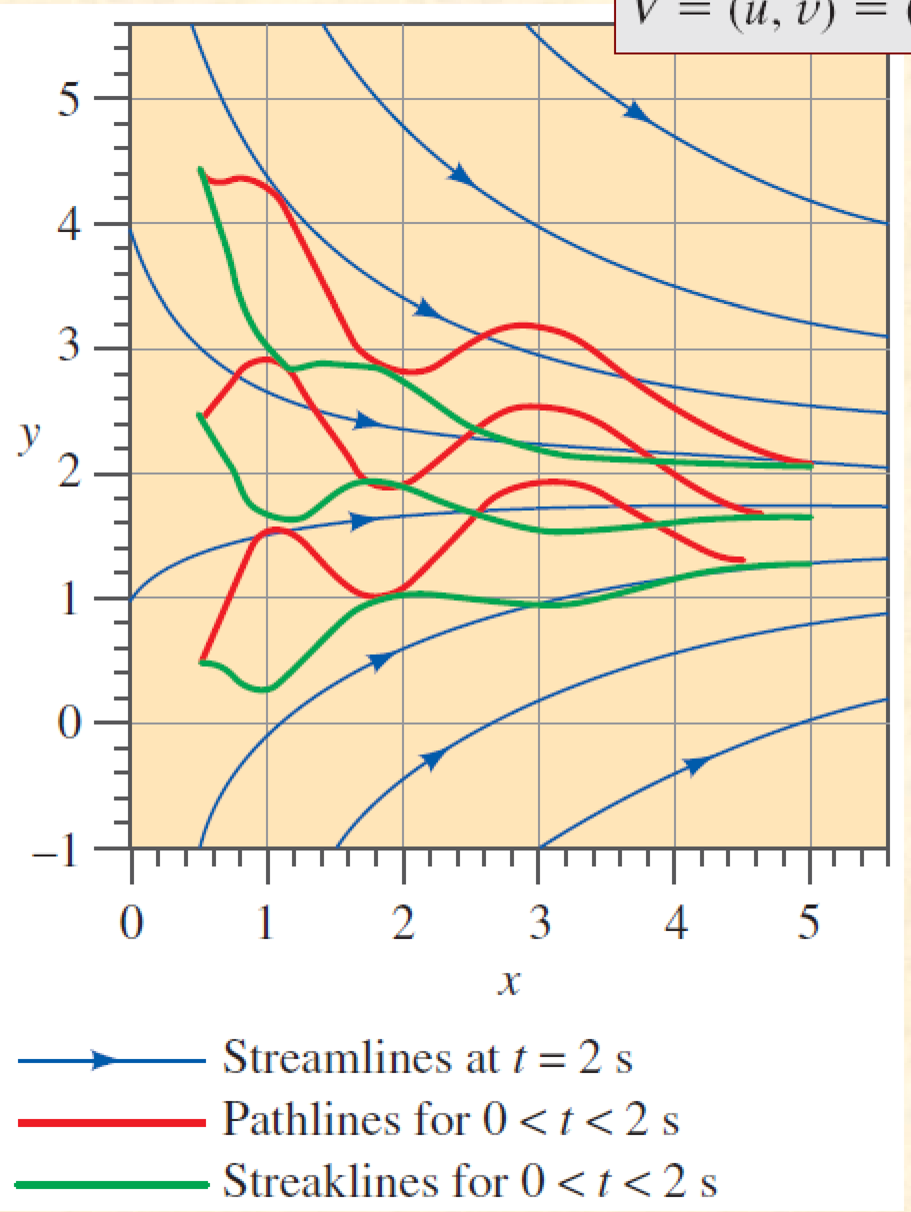
In a complex unsteady flow, the time integration must be performed numerically as the velocity field changes with time. When the locus of tracer particle locations at $t = t_{\text{present}}$ is connected by a smooth curve, the result is the desired streakline.

Tracer particle location at time t:
$$\vec{x} = \vec{x}_{\text{start}} + \int_{t_{\text{start}}}^{t} \vec{V}\,dt \qquad \textbf{(4–17)}$$

Comparison of Flow Patterns in an Unsteady Flow

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 + 2.5\sin(\omega t) - 0.8y)\vec{j}$$

An *unsteady*, incompressible, two-dimensional velocity field

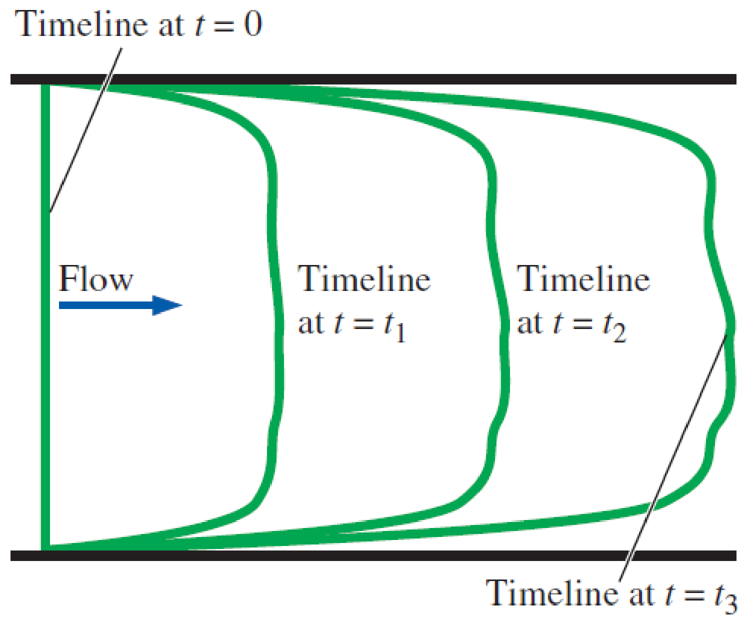


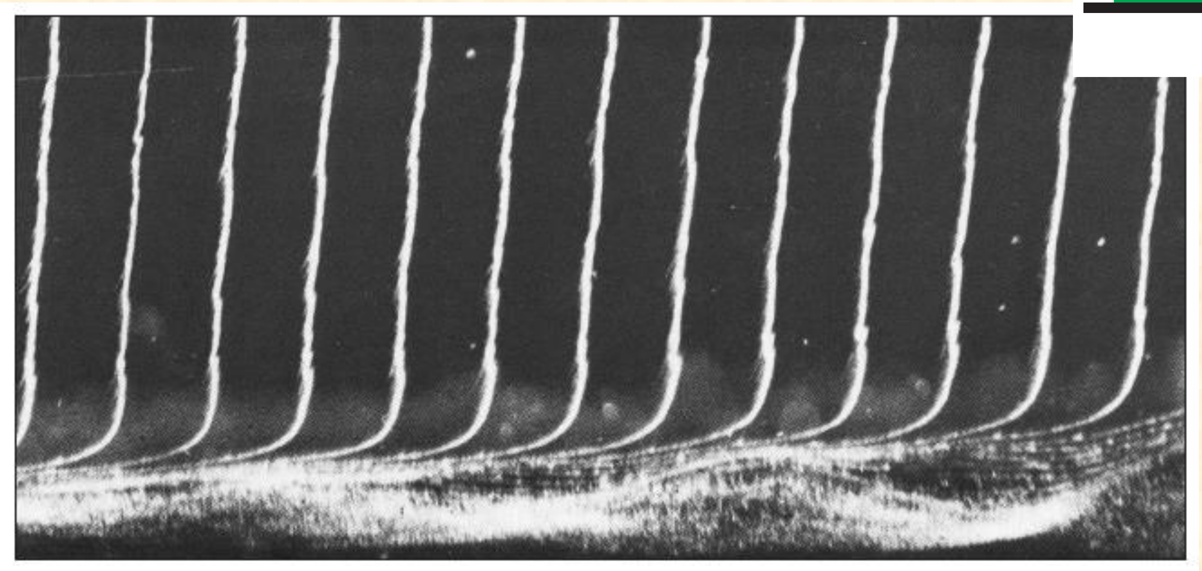
Streamlines, pathlines, and streaklines for the oscillating velocity field of Example 4–5. The streaklines and pathlines are wavy because of their integrated time history, but the streamlines are not wavy since they represent an instantaneous snapshot of the velocity field.

Timelines

Timeline: A set of adjacent fluid particles that were marked at the same (earlier) instant in time.

Timelines are particularly useful in situations where the uniformity of a flow (or lack thereof) is to be examined.





Timelines are formed by marking a line of fluid particles, and then watching that line move (and deform) through the flow field; timelines are shown at $t = 0$, t_1, t_2, and t_3.

Timelines produced by a hydrogen bubble wire are used to visualize the boundary layer velocity profile shape. Flow is from left to right, and the hydrogen bubble wire is located to the left of the field of view. Bubbles near the wall reveal a flow instability that leads to turbulence.

Refractive Flow Visualization Techniques

It is based on the **refractive property** of light waves.

The speed of light through one material may differ somewhat from that in another material, or even in the *same* material if its density changes. As light travels through one fluid into a fluid with a different index of refraction, the light rays bend (they are **refracted**).

Two primary flow visualization techniques that utilize the fact that the index of refraction in air (or other gases) varies with density: the **shadowgraph technique** and the **schlieren technique**.

Interferometry is a visualization technique that utilizes the related *phase change* of light as it passes through air of varying densities as the basis for flow visualization.

These techniques are useful for flow visualization in flow fields where density changes from one location in the flow to another, such as such as natural convection flows (temperature differences cause the density variations), mixing flows (fluid species cause the density variations), and supersonic flows (shock waves and expansion waves cause the density variations).

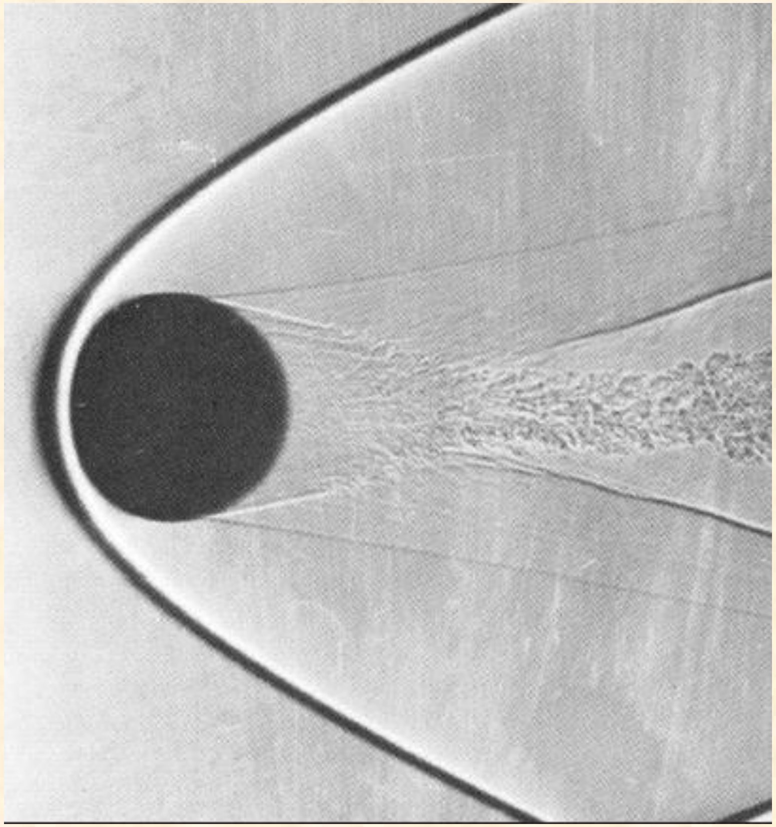
Unlike flow visualizations involving streaklines, pathlines, and timelines, the shadowgraph and schlieren methods do not require injection of a visible tracer (smoke or dye).

Rather, density differences and the refractive property of light provide the necessary means for visualizing regions of activity in the flow field, allowing us to “see the invisible.”

The image (a **shadowgram**) produced by the shadowgraph method is formed when the refracted rays of light rearrange the shadow cast onto a viewing screen or camera focal plane, causing bright or dark patterns to appear in the shadow.

The dark patterns indicate the location where the refracted rays *originate*, while the bright patterns mark where these rays *end up*, and can be misleading.

As a result, the dark regions are less distorted than the bright regions and are more useful in the interpretation of the shadowgram.



Shadowgram of a 14.3 mm sphere in free flight through air at Ma 3.0. A shock wave is clearly visible in the shadow as a dark band that curves around the sphere and is called a *bow wave* (see Chap. 12).

A shadowgram is not a true optical image; it is, after all, merely a shadow.

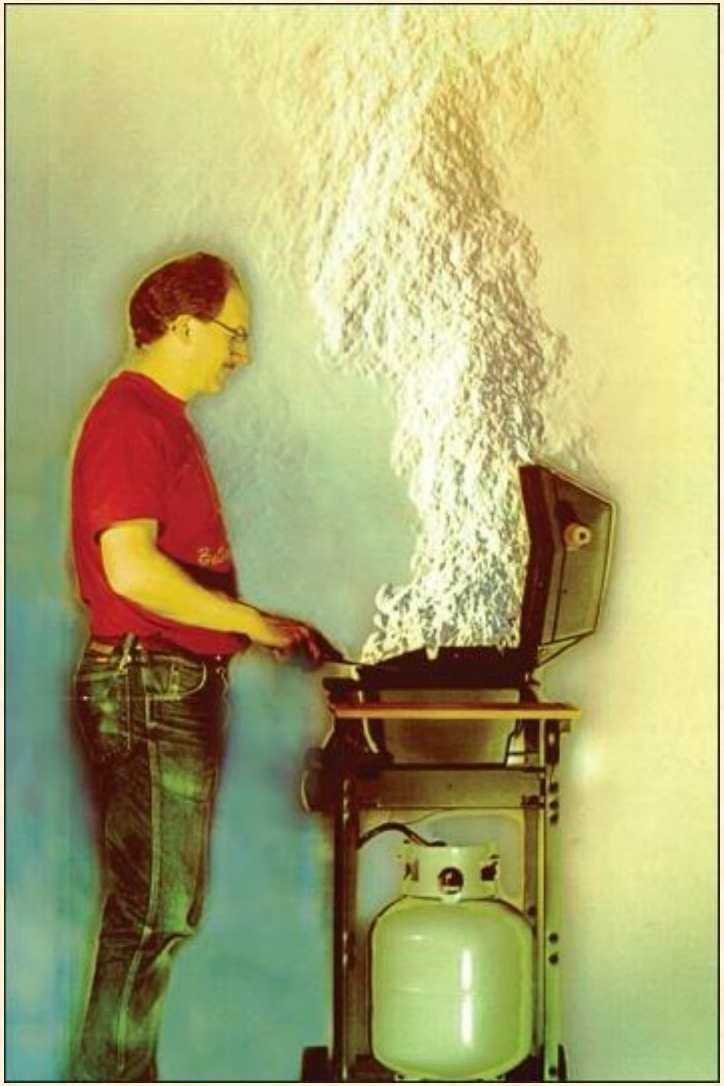
A **schlieren image**, involves lenses (or mirrors) and a knife edge or other cutoff device to block the refracted light and is a true focused optical image.

