$Q \times Z$, Q, Z both are integral domain

$(0,0) \neq) (1,0) \in Q \times Z$
$((0,0) \neq) (0,1) \in Q \times Z$
$(1,0)(0,1) = (0,0)$
$\therefore$ $Q \times Z$ is not integral domain.

$\boxed{\begin{matrix} \mathbb{C} \times Z_n \times \mathbb{R} \\ Q \times \mathbb{R} \\ Q \times \mathbb{C} \end{matrix}} \longrightarrow \overset{\neq 0}{(1,0,0)} \overset{\neq 0}{(0,1,0)} = 0$

are not integral domain

RESULT Z_n is AN INTEGRAL DOMAIN IFF $n = p$

eg Z_{45} [$n = 45 = 3 \times 15$ is not prime.
by result → so Z_{45} is not integral domain.
OR
$9, 5 \in Z_{45}$, $9, 5 \neq 0$ mod 45
but $9 \times 5 = 45$ mod $45 = 0$ so Z_{45} has zero divisor
so Z_{45} is not an integral domain.

RESULT : In zero divisor
$a \cdot b = 0$, the existence of b corresponds to a is not unique.

EX: $R = Z_{10}$
$2 \times 5 = 10 \bmod 10 = 0$
$2 \times 10 = 20 \bmod 10 = 0$

(... integral domain?)

Note T/F

If R_1 and R_2 are rings s.t. $R_1 \subseteq R_2$ and R_2 is an integral domain then R_1 is an integral domain
— (F) —— need not be true.

Counter $R_1 = \{0\} \to$ Ring, $R_2 = \mathbb{Q} \to$ Ring
$R_1 \subset R_2$ — R_2 is integral domain
but R_1 is not integral domain

Note

$$\boxed{\text{If } R_1 \text{ and } R_2 \text{ are CRU and } R_1 \subseteq R_2 \text{ and } R_2 \text{ is integral domain then } R_1 \text{ is also integral domain}}$$

$R_1 \subseteq R_2$ (R_2 ID) $\to$ also ID

Note If R_1 and R_2 are ring s.t. $R_1 \subseteq R_2$ and R_2 is not integral domain then R_1 is not integral domain —— (F)

Counter ① $\mathbb{Q} \times \{0\} \subseteq \mathbb{Q} \times \mathbb{R}$
$\mathbb{Q} \times \mathbb{R}$ → not integral domain
$\mathbb{Q} \times \{0\}$ → integral domain

② $2\mathbb{Z}_{10} = \{0, 2, 4, 6, 8\}$ is integral domain (6 → unity)
$2\mathbb{Z}_{10} \subset \mathbb{Z}_{10}$, $\mathbb{Z}_{10}$ is not integral domain
while $2\mathbb{Z}_{10}$ is integral domain.

Q $(\mathbb{Z}_n[i], +, \cdot)$ is integral domain or not?

$n = 4$: $(\mathbb{Z}_4[i], +, \cdot)$ → CRU
$n = 4 \neq p$
so not ID

$n = 5$: $(\mathbb{Z}_5[i], +, \cdot)$
$n = 5 = p$
$4 \nmid (5-3)$ so
$(\mathbb{Z}_5[i], +, \cdot)$ not ID

Counter: $(2+i)(2+4i)$
$= \cancel{4} + 10i \cancel{-4} = \underline{0}$

Result (need not be)
$(\mathbb{Z}_n[i], +, \cdot)$
if $n \neq p \Rightarrow$ has zero divisor
so not integral domain
if $n = p$,
① if $4 \mid p-3 \to$ has no zero divisor
so integral domain
② if $4 \nmid p-3 \to$ has zero divisor
→ not integral domain yes

RESULT $(Z_n[i],+,\cdot)$ is integral domain

if $n=p$ and p ~~$4\nmid(p-3)$~~ — $4/p-3$

$Z_4[i] \to$ ID.

RESULT For $Z_n[i]$

If $n \neq p$ then $Z_n[i]$ is not integral domain

if $n=p$

1) $4/p-3$ $Z_p[i]$ is integral domain

2) $4 \nmid p-3$ $Z_p[i]$ is not integral domain

$Z_3[i] \to$ ID

$Z_5[i] \to$ not ID

Ques $Z_{11}[i]$ is an integral domain? yes

$n=11=P$, $P-3=8$

$4/8$ so ID

Q $(Z_{41}[i],+,\cdot)$ is an ID? NO

$n=p=41$, $p-3=41-3=38$

$4 \nmid 38$ so not ID

Q $R \times Z \times Q[i]$

$(1,0,0) \times (0,1,0) = (0,0,0)$

so $R \times Z \times Q[i]$ has zero divisor so

not an integral domain

Q WOF value of n s.t. $\dfrac{Z[i]}{nZ[i]} \approx Z_n[i]$

is integral domain?

1) $n=2$ ✗	2) $n=13$ ✗	3) $n=19$ ✓	4) $n=7$ ✓
$n=2=p$ $4 \nmid p-4$ (not ID)	$n=13=p$ $p-3=10$ $4 \nmid 10$ so not ID	$n=19=p$ $p-3=16$ $4/16$ so ID	$n=7=p$ $p-3=4$ $4/p-3$ so ID