

Why we study field Extension?
Field Extension

motivation:-

$$x^2 + 1 = 0 \quad \mathbb{Q} \text{ field}$$

$$x = \pm i \notin \mathbb{Q}$$

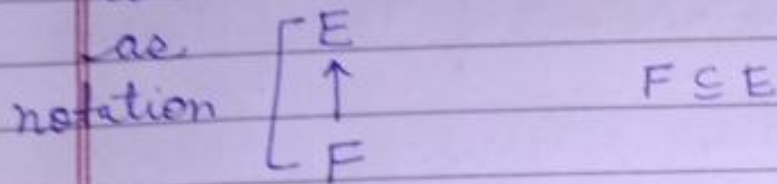
$x^2 + 1 \in \mathbb{Q}[x]$
 but roots do not belong in \mathbb{Q} .

$$x^2 + 1 = (x+i)(x-i)$$

$\mathbb{Q}[i] \quad \notin \mathbb{Q}[x] \quad \notin \mathbb{Q}[x]$

Definition:- If F is a subfield of E then E is said to be an extension field of F or simply extension of F .

We denote this extension



natural Question arises whether $x^2 + 1$ does not split into linear factors in any field other than $\mathbb{Q}[x]$.

s.t. $\mathbb{Q}[x] \subseteq E[x]$

Ex: \mathbb{R} is a subfield of \mathbb{C} hence \mathbb{C} is extension field of \mathbb{R} i.e.

$$\begin{array}{c} \mathbb{C} \\ | \\ \mathbb{R} \end{array}$$

other examples are

$$\begin{array}{cc} \mathbb{C} & \mathbb{R} \\ | & | \\ \mathbb{Q} & \mathbb{Q} \end{array} \text{ etc.}$$

Degree of Extension

Definition:- Let E be an extension of F . then E is vector space over F .
 $E(F) \rightarrow$ vector space over F .

The dimension of $E(F)$ is called the degree of extension and is denoted by $[E:F]$

$$d) [E:F] < \infty$$

that means if dimension of vector space $E(F)$ is finite then E is called

finite extension of F o.w. F Infinite extension.

Ex ①

Find degree of extensions of the following

① $[C:\mathbb{R}] = 2$

Solⁿ: $C[\mathbb{R}] \rightarrow$ v.s. $a+ib = a(1) + b(i) \quad \forall a, b \in \mathbb{R} \quad \dim = 2$

② $[\mathbb{R}:\mathbb{R}] \quad a \in \mathbb{R} \quad a = a(1) \quad \forall a \in \mathbb{R} \quad \text{basis} = \{1\}$

$\therefore [\mathbb{R}:\mathbb{R}] = 1$

③ $[C^2:C]$

④ $[\mathbb{R}:\mathbb{Q}]$

Solⁿ: ① $[C:\mathbb{R}]$

$a+ib \in C \quad a+ib = a(1) + b(i) \quad \forall a, b \in \mathbb{R}$

basis of $C(\mathbb{R}) = \{1, i\}$

$\therefore [C:\mathbb{R}] = 2$

In general $\dim C^n[\mathbb{R}] = 2n$

② $[\mathbb{R}:\mathbb{R}]$

basis = $\{1\} \quad \therefore [\mathbb{R}:\mathbb{R}] = 1$

③ $[C^2:C] = 2$

$(a+ib, c+id) \in C^2$

$(a+ib, c+id) = (a+ib)(1, 0) + (c+id)(0, 1)$

basis $\{(1, 0), (0, 1)\}$

(d) $[\mathbb{R}:\mathbb{Q}] = \infty$
 $a \in \mathbb{R}; a = a(1)$
 \downarrow
 \mathbb{Q}

none of 1 can not generate \mathbb{R} over \mathbb{Q} .

$[\mathbb{R}:\mathbb{Q}] = \infty$

notes:

$[\mathbb{C}:\mathbb{R}], [\mathbb{R}:\mathbb{R}], [\mathbb{C}^2:\mathbb{C}]$ are finite extensions, while $[\mathbb{R}:\mathbb{Q}]$ is infinite extension.

Splitting Fields

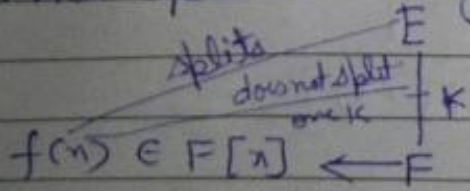
Definition Let E be an extension field of F i.e. E and let $f(x) \in F[x]$. We say f that $f(x)$ splits in E if

$f(x)$ can be factored as a product of linear factors in $E[x]$ i.e.

See \star
 E
 $|$
 $F \leftarrow f(x) \in F[x]$
 $\Rightarrow f(x)$
 splits over
 E
 iff $f(x) = (x-\alpha_1)(x-\alpha_2) \dots (x-\alpha_n)$
 $\in E[x]$

$f(x) = c(x-\alpha_1)(x-\alpha_2) \dots (x-\alpha_i), \alpha_i \in E$

we call E a splitting field for $f(x)$ over F if $f(x)$ splits in E but does not split in any proper subfield of E .



Examples: Find splitting field of

- ① $x^2+1 \in \mathbb{R}[x]$
- ② $x^2+1 \in \mathbb{Q}[x]$

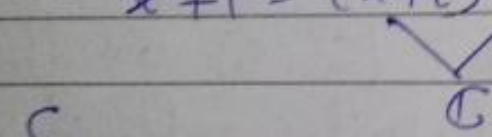
Solⁿ: (1) $x^2+1 \in \mathbb{R}[x]$

we have to find splitting field E of x^2+1 over \mathbb{R} but there does not exist any other field K such that $\mathbb{R} \subset K \subset E$

st: $\mathbb{R} \subset \mathbb{K} \subset E$

we know that

$$x^2+1 = (x+i)(x-i)$$



$\frac{\mathbb{C}}{\mathbb{R}}$ splitting field of $f(x) = x^2+1$ is \mathbb{C} .

degree of extension $[\mathbb{C}:\mathbb{R}] = 2$

(2)

$$x^2+1 \in \mathbb{Q}[x]$$

$$x^2+1 = (x+i)(x-i)$$

\downarrow \downarrow
 $-i$ i



$\mathbb{Q}[i] = \{a+ib : a, b \in \mathbb{Q}\}$, $\mathbb{C} = \{a+ib : a, b \in \mathbb{R}\}$
 $i \in \mathbb{Q}[i]$, $-i \in \mathbb{Q}[i]$

$$x^2+1 = (x+i)(x-i)$$

$\in \mathbb{Q}[i] \quad \in \mathbb{Q}[i]$

$\mathbb{Q} \subset \mathbb{Q}[i] \subset \mathbb{R} \subset \mathbb{C}$

x^2+1 splits over $\mathbb{Q}[i]$
 \therefore splitting field of x^2+1 is $\mathbb{Q}[i]$

$$[\mathbb{Q}[i] : \mathbb{Q}] = 2$$

$a+ib$
 $a, b \in \mathbb{Q}$