Schlieren imaging is more complicated to set up than is shadowgraphy but has a number of advantages.

A schlieren image does not suffer from optical distortion by the refracted light rays.

Schlieren imaging is also more sensitive to weak density gradients such as those caused by natural convection or by gradual phenomena like expansion fans in supersonic flow. Color schlieren imaging techniques have also been developed.

One can adjust more components in a schlieren setup.



Schlieren image of natural convection due to a barbeque grill.

Surface Flow Visualization Techniques

- The direction of fluid flow immediately above a solid surface can be visualized with **tufts**—short, flexible strings glued to the surface at one end that point in the flow direction.
- Tufts are especially useful for locating regions of flow separation, where the flow direction suddenly reverses.
- A technique called **surface oil visualization** can be used for the same purpose—oil placed on the surface forms streaks called **friction lines** that indicate the direction of flow.
- If it rains lightly when your car is dirty (especially in the winter when salt is on the roads), you may have noticed streaks along the hood and sides of the car, or even on the windshield.
- This is similar to what is observed with surface oil visualization.
- Lastly, there are pressure-sensitive and temperature-sensitive paints that enable researchers to observe the pressure or temperature distribution along solid surfaces.

4–3 ■ PLOTS OF FLUID FLOW DATA

Regardless of how the results are obtained (analytically, experimentally, or computationally), it is usually necessary to *plot* flow data in ways that enable the reader to get a feel for how the flow properties vary in time and/or space.

You are already familiar with *time plots*, which are especially useful in turbulent flows (e.g., a velocity component plotted as a function of time), and *xy*-plots (e.g., pressure as a function of radius).

In this section, we discuss three additional types of plots that are useful in fluid mechanics—

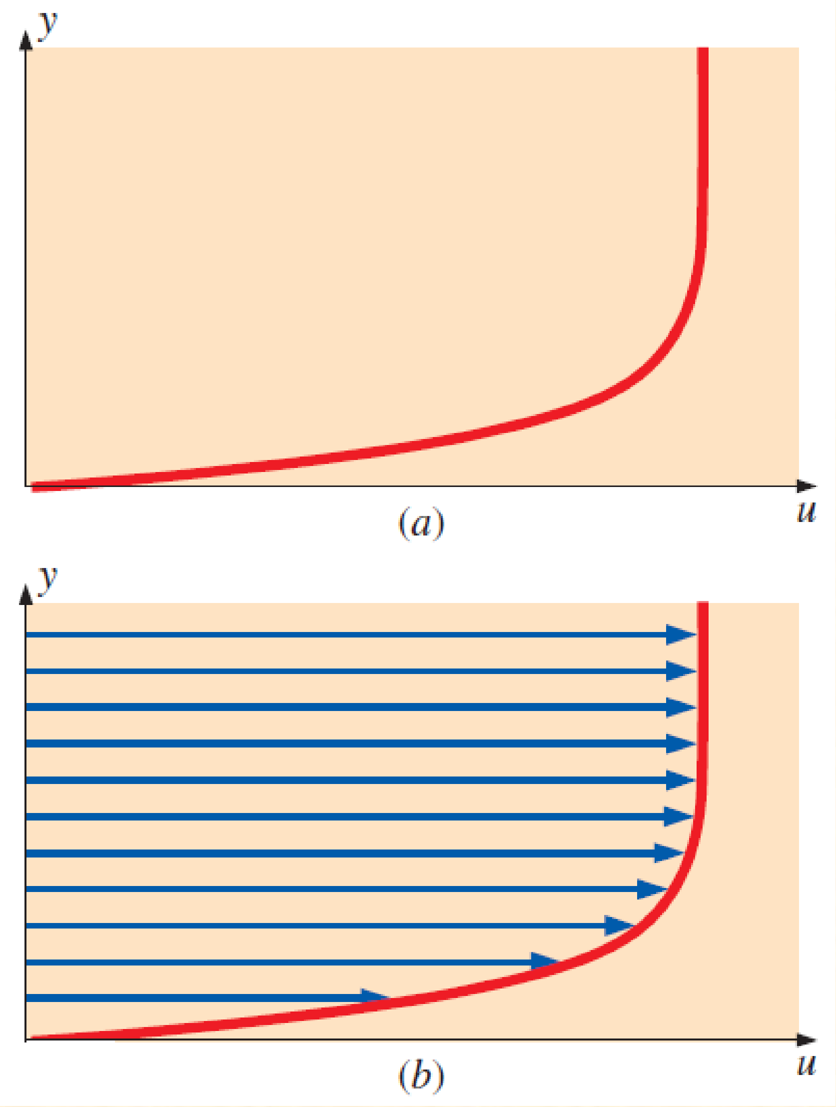
profile plots, vector plots, and contour plots.

Profile Plots

A **profile plot** indicates how the value of a scalar property varies along some desired direction in the flow field.

In fluid mechanics, profile plots of *any* scalar variable (pressure, temperature, density, etc.) can be created, but the most common one used in this book is the ***velocity profile plot***.

Since velocity is a vector quantity, we usually plot either the magnitude of velocity or one of the components of the velocity vector as a function of distance in some desired direction.



Profile plots of the horizontal component of velocity as a function of vertical distance; flow in the boundary layer growing along a horizontal flat plate: (*a*) standard profile plot and (*b*) profile plot with arrows.

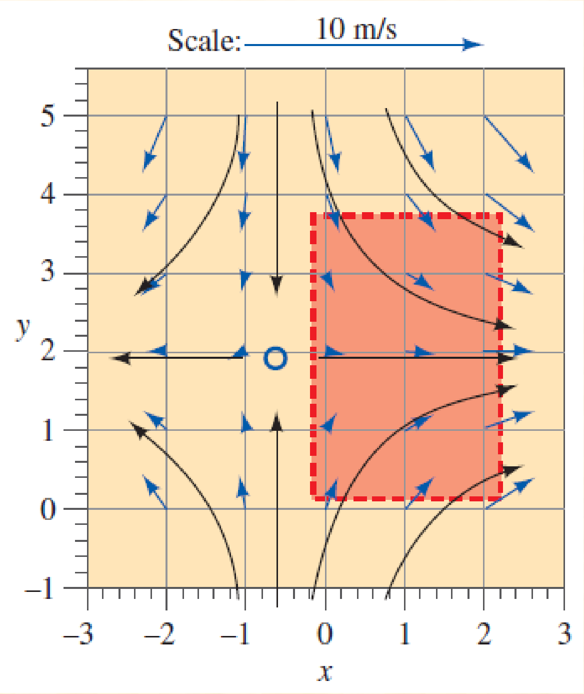
Vector Plots

A **vector plot** is an array of arrows indicating the magnitude and direction of a vector property at an instant in time.

Streamlines indicate the *direction* of the instantaneous velocity field, they do not directly indicate the *magnitude* of the velocity (i.e., the speed).

A useful flow pattern for both experimental and computational fluid flows is thus the vector plot, which consists of an array of arrows that indicate both magnitude *and* direction of an instantaneous vector property.

Vector plots can also be generated from experimentally obtained data (e.g., from PIV measurements) or numerically from CFD calculations.



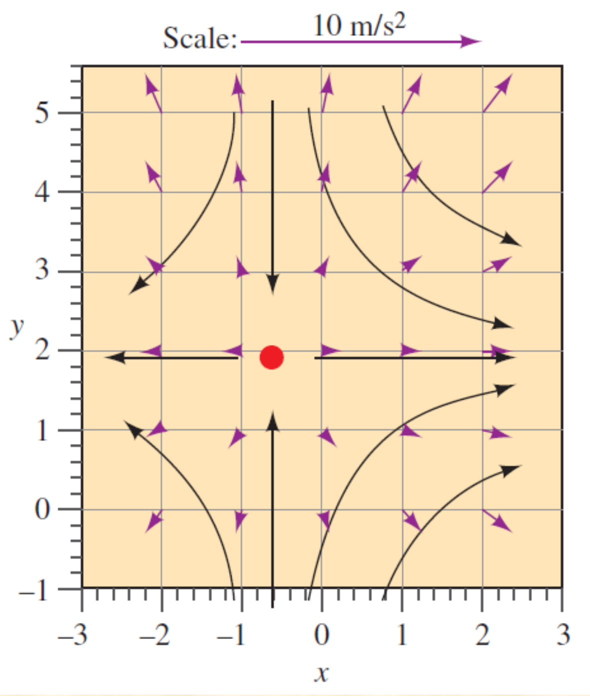
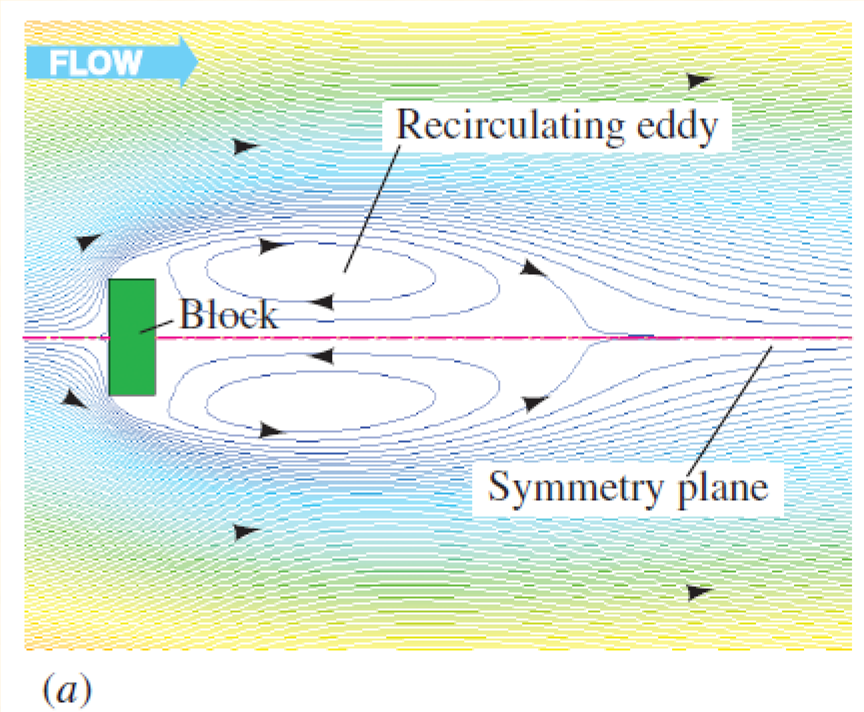
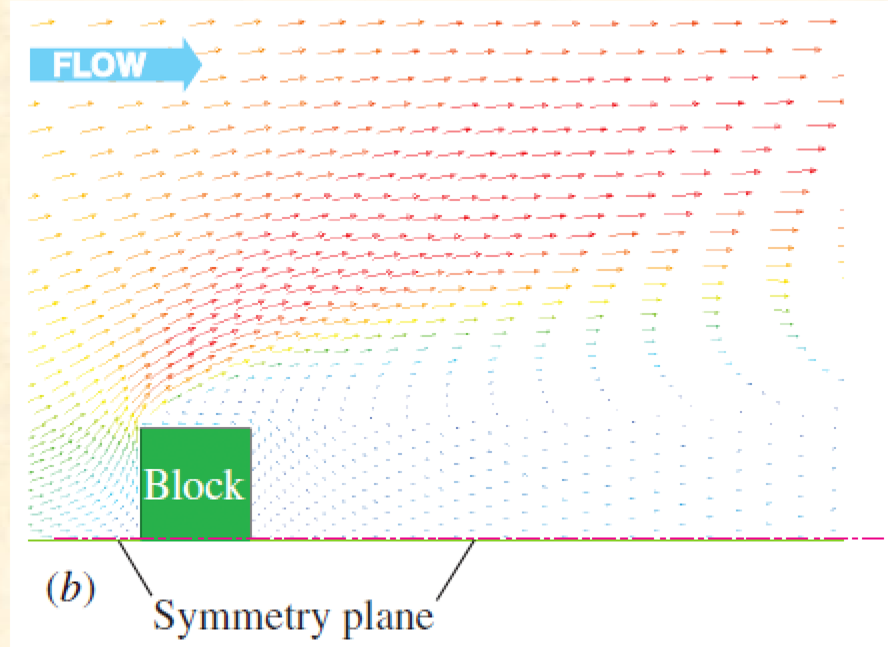
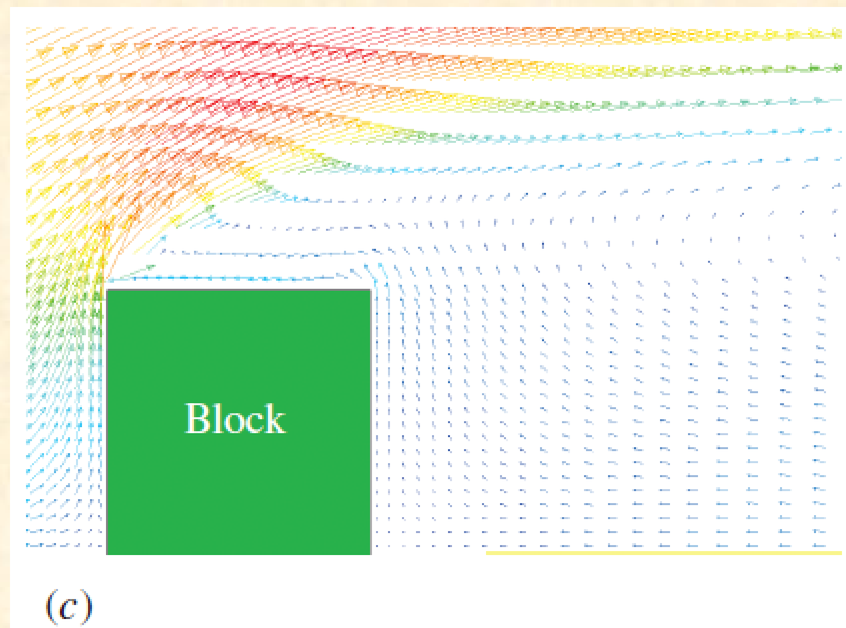


Fig. 4-4: Velocity vector plot
Fig. 4-14: Acceleration vector plot.
Both generated analytically.







Results of CFD calculations of a two-dimensional flow field consisting of free-stream flow impinging on a block of rectangular cross section.
(*a*) streamlines,
(*b*) velocity vector plot of the upper half of the flow, and
(*c*) velocity vector plot, close-up view revealing more details in the separated flow region.

