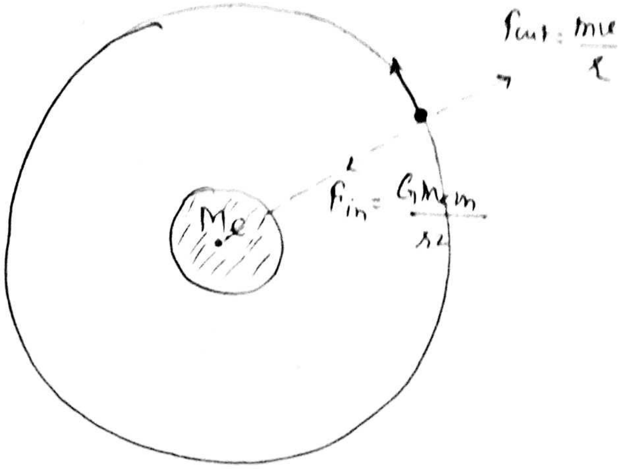
Let us consider a satellite of mass m travelling with velocity v in the plane of orbit.



The acceleration a, due to gravity at a distance r from centre of earth.

$$a = \mu / r^2 \quad km/sec^2$$

where μ is a constant called kepler's constant,

$$\mu = GM_e$$

G = universal gravitational constant

$$= 6.67 \times 10^{-11} \ Nm^2/kg^2$$

M_e = mass of earth.

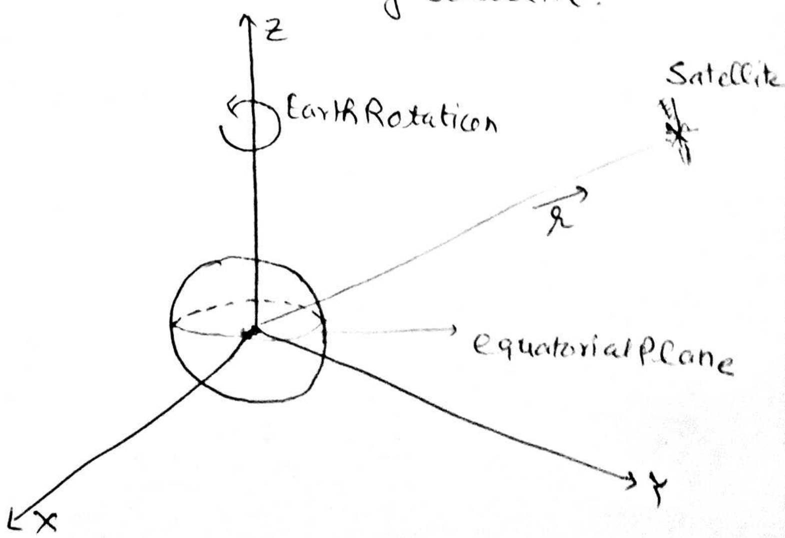
$$\mu = 3.86$$
$$= 3.986 \times 10^5 \ kg\ km^3/sec^2$$

There are two force working on satellite-

$F_{in} = \frac{GM_e m}{r^2}$ ⟶ Deu to gravity of earth &

$F_{out} = \frac{mv^2}{r}$ ⟶ Deu to rotational motion of Satellite.

Now the consider satellite in co-ordinate system as shown in the figure. The mass of Satellite with mass m located at a vector distance r from the centre of earth. The gravitation force F on.



the satellite given by.

$$\vec{F} = -\frac{G M_e m \vec{r}}{r^3} \quad \text{------------ (i)}$$

where M_e = mass of earth and G = universal gravitational Constant
$= 6.672 \times 10^{-11}\ Nm^2/kg^2$

But Force = mass × acceleration. Hence-

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} \quad \text{------------ (ii)}$$

Comparing equation (i) & (ii), we get

$$-\frac{G M_e m \vec{r}}{r^3} = \cancel{m}\frac{d^2\vec{r}}{dt^2}$$

$$\Rightarrow \quad -\frac{G M_e \vec{r}}{r^3} = \frac{d^2\vec{r}}{dt^2}$$

Or,
$$-\frac{\mu \vec{r}}{r^3} = \frac{d^2\vec{r}}{dt^2}$$

Or
$$\boxed{\frac{d^2\vec{r}}{dt^2} + \frac{\vec{r}}{r^3}\mu = 0} \quad \text{------------ (iii)}$$

Equation (iii) is a second-Order differential Equation and Solution of this differential equation involve six-constants called orbital element.

The orbit discribe by these orbital elements can be shown to lie in a plane and to have a constant angular momentum.

$$\frac{d^2\vec{r}}{dt^2} + \frac{\vec{r}}{r^3}\mu = 0$$

$$\Rightarrow \quad \vec{r} \times \frac{d^2\vec{r}}{dt^2} + \frac{\vec{r} \times \vec{r}}{r^3}\mu = 0$$

$$\Rightarrow \quad \vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0$$

Or $\left[\frac{d}{dt}\ \vec{r} \times \frac{d\vec{r}}{dt}\right] = 0$

$$\frac{d}{dt}\left[\vec{r} \times \frac{d\vec{r}}{dt}\right] = 0$$

$$\Rightarrow \quad \vec{r} \times \frac{d\vec{r}}{dt} = \text{Constant} = \vec{h}$$

$\vec{h}$ is a constant and called "Orbital angular momentum" of satellite.

