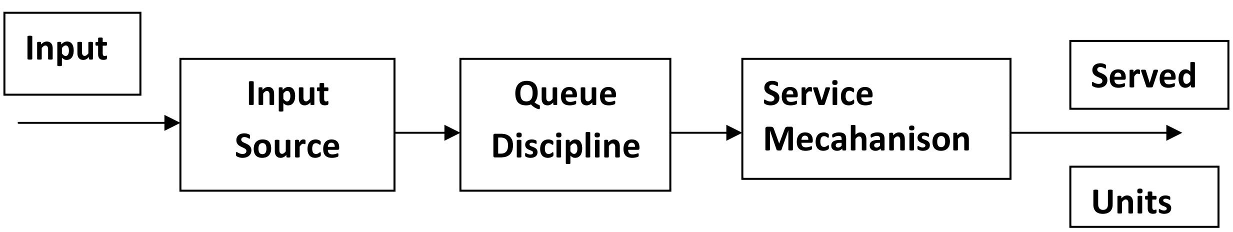
Queuing Theory

The Basic structure of queuing model

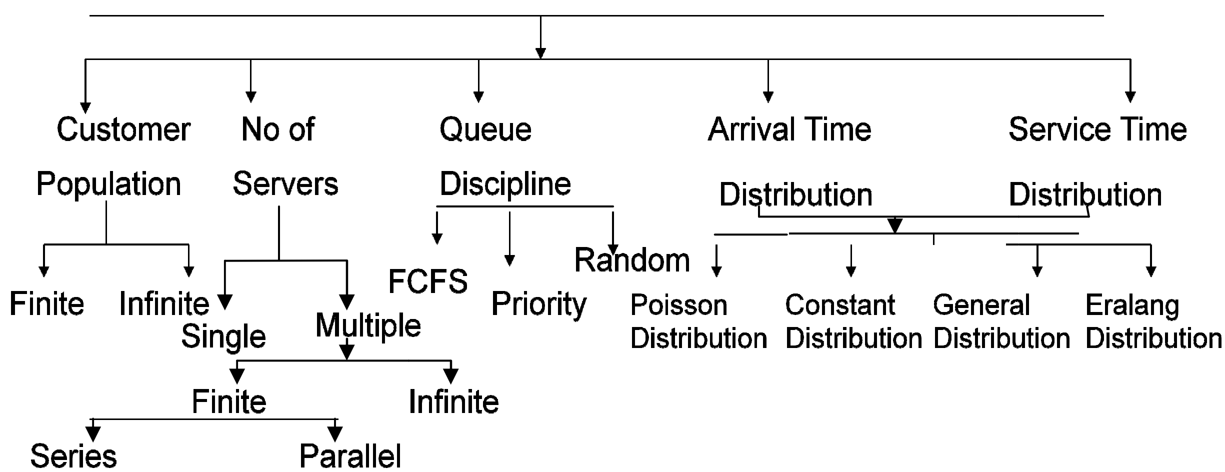
Introduction

Queues are a part of everyday life. We all wait in queues to buy a movie ticket, to make bank deposit, pay for groceries, mail a package, obtain a food in a cafeteria, to have ride in an amusement park and have become adjustment to wait but still get annoyed by unusually long waits.

The Queuing models are very helpful for determining how to operate a queuing system in the most effective way if too much service capacity to operate the system involves excessive costs. The models enable finding an appropriate balance between the cost of service and the amount of waiting.



Information required to solve the queuing problem:



Characteristics of the queuing system-:

(a) Input source

(b) Queue discipline

(c) Service mechanism

(a) Input source

One characteristic of the input source is the size. The size is the total number of units that might require service from time to time. It may be assumed to be finite or infinite.