Contour Plots

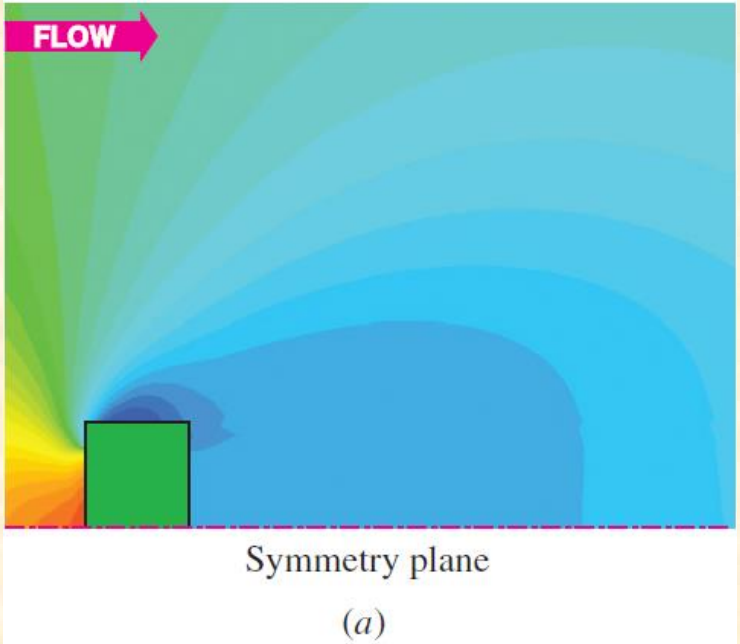
A **contour plot** shows curves of constant values of a scalar property (or magnitude of a vector property) at an instant in time.

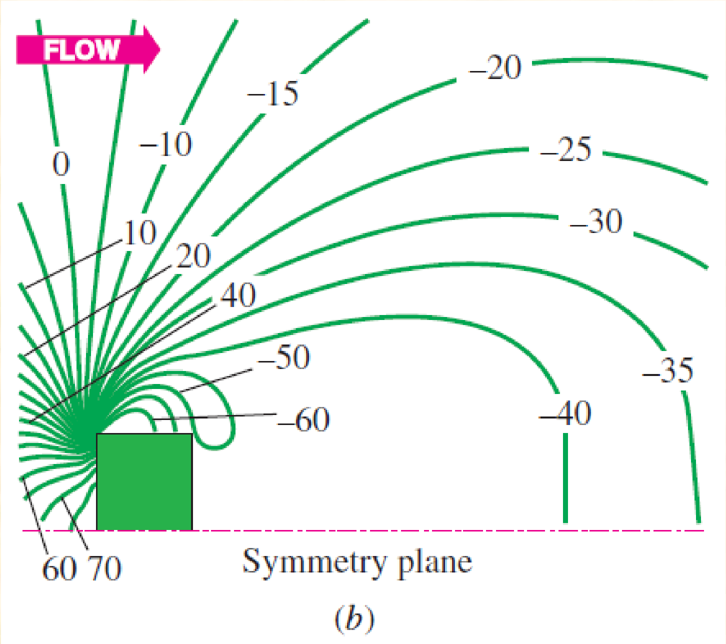
Contour plots (also called **isocontour plots**) are generated of pressure, temperature, velocity magnitude, species concentration, properties of turbulence, etc.

A contour plot can quickly reveal regions of high (or low) values of the flow property being studied.

A contour plot may consist simply of curves indicating various levels of the property; this is called a **contour line plot**.

Alternatively, the contours can be filled in with either colors or shades of gray; this is called a **filled contour plot**.





Contour plots of the pressure field due to flow impinging on a block, as produced by CFD calculations; only the upper half is shown due to symmetry; (*a*) filled color scale contour plot and (*b*) contour line plot where pressure values are displayed in units of Pa gage pressure.

4–4 ■ OTHER KINEMATIC DESCRIPTIONS

Types of Motion or Deformation of Fluid Elements

In fluid mechanics, an element may undergo four fundamental types of motion or deformation:

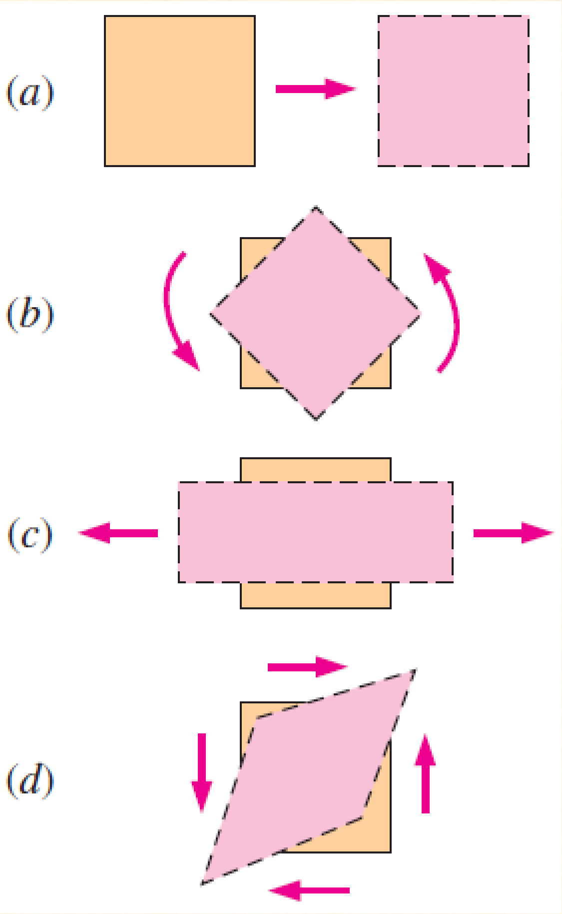
(*a*) **translation**, (*b*) **rotation**,
(*c*) **linear strain** (also called **extensional strain**), and
(*d*) **shear strain**.

All four types of motion or deformation usually occur simultaneously.

It is preferable in fluid dynamics to describe the motion and deformation of fluid elements in terms of *rates* such as

velocity (rate of translation),
angular velocity (rate of rotation),
linear strain rate (rate of linear strain), and
shear strain rate (rate of shear strain).

In order for these **deformation rates** to be useful in the calculation of fluid flows, we must express them in terms of velocity and derivatives of velocity.



Fundamental types of fluid element motion or deformation: (*a*) translation, (*b*) rotation, (*c*) linear strain, and (*d*) shear strain.

A vector is required in order to fully describe the rate of translation in three dimensions. The **rate of translation vector** is described mathematically as the **velocity vector.**

Rate of translation vector in Cartesian coordinates:

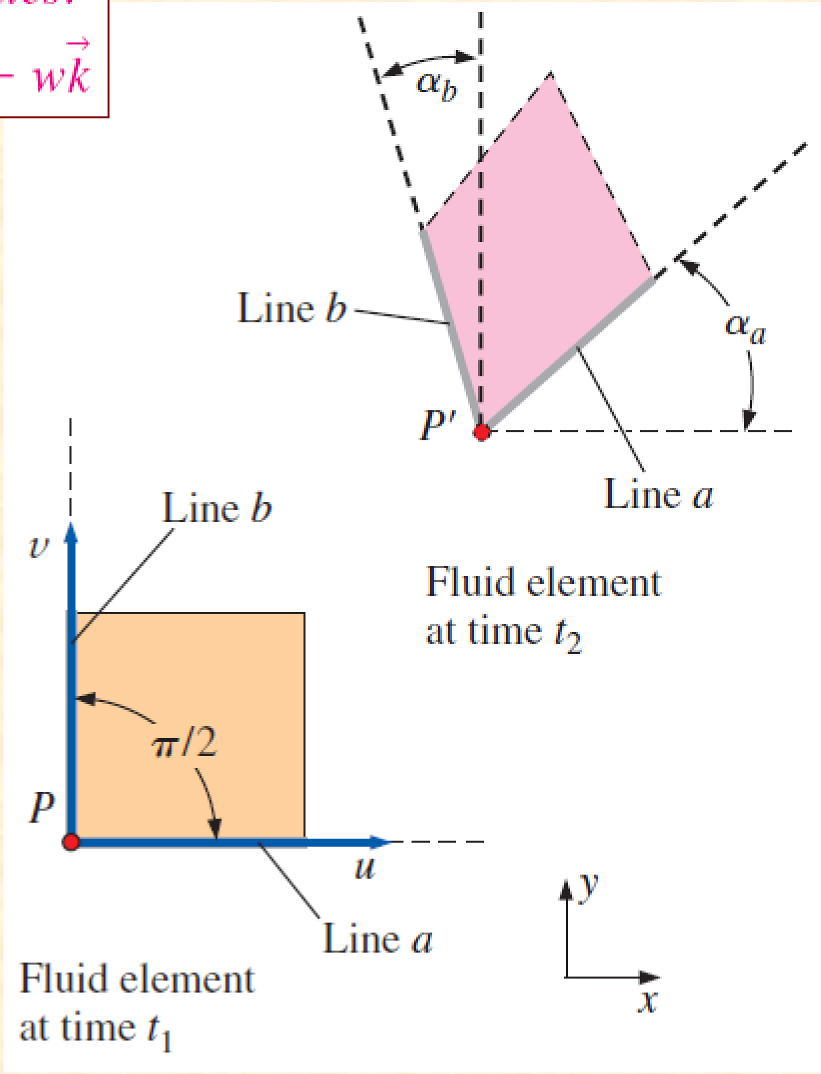
$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

Rate of rotation (angular velocity) at a point: *The average rotation rate of two initially perpendicular lines that intersect at that point*.

Rate of rotation of fluid element about point P

$$\omega = \frac{d}{dt}\left(\frac{\alpha_a + \alpha_b}{2}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

For a fluid element that translates and deforms as sketched, the *rate of rotation* at point *P* is defined as the average rotation rate of two initially perpendicular lines (lines *a* and *b*).



The **rate of rotation vector** is equal to the **angular velocity vector**.

Rate of rotation vector in Cartesian coordinates:

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

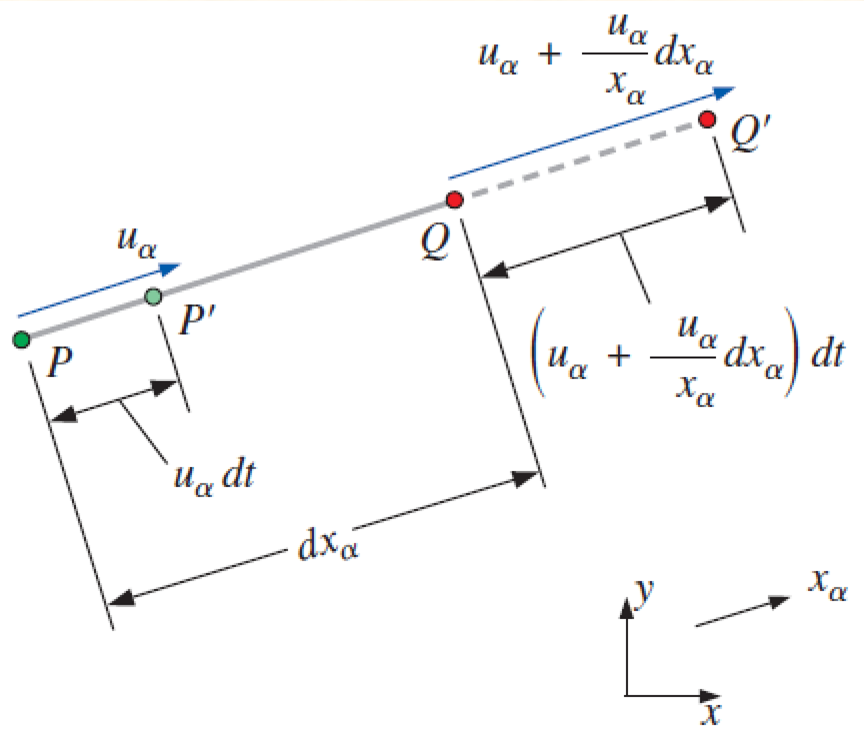
Linear strain rate: *The rate of increase in length per unit length*.

Mathematically, the linear strain rate of a fluid element depends on the initial orientation or direction of the line segment upon which we measure the linear strain.

Linear strain rate in Cartesian coordinates:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Linear strain rate in some arbitrary direction x_α is defined as the rate of increase in length per unit length in that direction. Linear strain rate would be *negative* if the line segment length were to *decrease*. Here we follow the increase in length of line segment PQ into line segment $P'Q'$, which yields a positive linear strain rate. Velocity components and distances are truncated to first-order since dx_α and dt are infinitesimally small.



Using the lengths marked in the figure, the linear strain rate in the x_a-direction is

$$\varepsilon_{\alpha\alpha} = \frac{d}{dt}\left(\frac{P'Q' - PQ}{PQ}\right)$$

$$\cong \frac{d}{dt}\left(\frac{\overbrace{\left(u_\alpha + \frac{\partial u_\alpha}{\partial x_\alpha} dx_\alpha\right) dt + dx_\alpha - u_\alpha\, dt}^{\text{Length of } P'Q' \text{ in the } x_\alpha\text{-direction}} \quad - \overbrace{dx_\alpha}^{\text{Length of } PQ \text{ in the } x_\alpha\text{-direction}}}{\underbrace{dx_\alpha}_{\text{Length of } PQ \text{ in the } x_\alpha\text{-direction}}}\right) = \frac{\partial u_\alpha}{\partial x_\alpha} \qquad \textbf{(4–22)}$$

Volumetric strain rate or **bulk strain rate:** The rate of increase of volume of a fluid element per unit volume.

This kinematic property is defined as *positive* when the volume *increases*.

Another synonym of volumetric strain rate is also called **rate of volumetric dilatation,** (the iris of your eye dilates (enlarges) when exposed to dim light).

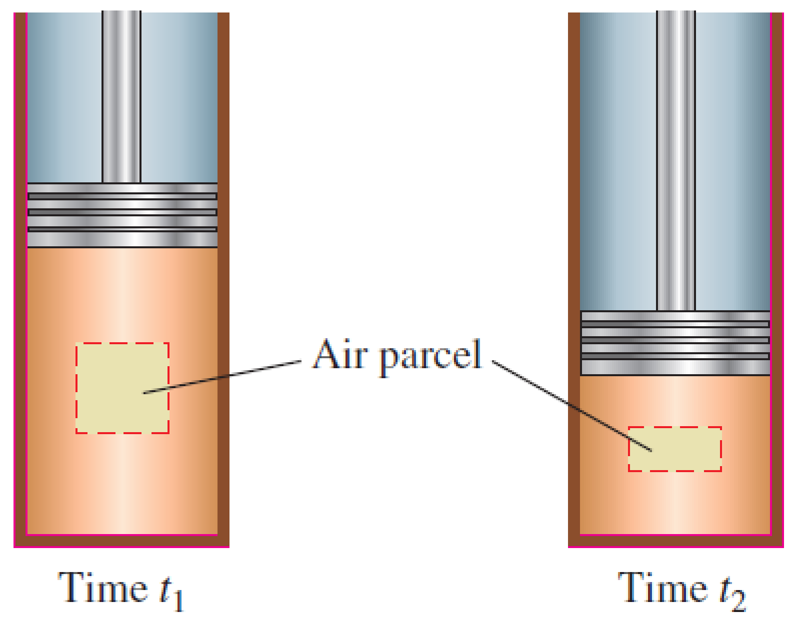
The volumetric strain rate is the sum of the linear strain rates in three mutually orthogonal directions.

