Zero Divisors:-

Ring$(R,+,\cdot)$ is CR

Let $(R,+,\cdot)$ is commutative ring. A non zero element $a \in R$ is said to be zero divisor if $\exists$ non zero element $b \in R$ s.t.

$a \cdot b = 0$.

the ring $(R,+,\cdot)$ has zero divisors:

$$\boxed{a \neq 0,\ b \neq 0,\ \Rightarrow ab = 0}$$

Ex ① Z_{18} has zero divisor.

$9 \neq 0,\ 9 \in Z_{18},\ 2 \neq 0,\ 2 \in Z_{18}$

$(9 \neq 0, 2 \neq 0)$ $9, 2 \in Z_{18} \Rightarrow 9 \times 2 = 18 \bmod 18 = 0$

$\therefore Z_{18}$ has zero divisors.

Ex ② $\boxed{R = Q \times Q[i]}$ has zero divisors

think lik this $\begin{cases} x = (1,0) \neq 0 & x \in Q \times Q[i] \\ y = (0,1) \neq 0 & y \in Q \times Q[i] \end{cases}$

$x \cdot y = (1,0)(0,1) = (0,0)$

$\therefore Q \times Q[i]$ has zero divisors.

Q $R = Z \times \mathbb{C} \times Q$ also has zero divisors.

take $0 \neq \begin{cases} (1,0,0) \in R \\ (0,1,0) \in R \\ (0,0,1) \in R \end{cases}$ but $(1,0,0)(0,1,0) = 0$

so $Z \times \mathbb{C} \times Q$ has zero divisors.

note $\boxed{\text{If } n = p \text{ then } Z_p \text{ has no zero divisors}}$

note $\boxed{Z / Z[i] / Q / Q[i] / R / C / Q(\sqrt{2}) / Z(\sqrt{2}) / Q(\sqrt{2})(\sqrt{3}) / Q(\omega) \text{ has no zero divisors.}}$

e.g $\mathbb{Z}_5, \mathbb{Z}_3, \mathbb{Z}_7, \mathbb{Z}_{11}$ has no zero divisors.

Q $(\mathbb{Z}_5[i], +, \cdot)$ has zero divisors?

$\mathbb{Z}_5[i] = \{a+ib \,;\, a, b \in \mathbb{Z}_5\}$

$2+i \neq 0,\ 2+i \in \mathbb{Z}_5(i)$

$2+4i \neq 0,\ 2+4i \in \mathbb{Z}_5(i)$

$(2+i)(2+4i) = \cancel{4} + 2i + 8i - \cancel{4} = 10i \bmod 5 = 0$

$\therefore \mathbb{Z}_5[i]$ has zero divisors.

note TRICK $\boxed{\text{If } 4 / p-3 \text{ then } \mathbb{Z}_p[i] \text{ has no zero divisors}}$

- $\mathbb{Z}_n[i]$ has zero-divisors when $n \neq p$
- $n = p$ — $4/p-3 \rightarrow$ no zero divisors
  — $4 \nmid p-3 \rightarrow$ zero divisors

e.g $\mathbb{Z}_{13}[i]$ has zero divisors

$\because n = 13 = p$, $4 / p-3$? $4 \nmid 10$

$\Rightarrow \mathbb{Z}_{13}[i]$ has zero divisors.

Units

Let $(R, +, \cdot)$ is ring. An element $a \in R$ is said to be unit of R if $\exists$ element $b \in R$ s.t. $\boxed{a \cdot b = \text{unity in } R}$

Set of all units of R is denoted by $U(R)$ and defined by $\boxed{U(R) = \{a \in R \,;\, a^{-1} \text{ exist in } R \text{ w.r.t multiplication}\}}$

e.g

$U(\mathbb{Z}) = \{1, -1\}$

$U(\mathbb{R}) = \mathbb{R} \setminus \{0\}$

- ☆ Let R be a finite non-zero commutative ring with unity then any non zero element of R is either zero divisor or unit element of R.

note If R is (not) finite commutative ring with unity then above result need not be true.

Counter $R = \mathbb{Z}$

e.g. $2 \in \mathbb{Z}$, 2 is not zero divisor.
2 is not unit element

it happens because $R = \mathbb{Z}$ is not finite CRU

note There exist ring R s.t. R has neither zero divisor nor unit element in R.

EX: $R = 2\mathbb{Z}$ is CR, $2\mathbb{Z} = \{0, \pm 2, \pm 4, \pm 6, \ldots$
R has no zero divisor, and no unit.

L-02 - INTEGRAL DOMAIN

ID ⇒ CRU

INTEGRAL DOMAIN A commutative ring with unity (CRU) is said to be integral domain if it has no zero divisor.

i.e $\boxed{a \neq 0, b \neq 0, a, b \in R \Rightarrow ab \neq 0}$

OR

if $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$

In integral domain, product of two non zero elements is never zero. If product zero then either of the element is zero.

Ex: ① $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}[i]$, $\mathbb{Z}[i]$ etc

here if $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

→ all are ID

all are CRU

② $\mathbb{Z}_{18}$ is integral domain? → NO

$9 \times 2 = 0$ but neither $9 = 0$ nor $2 = 0$ in $\mathbb{Z}_{18}$

so $\mathbb{Z}_{18}$ is not an integral domain.

Ex (3) $R = Z_3[i]$ is an integral domain? (yes)

$Z_3[i] = \{a+ib ; a, b \in Z_3\}$

$\because Z_3[i] \to$ CRU and $ab = 0 \Rightarrow a = 0$ or $b = 0$

$\Rightarrow Z_3[i]$ is an integral domain

My work: let $a_1+ib_1 \neq 0$, $c_1+id_1 \neq 0$

then T.S.T $(a_1+ib_1)(c_1+id_1) \neq 0$

by Result

$n = 3$ = prime

$4 \nmid p3$ so no zero divisor

$\Rightarrow Z_3[i]$ is integral domain.

V.V.V.V. Imp for counter examples $Q \times \{0\}$ ← good counter can be though in situations.

Ques Show that $\underbrace{\overset{CR}{Q} \times \overset{CR}{\{0\}}}_{CR}$ is an integral domain.

$Q \times \{0\} \to$ ID $\Downarrow$ CRO

Soln $\boxed{Q \times \{0\} = \{(a,0) ; a \in Q\}}$

Let $x = (a,0) \in Q \times \{0\}$

$y = (b,0) \in Q \times \{0\}$

s.t $xy = 0$, T.S.T. $x = 0$ or $y = 0$

take $xy = 0$

$\Rightarrow (a,0)(b,0) = 0 \Rightarrow (ab, 0) = 0 \Rightarrow ab = 0$

$\Rightarrow a = 0$ or $b = 0$

$\therefore (a,0) = 0$ or $(b,0) = 0$

$\Rightarrow x = 0$ or $y = 0$.

$\therefore$ $Q \times \{0\}$ is integral domain.

smly

$\left.\begin{array}{l} R \times \{0\} \\ C \times \{0\} \\ Z \times \{0\} \\ Z[i] \times \{0\} \\ Z \times \{0\} \times \{0\} \end{array}\right\}$ are integral domain $\Rightarrow$ CRU also

note: If R_1, R_2 both integral domain then $R_1 \times R_2$ never be integral domain

$R_1, R_2 = $ ID $\Longrightarrow R_1 \times R_2$ not ID