The customer assumption is that they generate according to 'Poisson Distribution' at a certain average rate

Therefore, the equivalent assumption is that they generate according to exponential distribution between consecutive arrivals. To solve the problems use & assume customer population as ∞

(b) Queue Discipline

A queue is characterized by maximum permissible number of units that it contains. Queues are called finite or infinite, according to whether number is finite or infinite. The service discipline refers to the order in which number of queues are selected for service.

Ex: It may be FIFO, random or priority; FIFO is usually assumed unless stated otherwise.

(c) Service mechanism

This consists of one or more service facilities each of which contains one or more parallel service channel. If there is more than one service facility, the arrival unit may receive the service from a sequence of service channels.

At a given facility, the arrival enters at the service facility and is completely served by that server. The time elapsed from the commencement of the service to its completion for an unit at the service facility is known as service time usually, service time follows as exponential distribution.

Classification of queuing models using kendal & Lee notations

Generally, any queuing models may be completely specified in the following symbolic form

$$a / b / c \; : \; d / e$$

a $\longrightarrow$ Type of distribution of inter – arrival time

b $\longrightarrow$ Type of distribution of inter – service time

c $\longrightarrow$ Number of servers

d $\longrightarrow$ Capacity of the system

e $\longrightarrow$ Queue discipline

M $\longrightarrow$ Arrival time follows Poisson distribution and service time follows an exponential distribution.

Model I : M / M / 1 : ∞ / FCFS

Where M Arrival time follows a Poisson distribution

M $\longrightarrow$ Service time follows a exponential distribution

1 $\longrightarrow$ Single service model

∞ $\longrightarrow$ Capacity of the system is infinite

FCFS $\longrightarrow$ Queue discipline is first come first served

RO

Formulas: 1. Utilization factor traffic intensity /

Utilization parameter / Busy period

$$\rho = \frac{\lambda}{\mu}$$

Where λ = Mean arrival rate ; μ= mean service rate

Note : $\mu > \lambda$ in single server model only

2. Probability that exactly zero units are in the system

$$P_o = 1 - \frac{\lambda}{\mu}$$

3. Probability that exactly ‘n’ units in the system

$$P_n = P_o \left(\frac{\lambda}{\mu} \right)^n$$

4. Probability that n or more units in the system

$$P_{n\ or\ more} = \left(\frac{\lambda}{\mu} \right)^n$$

more then ‘n’ means n should be n+1

5. Expected number of units in the queue / queue length

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

6. Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

7. Expected number of units in the system

$$L = L_q + \frac{\lambda}{\mu}$$

8. Expected waiting time in the system

$$W = W_q + \frac{1}{\mu}$$

9. Expected number of units in queue that from time to time – (OR) non - empty queue size

$$D = \frac{\mu}{\mu - \lambda}$$

10. Probability that an arrival will have to wait in the queue for service

Probability = 1 - P_o

11. Probability that an arrival will have to wait in the queue more than w (where w > o), the waiting time in the queue

$$\text{Probability} = \left(\frac{\lambda}{\mu}\right) e^{(\lambda-\mu)w}$$

12. Probability that an arrival will have to wait more than v(v > o) waiting time in the system is

$$= e^{(\lambda-\mu)^{v}}$$

13. Probability that an arrival will not have to wait in the queue for service = P_o

Model 1 - Problems

1. Arrivals at a telephone both are considered to be Poisson at an average time of 8 min between our arrival and the next. The length of the phone call is distributed exponentially, with a mean of 4 min.

Determine

(a) Expected fraction of the day that the phone will be in use.

(b) Expected number of units in the queue

(c) Expected waiting time in the queue.

(d) Expected number of units in the system.

(e) Expected waiting time in the system

(f) Expected number of units in queue that from time to time.

Solution:

The mean arrival rate = λ = 1/8 x 60 = 7.5 / hour.