Volumetric strain rate in Cartesian coordinates:

$$\frac{1}{V}\frac{DV}{Dt} = \frac{1}{V}\frac{dV}{dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The volumetric strain rate is zero in an incompressible flow.

Air being compressed by a piston in a cylinder; the volume of a fluid element in the cylinder decreases, corresponding to a negative rate of volumetric dilatation.



Shear strain rate at a point: *Half of the rate of decrease of the angle between two initially perpendicular lines that intersect at the point*.

Shear strain rate, initially perpendicular lines in the x- and y-directions:

$$\varepsilon_{xy} = -\frac{1}{2}\frac{d}{dt}\alpha_{a\text{-}b} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Shear strain rate in Cartesian coordinates:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Strain rate tensor in Cartesian coordinates:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

For a fluid element that translates and deforms as sketched, the *shear strain rate* at point *P* is defined as half of the rate of decrease of the angle between two initially perpendicular lines (lines *a* and *b*).

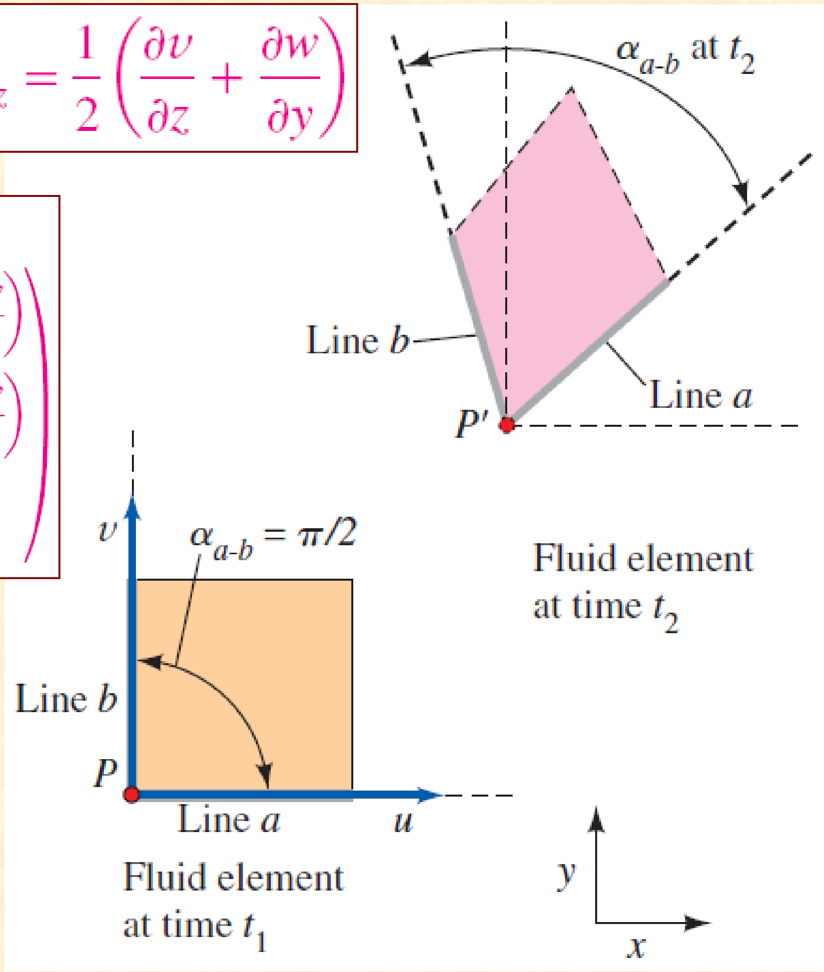
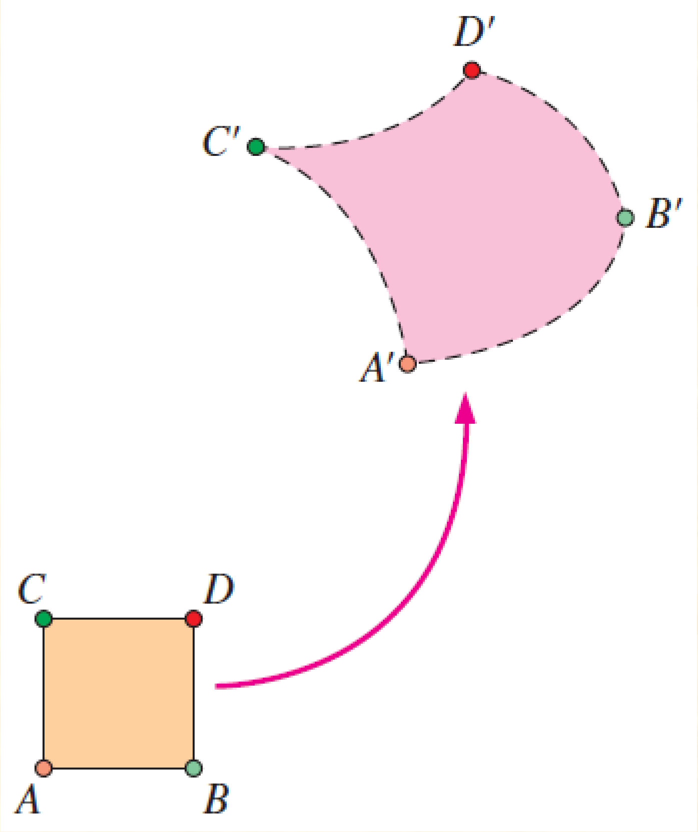


Figure shows a general (although two-dimensional) situation in a compressible fluid flow in which all possible motions and deformations are present simultaneously.

In particular, there is translation, rotation, linear strain, and shear strain.

Because of the compressible nature of the fluid flow, there is also volumetric strain (dilatation).

You should now have a better appreciation of the inherent complexity of fluid dynamics, and the mathematical sophistication required to fully describe fluid motion.



A fluid element illustrating translation, rotation, linear strain, shear strain, and volumetric strain.

EXAMPLE 4–6 **Calculation of Kinematic Properties in a Two-Dimensional Flow**

Consider the steady, two-dimensional velocity field of Example 4–1:

$$\vec{V} = (u, v) = (0.5 + 0.8\,x)\vec{i} + (1.5 - 0.8\,y)\vec{j} \quad (1)$$

where lengths are in units of m, time in s, and velocities in m/s. There is a stagnation point at (−0.625, 1.875) as shown in Fig. 4–41. Streamlines of the flow are also plotted in Fig. 4–41. Calculate the various kinematic properties, namely, the rate of translation, rate of rotation, linear strain rate, shear strain rate, and volumetric strain rate. Verify that this flow is incompressible.

SOLUTION We are to calculate several kinematic properties of a given velocity field and verify that the flow is incompressible.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional, implying no z-component of velocity and no variation of u or v with z.

Analysis By Eq. 4–19, the rate of translation is simply the velocity vector itself, given by Eq. 1. Thus,

Rate of translation: $$u = \mathbf{0.5 + 0.8x} \qquad v = \mathbf{1.5 - 0.8y} \qquad w = \mathbf{0} \quad (2)$$

The rate of rotation is found from Eq. 4–21. In this case, since $w = 0$ everywhere, and since neither u nor v vary with z, the only nonzero component of rotation rate is in the z-direction. Thus,

Rate of rotation: $$\vec{\omega} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = \frac{1}{2}(0 - 0)\vec{k} = \mathbf{0} \quad (3)$$

In this case, we see that there is no net rotation of fluid particles as they move about. (This is a significant piece of information, to be discussed in more detail later in this chapter and also in Chap. 10.)

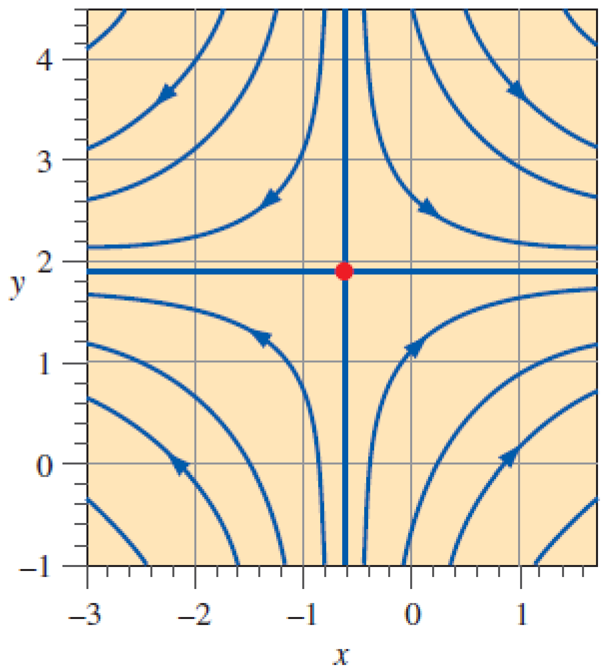


FIGURE 4–41
Streamlines for the velocity field of Example 4–6. The stagnation point is indicated by the red circle at $x = -0.625$ m and $y = 1.875$ m.

Linear strain rates can be calculated in any arbitrary direction using Eq. 4–23. In the *x*-, *y*-, and *z*-directions, the linear strain rates are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \mathbf{0.8\ s^{-1}} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \mathbf{-0.8\ s^{-1}} \qquad \varepsilon_{zz} = \mathbf{0} \tag{4}$$

Thus, we predict that fluid particles *stretch* in the *x*-direction (positive linear strain rate) and *shrink* in the *y*-direction (negative linear strain rate). This is illustrated in Fig. 4–42, where we have marked an initially square parcel of fluid centered at (0.25, 4.25). By integrating Eqs. 2 with time, we calculate the location of the four corners of the marked fluid after an elapsed time of 1.5 s. Indeed this fluid parcel has stretched in the *x*-direction and has shrunk in the *y*-direction as predicted.

Shear strain rate is determined from Eq. 4–26. Because of the two-dimensionality, nonzero shear strain rates can occur only in the *xy*-plane. Using lines parallel to the *x*- and *y*-axes as our initially perpendicular lines, we calculate ε_{xy},

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(0 + 0) = \mathbf{0} \tag{5}$$

Thus, there is no shear strain in this flow, as also indicated by Fig. 4–42. Although the sample fluid particle deforms, it remains rectangular; its initially 90° corner angles remain at 90° throughout the time period of the calculation.

Finally, the volumetric strain rate is calculated from Eq. 4–24:

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = (0.8 - 0.8 + 0)\ s^{-1} = \mathbf{0} \tag{6}$$

Since the volumetric strain rate is zero everywhere, we can say definitively that fluid particles are neither dilating (expanding) nor shrinking (compressing) in volume. Thus, **we verify that this flow is indeed incompressible.** In Fig. 4–42, the area of the shaded fluid particle (and thus its volume since it is a 2-D flow) remains constant as it moves and deforms in the flow field.

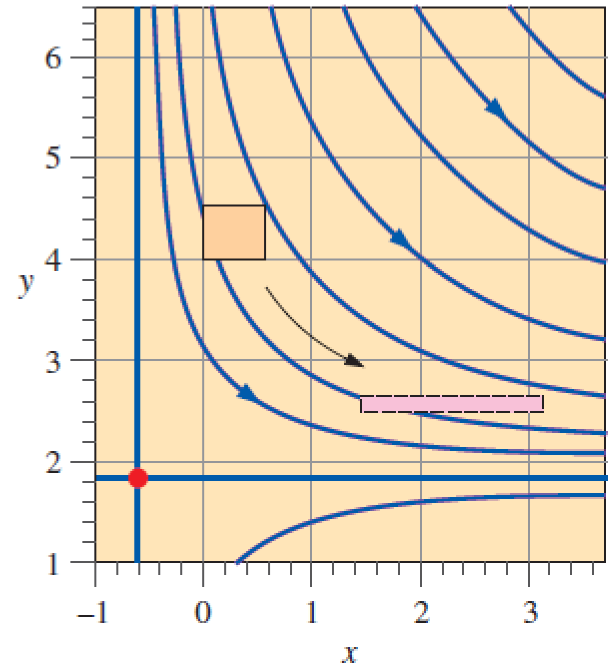


FIGURE 4–42
Deformation of an initially square parcel of marked fluid subjected to the velocity field of Example 4–6 for a time period of 1.5 s. The stagnation point is indicated by the red circle at $x = -0.625$ m and $y = 1.875$ m, and several streamlines are plotted.