For solving equation(iii) select a different set of co-ordinates such that unit vectors in the three axes are constant. This co-ordinate system uses the plane of satellite's orbit as the refference plane.

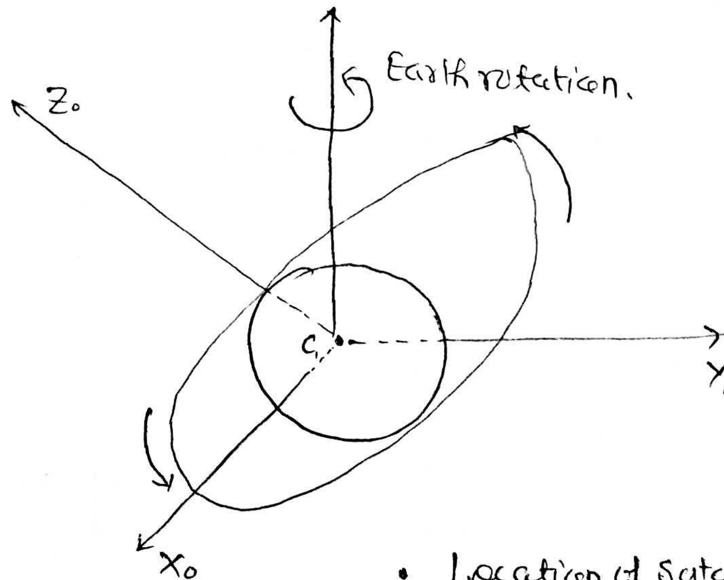
Now the equation(iii) can be written as-

in the terms of (x_0, y_0, z_0) -

$$\left(\frac{d^2 x_0}{dt^2}\right)\hat{x}_0 + \left(\frac{d^2 y_0}{dt^2}\right)\hat{y}_0 + \frac{\mu\,(x_0 \hat{x}_0 + y_0 \hat{y}_0)}{(x_0^2 + y_0^2)^{3/2}} = 0 \quad \text{------------ (iv)}$$

We can solve equation (iv) easily in polar co-ordinate system rather than certissian co-ordinate system.

The polar co-ordinate system shown in the figure and using transformations are given bellow-



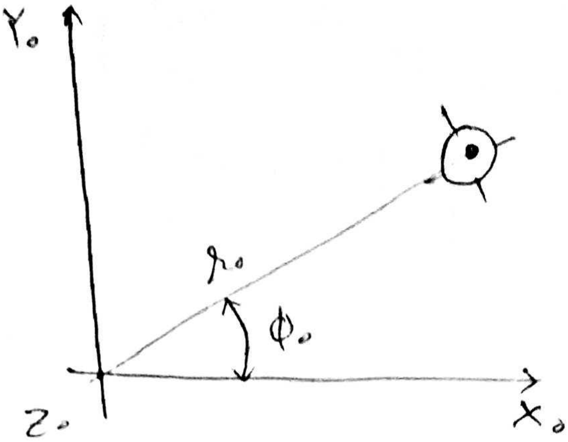
• Location of satellite in co-ordinate system. (X_0, Y_0, Z_0) •

$x_0 = r_0 \cos\phi_0$

$y_0 = r_0 \sin\phi_0$

$\hat{x}_0 = \hat{r}_0 \cos\phi_0 - \hat{\phi}_0 \sin\phi_0$

$\hat{y}_0 = \hat{\phi}_0 \cos\phi_0 + \hat{r}_0 \sin\phi_0$



→ Polar Co-ordinate System in the plane of satellite ←

Converting equation (iii) in the terms of polar co-ordinate System and equating vector components of r_0 & ϕ_0, the equation (iv) becomes -

$$\frac{d^2 r_0}{dt^2} - r_0\left(\frac{d\phi_0}{dt}\right) = -\frac{\mu}{r_0^2} \quad \text{------ (vi)}$$

and,

$$r_0\left(\frac{d^2\phi_0}{dt^2}\right) + 2\left(\frac{dr_0}{dt}\right)\left(\frac{d\phi_0}{dt}\right) = 0 \quad \text{------ (vi)}$$

using standard mathematical procedures, we can develop an equation for the radius of the satellite's orbit, r_0 -

$$\boxed{r_0 = \frac{p}{1 + e\cos(\phi_0 - \theta_0)}} \quad \text{------ (*)}$$

where θ_0 = Constant

e = eccentricity of the ellipse

p = Semilatus rectum of ellipse.

$p = \frac{h^2}{\mu}$

h = magnitude of orbital angular momentum of the Satlelite Satellite.

$e = \frac{h^2 c}{\mu}$