The mean service = μ = x 60 = 15 / hour.

a) Fraction of the day that the phone will be in use

$$\rho = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

(b) The expected number pf units in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{7.5^2}{15(15 - 7.5)}$$

$$L_q = 0.5\,(units)\,person$$

(c) Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{0.5}{7.5} = 0.066\ hrs$$

(d) Expected number of units in the system:-

$L = L_q + \lambda / \mu$

$= 0.5+0.5$

$L = 1$ person

(e) Expected waiting time in the system

$$W = Wq + \frac{1}{\mu}$$

$$= \ 0.066\ + \frac{1}{15}\ =\ 0.133$$

(f) Expected number of units in the queue that form from time to time:-

$$D = \frac{\mu}{\mu - \lambda}$$

$$= \ \frac{15}{15 - 7.5}\ = 2\ \ persons$$

2) In a self service store with one cashier, 8 customers arrive on an average of every 5 mins. and the cashier can serve 10 in 5 mins. If both arrival and service time are exponentially distributed, then determine

a) Average number of customer waiting in the queue for average.

b) Expected waiting time in the queue

c) What is the probability of having more than 6 customers In the system

Solution:

Mean arrival rate =λ= =1.6 x 60

= 96 / hour

Mean service rate = μ = x 60

= 120 / hour.

(a) Average number of customers waiting in queue for service

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{96^2}{120(120-96)}$$

$$L_q = 3.2\ customers$$

(b) Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{3.2}{96} = 0.033$$

(c) Probability of having more than 6 customers in the system

$$P_{6\ or\ more} = \left(\frac{\lambda}{\mu}\right)^n \qquad where \quad n = 7$$

$$= \left(\frac{96}{120}\right)^7 = 0.209$$

3) Consider a box office ticket window being manned by a single server. Customer arrives to purchase ticket according to Poisson input process with a mean rate of 30/hr. the time required to serve a customer has an ED with a mean of 90 seconds determine:

(a) Mean queue length.
(b) Mean waiting time in the system.
(c) The probability of the customer waiting in the queue for more than 10min.

(d) The fraction of the time for which the server is busy.

Solution:

The mean arrival rate = $\lambda = 30 / hr$

The mean service rate

$$\mu = \frac{1}{90} \times 60 \times 60$$
$$= 40 / hr$$

(a) Mean queue length

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \quad \frac{30^2}{40(40-30)} \quad = 2.25\ customers$$

(b) Mean waiting time in the system

$$W = W_q + \frac{1}{\mu}$$
$$= \frac{L_q}{\lambda} + \frac{1}{\mu}$$
$$= \frac{2.25}{30} + \frac{1}{40}$$
$$= 0.1 hr$$

(c) Probability of the customer waiting in queue for more than 10min.

$$W\frac{10}{60} \quad = 1/6hour$$

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^{e^{(\lambda-\mu)w}}$$

$$= \left(\frac{30}{40}\right)^{e^{(30-40)1/6}}$$

$$P_{ro} = 0.1416$$

(d) Fraction of time the serve is busy

$$\rho = \frac{\lambda}{\mu}$$

$$= \frac{40}{30}$$

$$= 0.75hr$$

4) A T.V repairman repair the sets in the order in which they arrive and expects that the time required to repair a set has an ED with mean 30mins. The sets arrive in a Poisson fashion at an average rate of 10/8 hrs a day.

(a) What is the expected idle time / day for the repairman? (0.375x8)

b) How many TV sets will be there awaiting for the repair? (1.04)

Solution

Mean arrival rate = λ = $\frac{10}{8}$ hours

Mean service rate = μ = $\frac{1}{30}$ x 60 = 2 hours

(a) Expected idle time / day of the repair

Busy Period = $\frac{\lambda}{\mu}$ = $\frac{1.25}{2}$ = 0.625 hour

$$\therefore idle\ time = P_o = 1 - \frac{\lambda}{\mu} = 1 - 0.625 = 0.375$$

$$\therefore idle\ time / day = 0.375 \times 8 = 3\ hrs / day$$

(b) Number of T.V sets awaiting for the repair:-

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{1.25^2}{2(2 - 1.25)} = 1.04$$