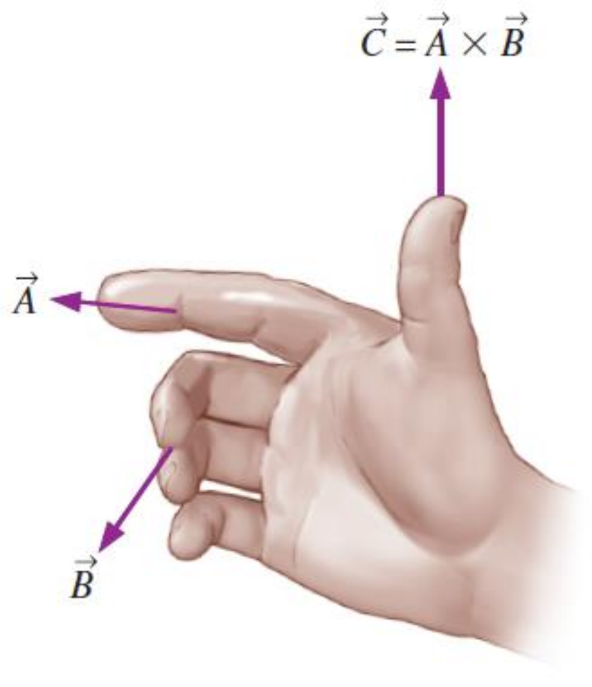
4–5 ■ VORTICITY AND ROTATIONALITY

Another kinematic property of great importance to the analysis of fluid flows is the **vorticity vector**, defined mathematically as the curl of the velocity vector

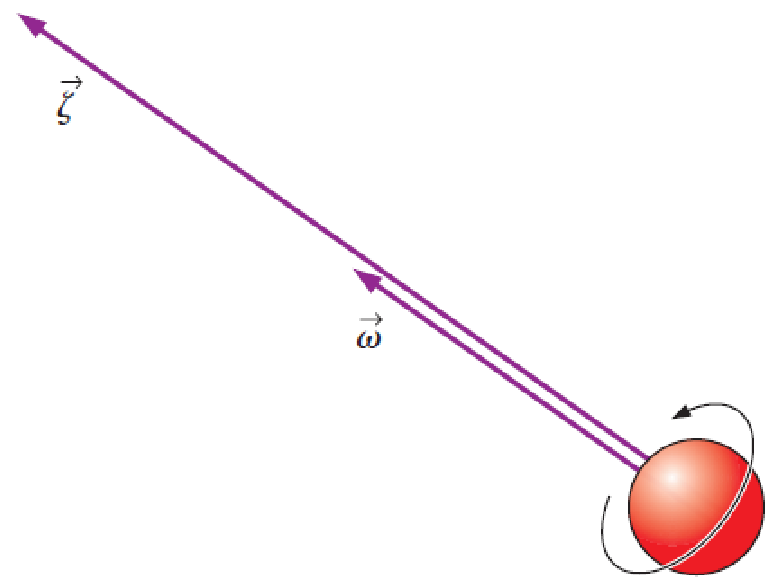
Vorticity vector: $$\vec{\zeta} = \vec{\nabla} \times \vec{V} = \text{curl}(\vec{V})$$

Rate of rotation vector: $$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V} = \frac{1}{2}\text{curl}(\vec{V}) = \frac{\vec{\zeta}}{2}$$

Vorticity is equal to twice the angular velocity of a fluid particle



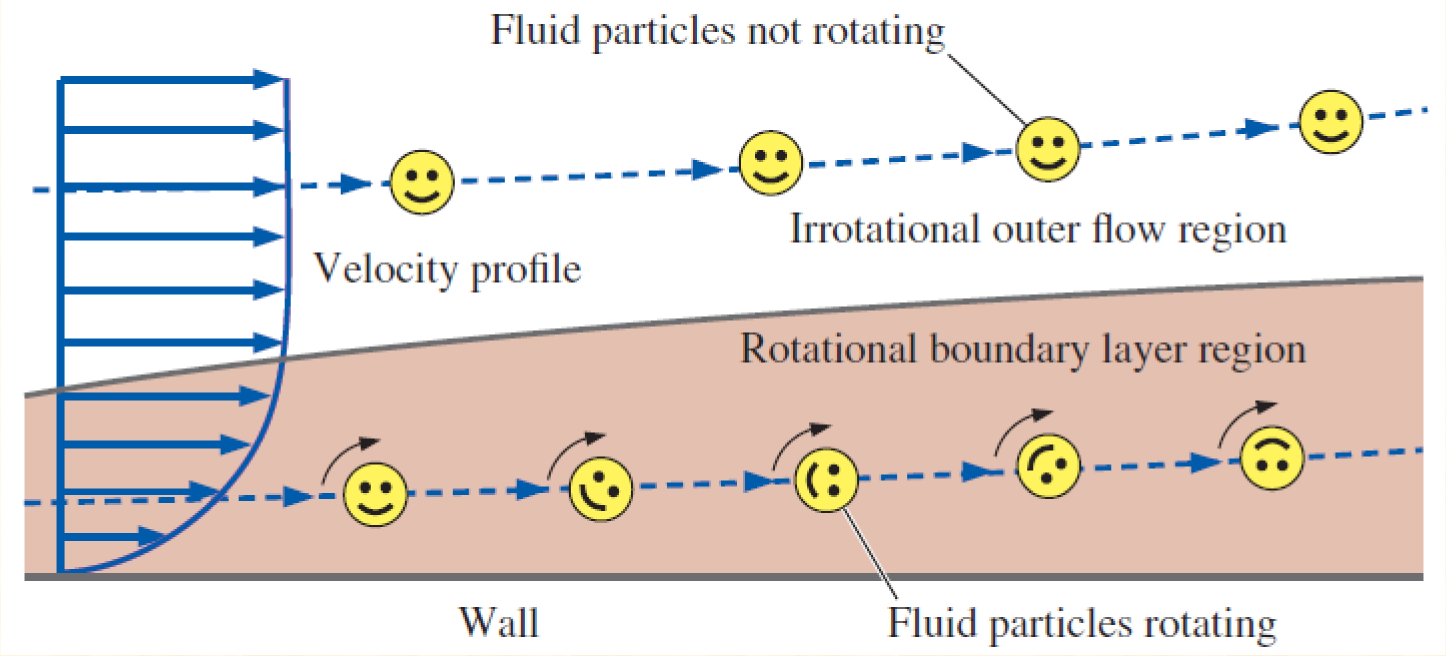
The direction of a vector cross product is determined by the right-hand rule.



The *vorticity vector* is equal to twice the angular velocity vector of a rotating fluid particle.

- If the vorticity at a point in a flow field is nonzero, the fluid particle that happens to occupy that point in space is rotating; the flow in that region is called **rotational**.
- Likewise, if the vorticity in a region of the flow is zero (or negligibly small), fluid particles there are not rotating; the flow in that region is called **irrotational**.
- Physically, fluid particles in a rotational region of flow rotate end over end as they move along in the flow.

The difference between rotational and irrotational flow: fluid elements in a rotational region of the flow rotate, but those in an irrotational region of the flow do not.

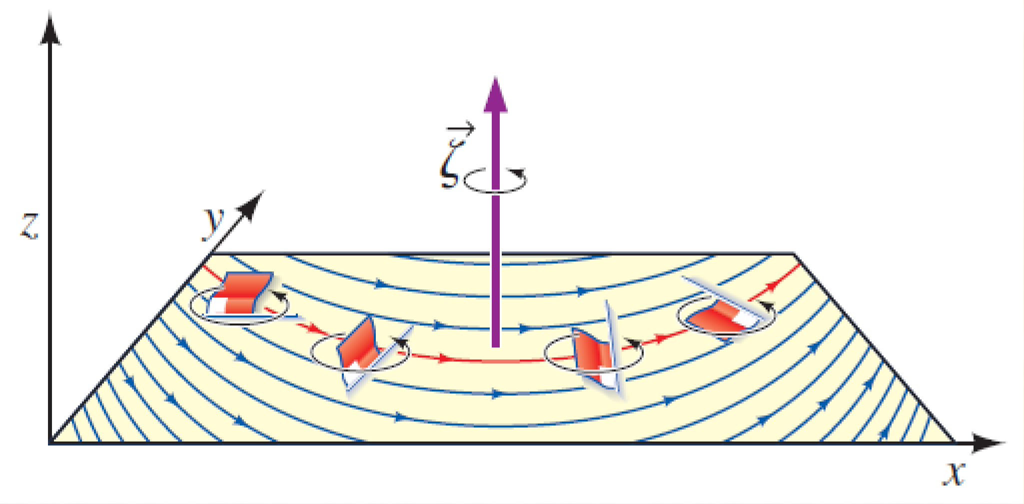


Vorticity vector in Cartesian coordinates:

$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Two-dimensional flow in Cartesian coordinates:

$$\vec{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$



For a two-dimensional flow in the *xy*-plane, the vorticity vector always points in the *z*- or *z*-direction. In this illustration, the flag-shaped fluid particle rotates in the counterclockwise direction as it moves in the *xy*-plane; its vorticity points in the positive *z*-direction as shown.

EXAMPLE 4–7 **Vorticity Contours in a Two-Dimensional Flow**

Consider the CFD calculation of two-dimensional free-stream flow impinging on a block of rectangular cross section, as shown in Figs. 4–33 and 4–34. Plot vorticity contours and discuss.

SOLUTION We are to calculate the vorticity field for a given velocity field produced by CFD and then generate a contour plot of vorticity.

Analysis Since the flow is two-dimensional, the only nonzero component of vorticity is in the *z*-direction, normal to the page in Figs. 4–33 and 4–34. A contour plot of the *z*-component of vorticity for this flow field is shown in Fig. 4–47. The blue region near the upper-left corner of the block indicates large negative values of vorticity, implying *clockwise* rotation of fluid particles in that region. This is due to the large velocity gradients encountered in this portion of the flow field; the boundary layer separates off the wall at the corner of the body and forms a thin **shear layer** across which the velocity changes rapidly. The concentration of vorticity in the shear layer diminishes as vorticity diffuses downstream. The small red region near the top right corner of the block represents a region of *positive* vorticity (counterclockwise rotation)—a secondary flow pattern caused by the flow separation.

Discussion We expect the magnitude of vorticity to be highest in regions where spatial derivatives of velocity are high (see Eq. 4–30). Close examination reveals that the blue region in Fig. 4–47 does indeed correspond to large velocity gradients in Fig. 4–33. Keep in mind that the vorticity field of Fig. 4–47 is time-averaged. The instantaneous flow field is in reality turbulent and unsteady, and vortices are shed from the bluff body.

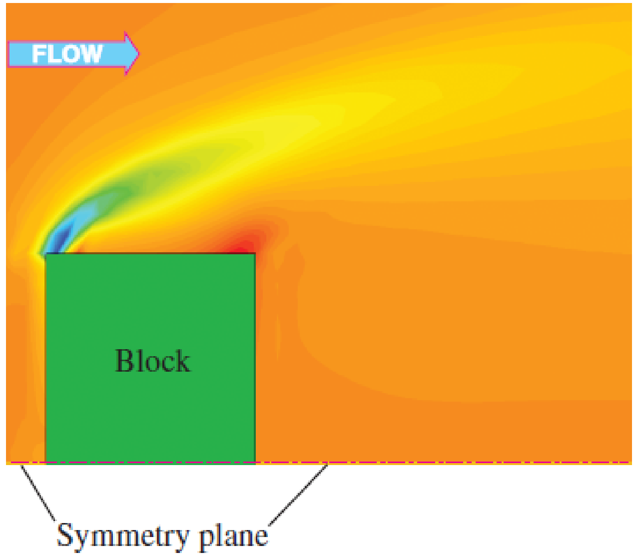


FIGURE 4–47
Contour plot of the vorticity field ζ_z due to flow impinging on a block, as produced by CFD calculations; only the upper half is shown due to symmetry. Blue regions represent large negative vorticity, and red regions represent large positive vorticity.

EXAMPLE 4–8 Determination of Rotationality in a Two-Dimensional Flow

Consider the following steady, incompressible, two-dimensional velocity field:

$$\vec{V} = (u, v) = x^2\vec{i} + (-2xy - 1)\vec{j} \qquad (1)$$

Is this flow rotational or irrotational? Sketch some streamlines in the first quadrant and discuss.

SOLUTION We are to determine whether a flow with a given velocity field is rotational or irrotational, and we are to draw some streamlines in the first quadrant.

Analysis Since the flow is two-dimensional, Eq. 4–31 is applicable. Thus,

$$\text{Vorticity:} \qquad \vec{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = (-2y - 0)\vec{k} = -2y\vec{k} \qquad (2)$$

Since the vorticity is nonzero, this flow is **rotational.** In Fig. 4–48 we plot several streamlines of the flow in the first quadrant; we see that fluid moves downward and to the right. The translation and deformation of a fluid parcel is also shown: at $\Delta t = 0$, the fluid parcel is square, at $\Delta t = 0.25$ s, it has moved and deformed, and at $\Delta t = 0.50$ s, the parcel has moved farther and is further deformed. In particular, the right-most portion of the fluid parcel moves faster to the right and faster downward compared to the left-most portion, stretching the parcel in the *x*-direction and squashing it in the vertical direction. It is clear that there is also a net *clockwise* rotation of the fluid parcel, which agrees with the result of Eq. 2.

Discussion From Eq. 4–29, individual fluid particles rotate at an angular velocity equal to $\vec{\omega} = -y\vec{k}$, half of the vorticity vector. Since $\vec{\omega}$ is not constant, this flow is *not* solid-body rotation. Rather, $\vec{\omega}$ is a linear function of *y*. Further analysis reveals that this flow field is incompressible; the area (and volume) of the shaded regions representing the fluid parcel in Fig. 4–48 remains constant at all three instants in time.

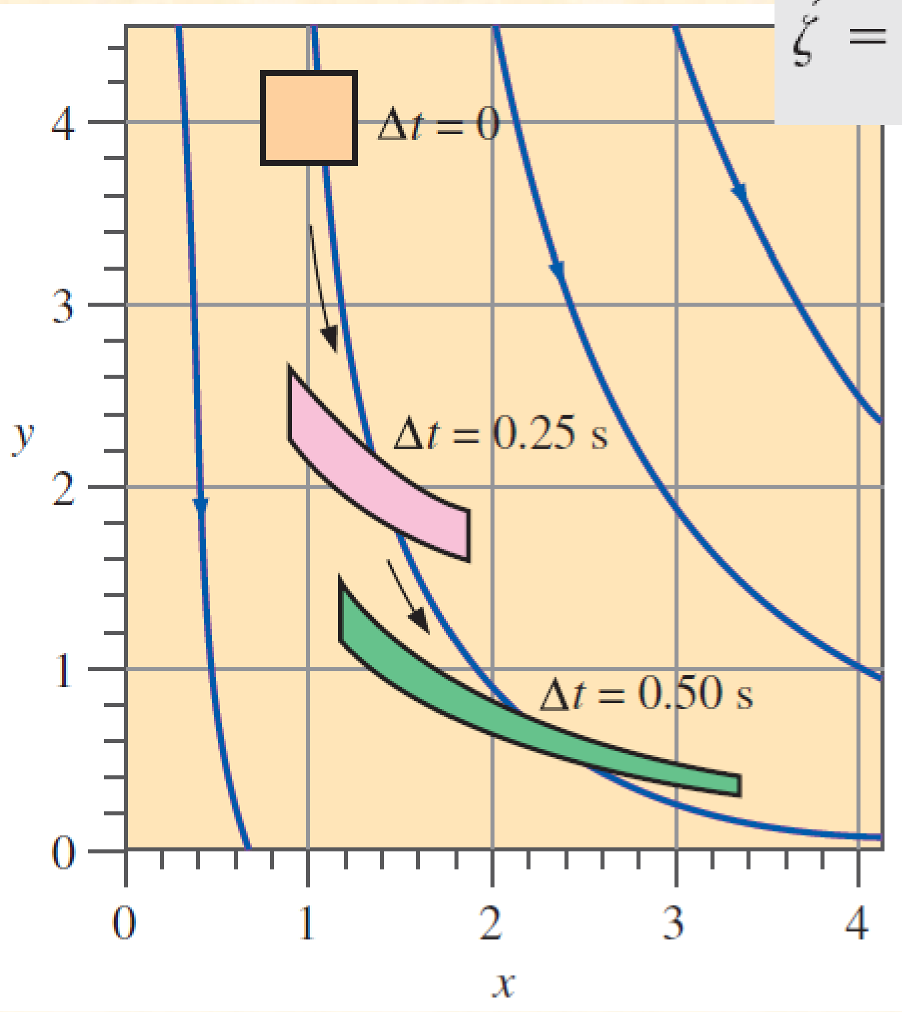
Determination of Rotationality in a Two-Dimensional Flow

steady, incompressible, two-dimensional velocity field:

$$\vec{V} = (u, v) = x^2\vec{i} + (-2xy - 1)\vec{j}$$

Vorticity:

$$\vec{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = (-2y - 0)\vec{k} = -2y\vec{k}$$



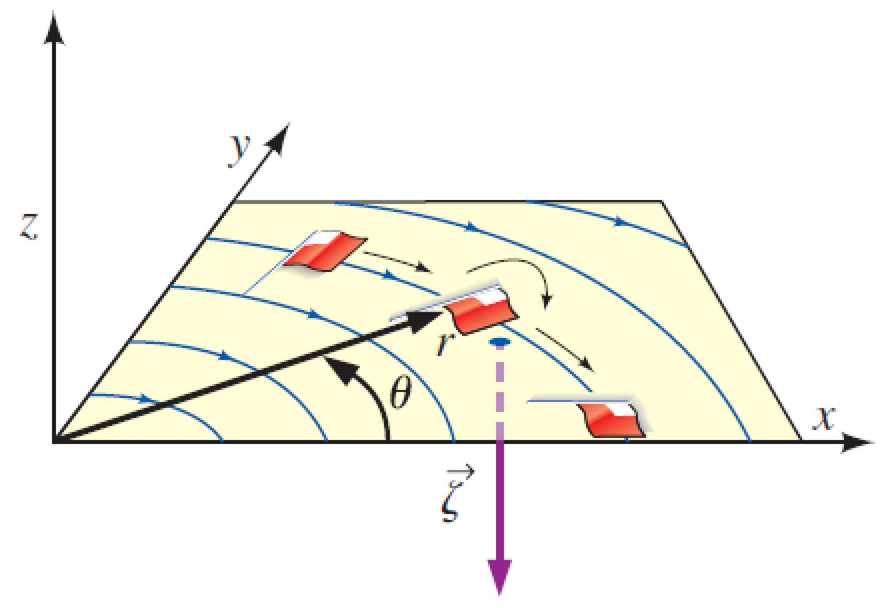
Deformation of an initially square fluid parcel subjected to the velocity field of Example 4–8 for a time period of 0.25 s and 0.50 s. Several streamlines are also plotted in the first quadrant. It is clear that this flow is *rotational*.

Vorticity vector in cylindrical coordinates:

$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z$$

Two-dimensional flow in cylindrical coordinates:

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{k}$$



For a two-dimensional flow in the $r\theta$-plane, the vorticity vector always points in the z (or z) direction. In this illustration, the flag-shaped fluid particle rotates in the clockwise direction as it moves in the *r*u-plane; its vorticity points in the z-direction as shown.

Comparison of Two Circular Flows

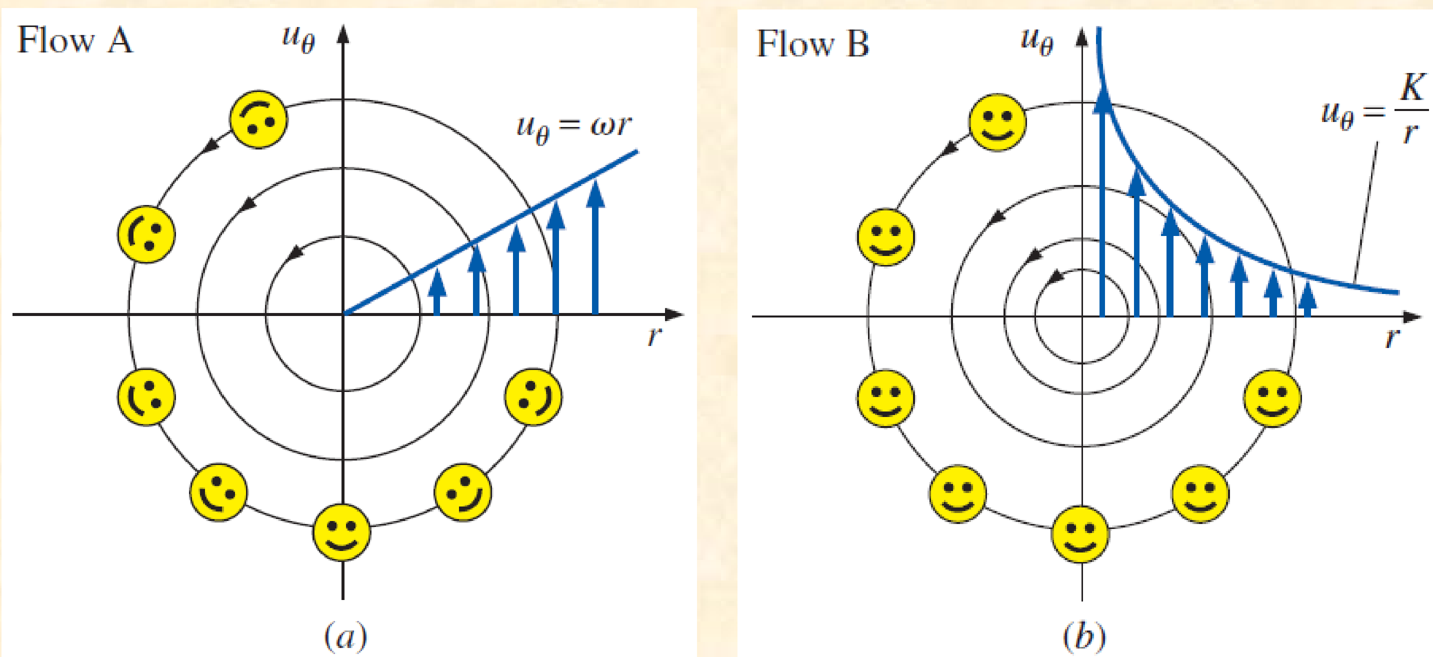
Flow A—solid-body rotation: $u_r = 0$ and $u_\theta = \omega r$

Flow B—line vortex: $u_r = 0$ and $u_\theta = \dfrac{K}{r}$

Flow A—solid-body rotation: $$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{k} = 2\omega\vec{k}$$

Flow B—line vortex: $$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{k} = 0$$

Streamlines and velocity profiles for (*a*) flow A, solid-body rotation and (*b*) flow B, a line vortex. Flow A is rotational, but flow B is irrotational everywhere except at the origin.



A simple analogy can be made between flow A and a merry-go-round or roundabout, and flow B and a Ferris wheel.

As children revolve around a roundabout, they also rotate at the same angular velocity as that of the ride itself. This is analogous to a rotational flow.

In contrast, children on a Ferris wheel always remain oriented in an upright position as they trace out their circular path. This is analogous to an irrotational flow.

A simple analogy: (*a*) *rotational* circular flow is analogous to a roundabout, while (*b*) *irrotational* circular flow is analogous to a Ferris wheel.

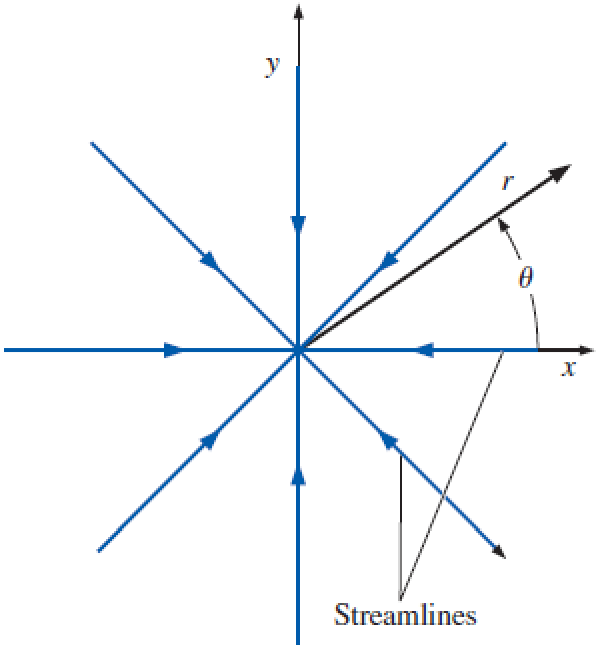


FIGURE 4–52
Streamlines in the $r\theta$-plane for the case of a line sink.

EXAMPLE 4–9 Determination of Rotationality of a Line Sink

A simple two-dimensional velocity field called a **line sink** is often used to simulate fluid being sucked into a line along the z-axis. Suppose the volume flow rate per unit length along the z-axis, $\dot{V}/L$, is known, where $\dot{V}$ is a negative quantity. In two dimensions in the $r\theta$-plane,

Line sink:
$$u_r = \frac{\dot{V}}{2\pi L}\frac{1}{r} \quad \text{and} \quad u_\theta = 0 \tag{1}$$

Draw several streamlines of the flow and calculate the vorticity. Is this flow rotational or irrotational?

SOLUTION Streamlines of the given flow field are to be sketched and the rotationality of the flow is to be determined.

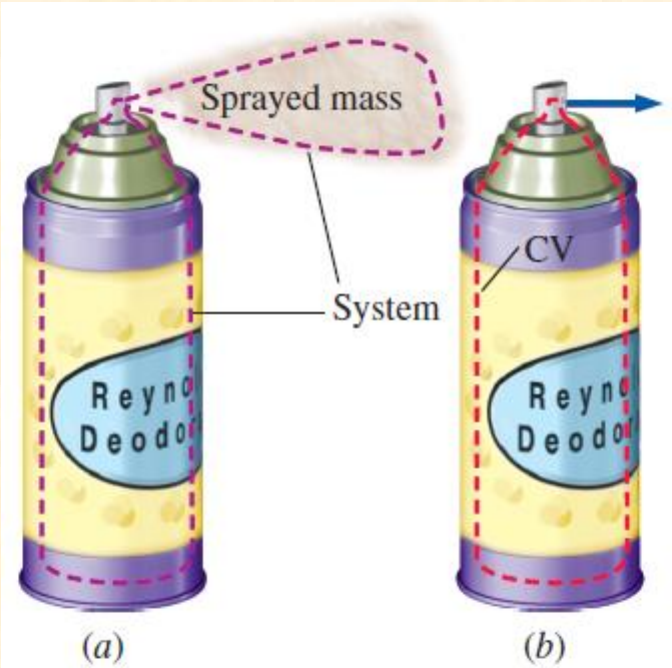
Analysis Since there is only radial flow and no tangential flow, we know immediately that all streamlines must be rays into the origin. Several streamlines are sketched in Fig. 4–52. The vorticity is calculated from Eq. 4–33:

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial}{\partial\theta}u_r\right)\vec{k} = \frac{1}{r}\left(0 - \frac{\partial}{\partial\theta}\left(\frac{\dot{V}}{2\pi L}\frac{1}{r}\right)\right)\vec{k} = 0 \tag{2}$$

Since the vorticity vector is everywhere zero, this flow field is **irrotational**.

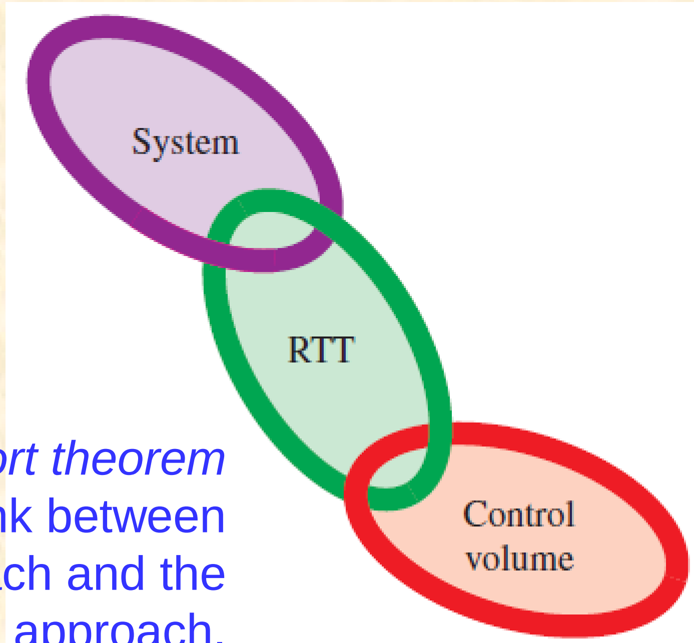
Discussion Many practical flow fields involving suction, such as flow into inlets and hoods, can be approximated quite accurately by assuming irrotational flow (Heinsohn and Cimbala, 2003).

4–6 ■ THE REYNOLDS TRANSPORT THEOREM

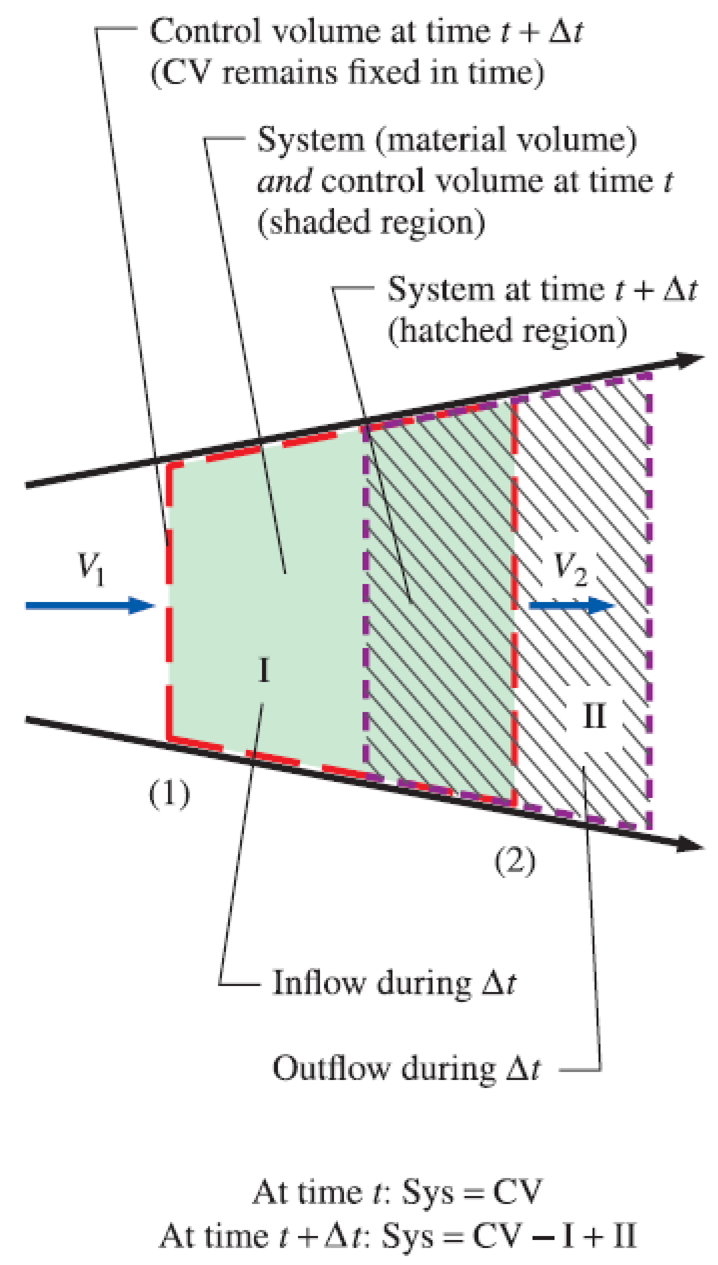


Two methods of analyzing the spraying of deodorant from a spray can:
(***a***) We follow the fluid as it moves and deforms. This is the *system approach*—no mass crosses the boundary, and the total mass of the system remains fixed.
(***b***) We consider a fixed interior volume of the can. This is the *control volume approach*—mass crosses the boundary.

The relationship between the time rates of change of an extensive property for a system and for a control volume is expressed by the **Reynolds transport theorem (RTT)**.



The *Reynolds transport theorem* (RTT) provides a link between the system approach and the control volume approach.



$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} - \dot{B}_{in} + \dot{B}_{out}$$

The time rate of change of the property *B* of the system is equal to the time rate of change of *B* of the control volume plus the net flux of *B* out of the control volume by mass crossing the control surface.

This equation applies at any instant in time, where it is assumed that the system and the control volume occupy the same space at that particular instant in time.

A moving *system* (hatched region) and a fixed *control volume* (shaded region) in a diverging portion of a flow field at times *t* and *t*+Δ*t*. The upper and lower bounds are streamlines of the flow.

Let B represent any **extensive property** (such as mass, energy, or momentum), and let $b = B/m$ represent the corresponding **intensive property**. Noting that extensive properties are additive, the extensive property B of the system at times t and $t + \Delta t$ is expressed as

$$B_{\text{sys},\,t} = B_{\text{CV},\,t} \qquad \text{(the system and CV concide at time } t\text{)}$$

$$B_{\text{sys},\,t+\Delta t} = B_{\text{CV},\,t+\Delta t} - B_{\text{I},\,t+\Delta t} + B_{\text{II},\,t+\Delta t}$$

Subtracting the first equation from the second one and dividing by Δt gives

$$\frac{B_{\text{sys},\,t+\Delta t} - B_{\text{sys},\,t}}{\Delta t} = \frac{B_{\text{CV},\,t+\Delta t} - B_{\text{CV},\,t}}{\Delta t} - \frac{B_{\text{I},\,t+\Delta t}}{\Delta t} + \frac{B_{\text{II},\,t+\Delta t}}{\Delta t}$$

Taking the limit as $\Delta t \to 0$, and using the definition of derivative, we get

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{CV}}}{dt} - \dot{B}_{\text{in}} + \dot{B}_{\text{out}} \qquad \textbf{(4–38)}$$

or

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{CV}}}{dt} - b_1\rho_1 V_1 A_1 + b_2\rho_2 V_2 A_2$$

since

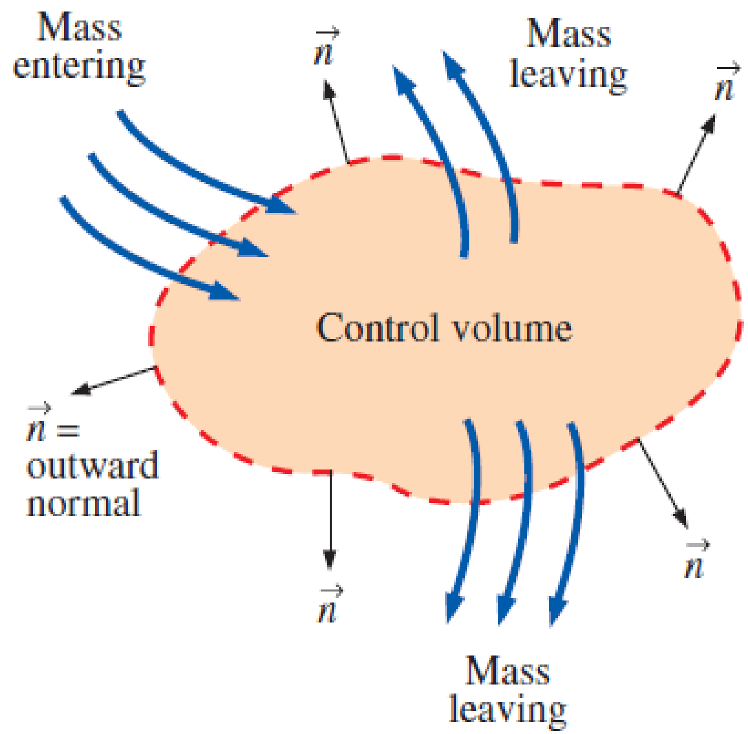
$$B_{\text{I},\,t+\Delta t} = b_1 m_{\text{I},\,t+\Delta t} = b_1\rho_1 V_{\text{I},\,t+\Delta t} = b_1\rho_1 V_1\,\Delta t\,A_1$$

$$B_{\text{II},\,t+\Delta t} = b_2 m_{\text{II},\,t+\Delta t} = b_2\rho_2 V_{\text{II},\,t+\Delta t} = b_2\rho_2 V_2\,\Delta t\,A_2$$

and

$$\dot{B}_{\text{in}} = \dot{B}_{\text{I}} = \lim_{\Delta t\to 0}\frac{B_{\text{I},\,t+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{b_1\rho_1 V_1\,\Delta t\,A_1}{\Delta t} = b_1\rho_1 V_1 A_1$$

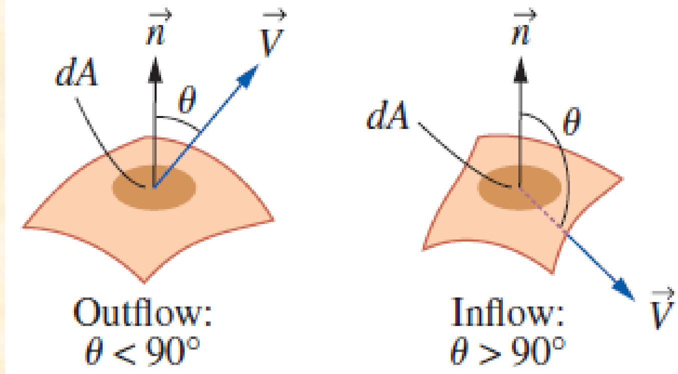
$$\dot{B}_{\text{out}} = \dot{B}_{\text{II}} = \lim_{\Delta t\to 0}\frac{B_{\text{II},\,t+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{b_2\rho_2 V_2\,\Delta t\,A_2}{\Delta t} = b_2\rho_2 V_2 A_2$$



$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$

The integral of $b\rho\vec{V} \cdot \vec{n}\, dA$ over the control surface gives the net amount of the property B flowing out of the control volume (into the control volume if it is negative) per unit time.

$$B_{CV} = \int_{CV} \rho b\, dV$$



$\vec{V} \cdot \vec{n} = |\vec{V}||\vec{n}| \cos\theta = V\cos\theta$

If $\theta < 90°$, then $\cos\theta > 0$ (outflow).

If $\theta > 90°$, then $\cos\theta < 0$ (inflow).

If $\theta = 90°$, then $\cos\theta = 0$ (no flow).

Outflow and inflow of mass across the differential area of a control surface.

RTT, fixed CV:

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$

Alternate RTT, fixed CV:

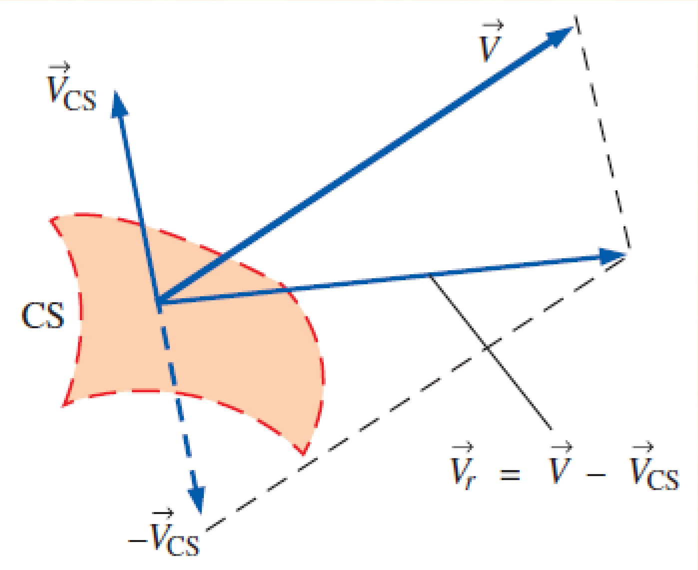
$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$

Relative velocity: $\vec{V}_r = \vec{V} - \vec{V}_{CS}$

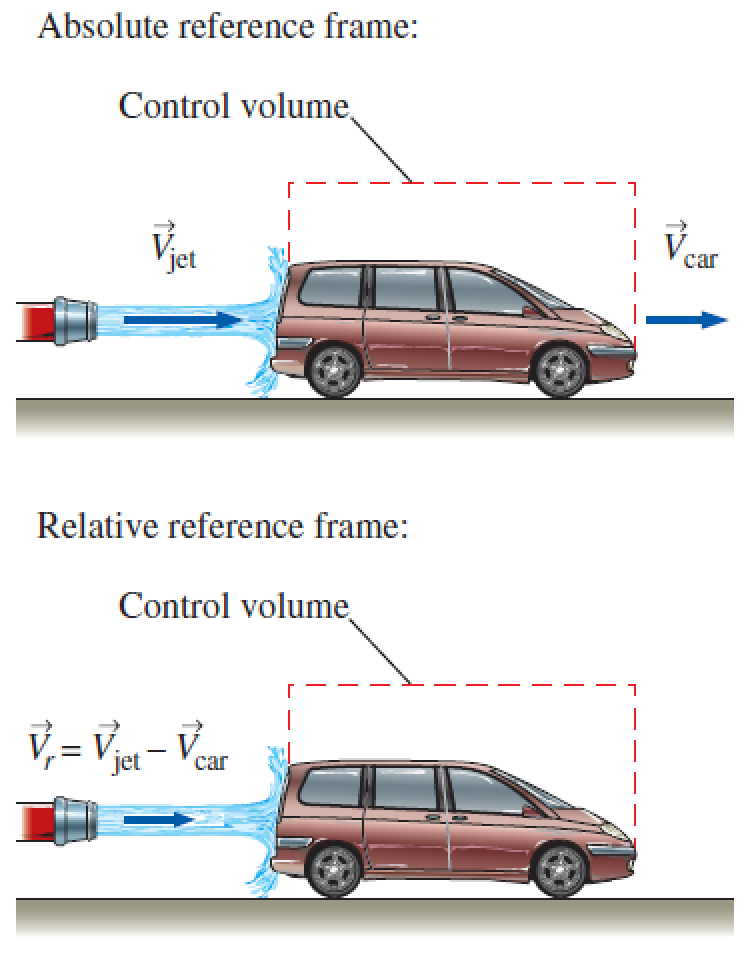
RTT, nonfixed CV: $$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA$$

RTT, steady flow: $$\frac{dB_{sys}}{dt} = \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA$$

Reynolds transport theorem applied to a control volume moving at constant velocity.



Relative velocity crossing a control surface is found by vector addition of the absolute velocity of the fluid and the negative of the local velocity of the control surface.



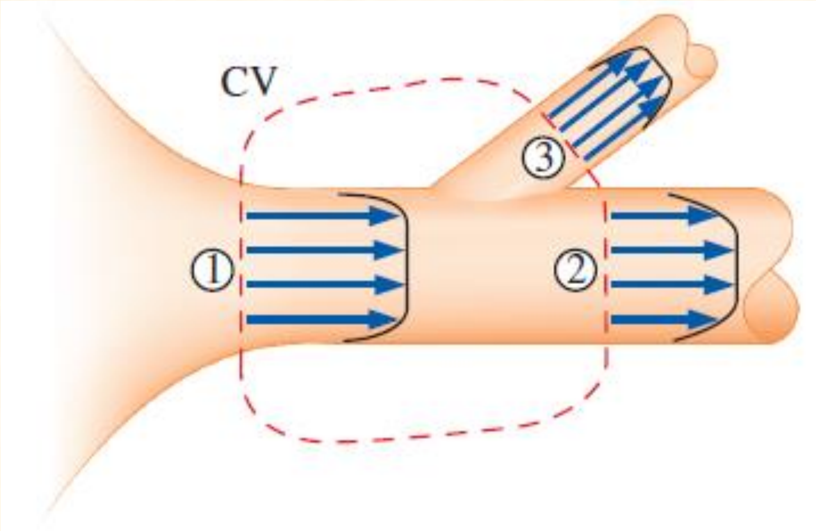
$$\int_A \rho b \vec{V}_r \cdot \vec{n}\, dA \cong b_{avg} \int_A \rho \vec{V}_r \cdot \vec{n}\, dA = b_{avg} \dot{m}_r$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \int_{CV} \rho b\, dV + \sum_{out} \underbrace{\dot{m}_r b_{avg}}_{\text{for each outlet}} - \sum_{in} \underbrace{\dot{m}_r b_{avg}}_{\text{for each inlet}}$$

Approximate RTT for well-defined inlets and outlets:

$$\dot{m}_r \approx \rho_{avg} \dot{V}_r = \rho_{avg} V_{r,\,avg} A$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \int_{CV} \rho b\, dV + \sum_{out} \underbrace{\rho_{avg} b_{avg} V_{r,\,avg} A}_{\text{for each outlet}} - \sum_{in} \underbrace{\rho_{avg} b_{avg} V_{r,avg} A}_{\text{for each inlet}}$$



An example control volume in which there is one well-defined inlet (1) and two well-defined outlets (2 and 3). In such cases, the control surface integral in the RTT can be more conveniently written in terms of the average values of fluid properties crossing each inlet and outlet.

Alternate Derivation of the Reynolds Transport Theorem

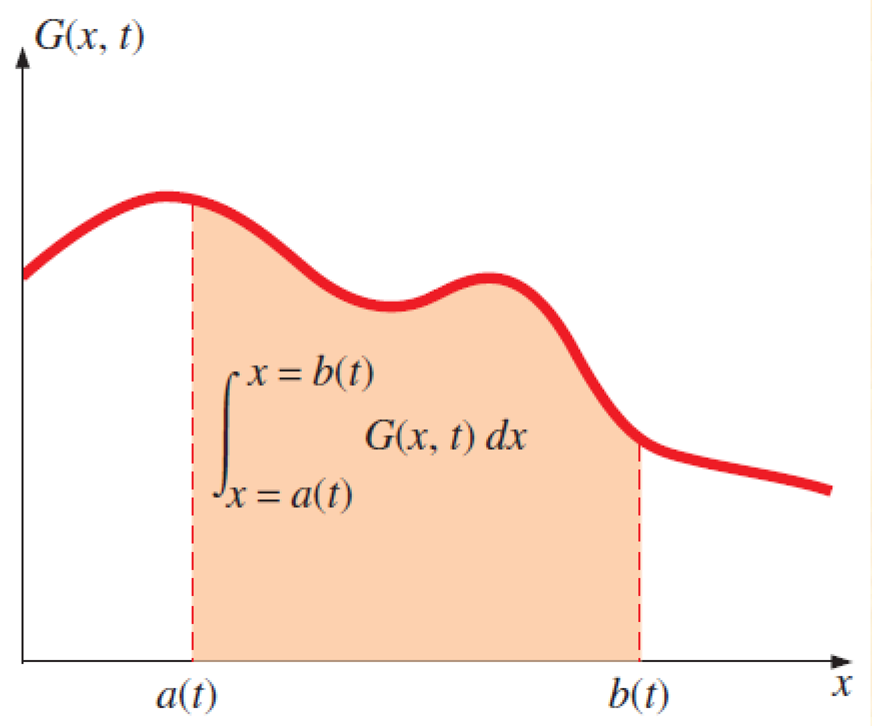
One-dimensional Leibniz theorem:

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} G(x,t)\,dx = \int_a^b \frac{\partial G}{\partial t}\,dx + \frac{db}{dt}G(b,t) - \frac{da}{dt}G(a,t)$$

A more elegant mathematical derivation of the Reynolds transport theorem is possible through use of the **Leibniz theorem**

The Leibniz theorem takes into account the change of limits $a(t)$ and $b(t)$ with respect to time, as well as the unsteady changes of integrand $G(x, t)$ with time.

The *one-dimensional Leibniz theorem* is required when calculating the time derivative of an integral (with respect to x) *for which the limits of the* integral are functions of time.



EXAMPLE 4–10 **One-Dimensional Leibniz Integration**

Reduce the following expression as far as possible:

$$F(t) = \frac{d}{dt}\int_{x=0}^{x=Ct} e^{-x^2}\,dx \tag{1}$$

SOLUTION *F*(*t*) is to be evaluated from the given expression.

Analysis We could try integrating first and then differentiating, but since Eq. 1 is of the form of Eq. 4–49, we use the one-dimensional Leibniz theorem. Here, $G(x, t) = e^{-x^2}$ (G is not a function of time in this simple example). The limits of integration are $a(t) = 0$ and $b(t) = Ct$. Thus,

$$F(t) = \int_a^b \underbrace{\frac{\partial G}{\partial t}}_{0} dx + \underbrace{\frac{db}{dt}}_{C} \underbrace{G(b, t)}_{e^{-b^2}} - \underbrace{\frac{da}{dt}}_{0} G(a, t) \quad \rightarrow \quad \mathbf{F(t) = Ce^{-C^2t^2}} \tag{2}$$

Discussion You are welcome to try to obtain the same solution without using the Leibniz theorem.

In three dimensions, the Leibniz theorem for a *volume* integral is

Three-dimensional Leibniz theorem:

$$\frac{d}{dt}\int_{V(t)} G(x, y, z, t)\, dV = \int_{V(t)} \frac{\partial G}{\partial t}\, dV + \int_{A(t)} G\vec{V}_A \cdot \vec{n}\, dA \qquad \textbf{(4–50)}$$

where $V(t)$ is a moving and/or deforming volume (a function of time), $A(t)$ is its surface (boundary), and $\vec{V}_A$ is the absolute velocity of this (moving) surface (Fig. 4–62). Equation 4–50 is valid for *any* volume, moving and/or deforming arbitrarily in space and time. For consistency with the previous analyses, we set integrand G to ρb for application to fluid flow,

Three-dimensional Leibniz theorem applied to fluid flow:

$$\frac{d}{dt}\int_{V(t)} \rho b\, dV = \int_{V(t)} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{A(t)} \rho b\vec{V}_A \cdot \vec{n}\, dA \qquad \textbf{(4–51)}$$

If we apply the Leibniz theorem to the special case of a **material volume** (a system of fixed identity moving with the fluid flow), then $\vec{V}_A = \vec{V}$ everywhere on the material surface since it moves *with* the fluid. Here $\vec{V}$ is the local fluid velocity, and Eq. 4–51 becomes

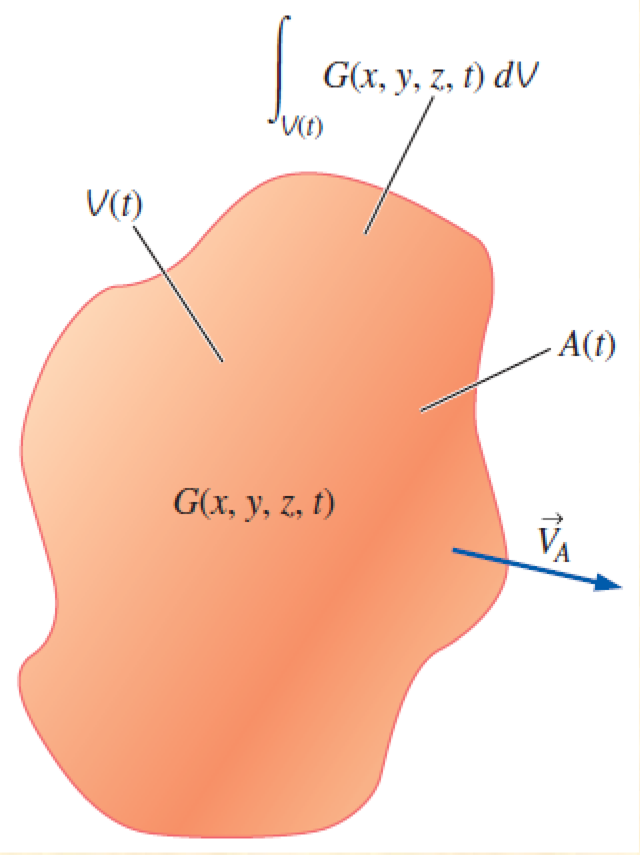
Three-dimensional Leibniz theorem:

$$\frac{d}{dt}\int_{V(t)} G(x, y, z, t)\, dV = \int_{V(t)} \frac{\partial G}{\partial t}\, dV + \int_{A(t)} G\vec{V}_A \cdot \vec{n}\, dA$$

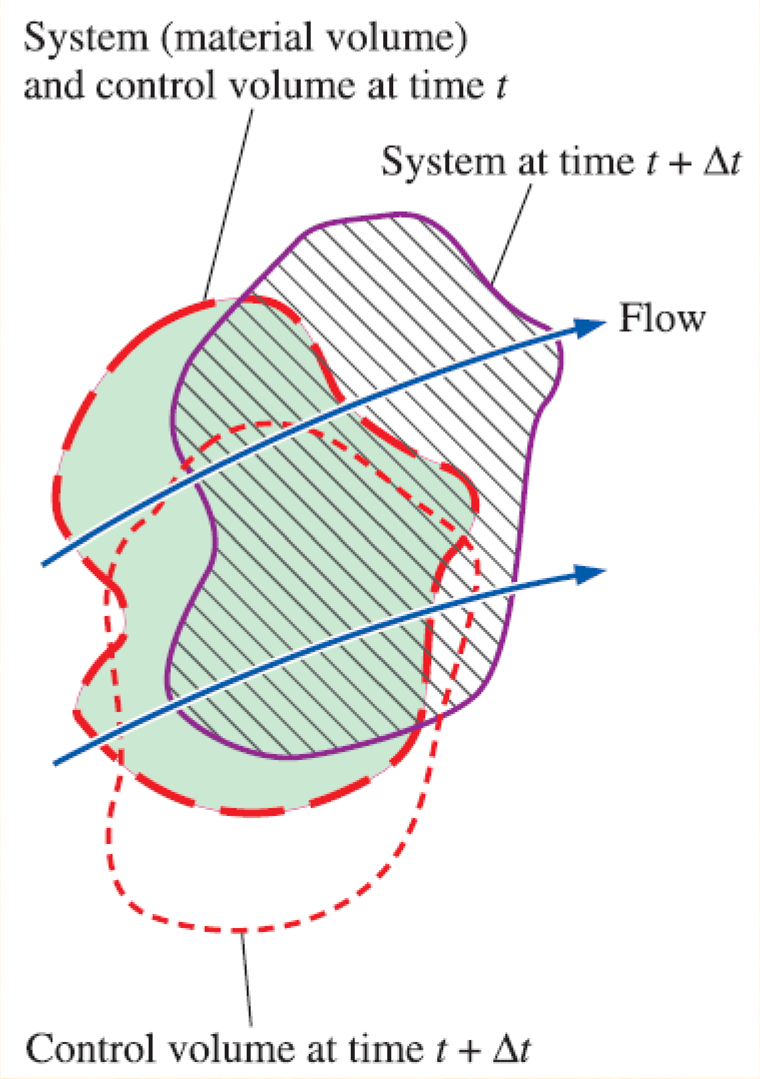
Three-dimensional Leibniz theorem applied to fluid flow:

$$\frac{d}{dt}\int_{V(t)} \rho b\, dV = \int_{V(t)} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{A(t)} \rho b\vec{V}_A \cdot \vec{n}\, dA$$

The *three-dimensional Leibniz theorem* is required when calculating the time derivative of a volume integral for which the volume itself moves and/or deforms with time. It turns out that the three-dimensional form of the Leibniz theorem can be used in an alternative derivation of the Reynolds transport theorem.



$$\textit{General RTT, nonfixed CV:} \quad \frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$



The material volume (system) and control volume occupy the same space at time *t* (the blue shaded area), but move and deform differently. At a later time they are *not* coincident.

EXAMPLE 4–11 Reynolds Transport Theorem in Terms of Relative Velocity

Beginning with the Leibniz theorem and the general Reynolds transport theorem for an arbitrarily moving and deforming control volume, Eq. 4–53, prove that Eq. 4–44 is valid.

SOLUTION Equation 4–44 is to be proven.

Analysis The general three-dimensional version of the Leibniz theorem, Eq. 4–50, applies to *any* volume. We choose to apply it to the control volume of interest, which can be moving and/or deforming differently than the material volume (Fig. 4–63). Setting G to ρb, Eq. 4–50 becomes

$$\frac{d}{dt}\int_{CV} \rho b \, dV = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b \vec{V}_{CS} \cdot \vec{n}\, dA \tag{1}$$

We solve Eq. 4–53 for the control volume integral,

$$\int_{CV} \frac{\partial}{\partial t}(\rho b)\, dV = \frac{dB_{sys}}{dt} - \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA \tag{2}$$

Substituting Eq. 2 into Eq. 1, we get

$$\frac{d}{dt}\int_{CV} \rho b \, dV = \frac{dB_{sys}}{dt} - \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA + \int_{CS} \rho b \vec{V}_{CS} \cdot \vec{n}\, dA \tag{3}$$

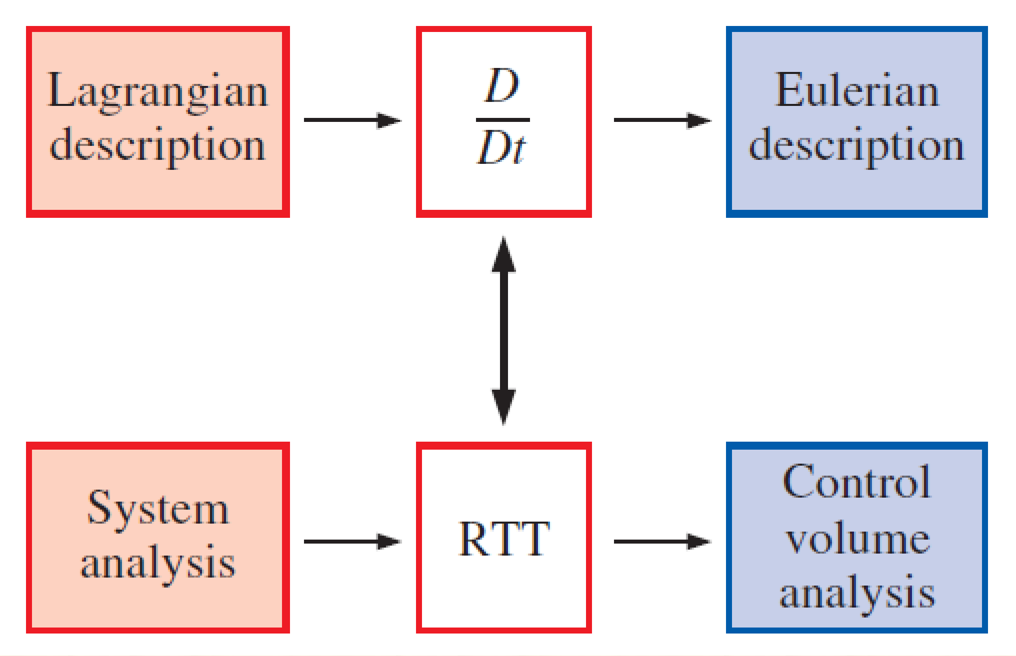
Combining the last two terms and rearranging,

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b (\vec{V} - \vec{V}_{CS}) \cdot \vec{n}\, dA \tag{4}$$

But recall that the relative velocity is defined by Eq. 4–43. Thus,

RTT in terms of relative velocity:
$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r \cdot \vec{n}\, dA \tag{5}$$

Relationship between Material Derivative and RTT



The Reynolds transport theorem for finite volumes (integral analysis) is analogous to the material derivative for infinitesimal volumes (differential analysis). In both cases, we transform from a Lagrangian or system viewpoint to an Eulerian or control volume viewpoint.

While the Reynolds transport theorem deals with finite-size control volumes and the material derivative deals with infinitesimal fluid particles, the same fundamental physical interpretation applies to both.

Just as the material derivative can be applied to any fluid property, scalar or vector, the Reynolds transport theorem can be applied to any scalar or vector property as well.

Summary

- Lagrangian and Eulerian Descriptions
  - ✓ Acceleration Field
  - ✓ Material Derivative
- Flow Patterns and Flow Visualization
  - ✓ Streamlines and Streamtubes, Pathlines,
  - ✓ Streaklines, Timelines
  - ✓ Refractive Flow Visualization Techniques
  - ✓ Surface Flow Visualization Techniques
- Plots of Fluid Flow Data
  - ✓ Vector Plots, Contour Plots
- Other Kinematic Descriptions
  - ✓ Types of Motion or Deformation of Fluid Elements
- Vorticity and Rotationality
  - ✓ Comparison of Two Circular Flows
- The Reynolds Transport Theorem
  - ✓ Alternate Derivation of the Reynolds Transport Theorem
  - ✓ Relationship between Material Derivative and RTT